APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 17, no. 2(2016), 185-198
doi:10.4995/agt.2016.5660

Best proximity point for $\mathcal{Z}$-contraction and Suzuki type $\mathcal{Z}$-contraction mappings with an application to fractional calculus

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Abstract

In this article, we introduced the best proximity point theorems for $\mathcal{Z}$-contraction and Suzuki type $\mathcal{Z}$-contraction in the setting of complete metric spaces. Also by the help of weak P-property and P-property, we proved existence and uniqueness of best proximity point. There is a simple example to show the validity of our results. Our results extended and unify many existing results in the literature. Moreover, an application to fractional order functional differential equation is discussed.

2010 MSC: *54H25; 34A08.*

Keywords: *best proximity point; weak P-property; Suzuki type $\mathcal{Z}$-contraction; functional differential equation.*

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Received 26 April 2016 – Accepted 23 June 2016

1. Introduction

When we study about fixed points of different mappings satisfying certain conditions, then it is observed that this theory has enormous applications in various branches of mathematics and mathematical sciences and hence become the source of inspiration for many researchers and mathematicians working in the metric fixed point theory (see for instant [5, 16, 12, 26]). When a self mapping in a metric space has no fixed points, then it could be interesting to study the existence and uniqueness of some points that minimize the distance between the point and its corresponding image. These points are known as best proximity points. Best proximity points theorems for several types of non-self mappings have been derived in [1], [2], [3], [6], [7], [8], [10], [9] and [24]. The best proximity points were introduced by [13] and modified by Sadiq Basha in [7]. The results about best proximity point theory have been found very briefly in the work of [6] to [9]. Now, after the new generalization of Banach contraction principle given by khoj. et al. in [15] by defining a notion of $\mathcal{Z}$-contraction, after that Kumam et. al. in [16] introduced Suzuki type $\mathcal{Z}$-contraction and unified many fixed point results. Some recent contribution in this field can be found in ([18, 17, 4, 20, 21, 22, 23]). Because of its importance in nonlinear analysis, we extend these generalizations and contractions to find out the unique best proximity point in metric spaces and introduced these notions for non self mappings in the light of Yaq. et al. [25] by using some suitable properties. Some examples and an application to fractional order functional differential equation is given to illustrate the usability of new theory.

2. Preliminaries

In this section, we collect some notations and notions which will be used throughout the rest of this work.
Let A and B be two nonempty subsets of a metric space (X, d). We will use the following notations:

$$d(A, B) := \inf\{d(a, b) : a \in A, b \in B\};$$
$$A_0 := \{a \in A : d(a, b) = d(A, B) \text{ for some } b \in B\};$$
$$B_0 := \{b \in B : d(a, b) = d(A, B) \text{ for some } a \in A\}.$$

Definition 2.1. An element $x^* \in A$ is said to be a best proximity point of the non-self-mapping $T : A \to B$ if it satisfies the condition that $d(x^*, Tx^*) = d(A, B)$.

Remark 2.2. It can be observed that a best proximity reduces to a fixed point if the underlying mapping is a self-mapping.

Definition 2.3 ([15])**.** Let $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a mapping, then ζ is called a simulation function if it satisfies the following conditions:

(1) $\zeta(0, 0) = 0$;
(2) $\zeta(t, s) < s - t$ for $t, s > 0$;

(3) if $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0,$$

then

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

We denote the set of all simulation functions by $\mathcal{Z}$.

Definition 2.4 ([15]). Let (X, d) be a metric space, $F : X \to X$ is a mapping and $\zeta \in \mathcal{Z}$. Then F is called a $\mathcal{Z}$-contraction with respect to ζ if the following condition holds:

$$\zeta(d(Fx, Fy), d(x, y)) \geq 0 \tag{2.1}$$

where $x, y \in X$, with $x \neq y$.

Definition 2.5 ([16]). Let (X, d) be a metric space, $F : X \to X$ is a mapping and $\zeta \in \mathcal{Z}$. Then F is called a Suzuki type $\mathcal{Z}$-contraction with respect to ζ if the following condition holds:

$$\frac{1}{2}d(x, Fx) < d(x, y) \Rightarrow \zeta(d(Fx, Fy), d(x, y)) \geq 0 \tag{2.2}$$

where $x, y \in X$, with $x \neq y$.

Definition 2.6 ([19]). Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \phi$. Then the pair (A, B) is said to have the P-property if and only if

$$d(x_1, y_1) = d(A, B) \textit{ and } d(x_2, y_2) = d(A, B) \Rightarrow d(x_1, x_2) = d(y_1, y_2),$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

Definition 2.7 ([27]). Let (A,B) be a pair of nonempty subsets of a metric space (X,d) with $A_0 \neq \varnothing$. Then the pair (A,B) is said to have weak P-property if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$

$$\left.\begin{array}{ll} d(x_1, y_1) & = d(A, B) \\ d(x_2, y_2) & = d(A, B) \end{array}\right\} \Rightarrow d(x_1, x_2) \leq d(y_1, y_2).$$

Theorem 2.8 ([16]). *Let (X, d) be a complete metric space. Define a mapping $F : X \to X$ satisfying the following conditions:*

(1) *F is Suzuki type $\mathcal{Z}$-contraction with respect to ζ;*
(2) *for every bounded Picard sequence there exists a natural number k such that $\frac{1}{2}d(x_{m_k}, x_{m_k+1}) < d(x_{m_k}, x_{n_k})$ for $m_k > n_k \geq k$.*

Then there exists unique fixed point in X and the Picard iteration sequence $\{x_n\}$ defined by

$$x_n = Fx_{n-1}, n = 1, 2, ...$$

converges to a fixed point of F,

Remark 2.9 ([15]). Every $\mathcal{Z}$-contraction is contractive and hence Banach contraction.

Theorem 2.10 ([5]). *Let (X, d) be a complete metric space. Then every contraction mapping has a unique fixed point. It is known as Banach contraction principle.*

3. Main Results

In this section, we will introduced the notion of generalized contraction principle for non self mappings by combining Suzuki and $\mathcal{Z}$- contraction mappings and will find the unique best proximity point.

Definition 3.1. Let (X, d) be a metric space, $F : A \to B$ is a mapping and $\zeta \in \mathcal{Z}$. Then F is called a $\mathcal{Z}$-contraction with respect to ζ if the following condition holds:

$$\zeta(d(Fx, Fy), d(x, y)) \geq 0 \tag{3.1}$$

where $A, B \subseteq X$ and $x, y \in A$, with $x \neq y$.

Definition 3.2. Let (X, d) be a metric space, $F : A \to B$ is a mapping and $\zeta \in \mathcal{Z}$. Then F is called a Suzuki type $\mathcal{Z}$-contraction with respect to ζ if the following condition holds:

$$\frac{1}{2}d(x, Fx) < d(x, y) \Rightarrow \zeta(d(Fx, Fy), d(x, y)) \geq 0 \tag{3.2}$$

where $A, B \subseteq X$ and $x, y \in A$, with $x \neq y$.

Remark 3.3. Since the definition of simulation function implies that $\zeta(t, s) < 0$ for all $t \geq s > 0$. Therefore F is Suzuki type $\mathcal{Z}$ contraction with respect to ζ, then

$$\frac{1}{2}d(x, Fx) < d(x, y) \Rightarrow d(Fx, Fy) < d(x, y)$$

for any distinct $x, y \in A$.

Remark 3.4. Every Suzuki type $\mathcal{Z}$-contraction is also a $\mathcal{Z}$-contraction.

Now, we are in a position to prove best proximity point theorems for $\mathcal{Z}$ and Suzuki type $\mathcal{Z}$-contractions in metric spaces.

Theorem 3.5. *Let (A, B) be the pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Define a mapping $F : A \to B$ satisfying the following conditions:*

(1) *F is $\mathcal{Z}$-contraction with $F(A_0) \subseteq B_0$;*
(2) *the pair (A, B) has weak P-property.*

Then there exists unique best proximity point in A and the iteration sequence $\{x_{2n}\}$ defined by

$$x_{2n+1} = Fx_{2n}, \quad d(x_{2n+2}, x_{2n+1}) = d(A, B), \quad n = 0, 1, 2, ...$$

converges, to x^*, for every $x_0 \in A_0$.

Proof. First of all, we have to show that B_0 is closed. For this, let us take $\{y_n\} \subseteq B_0$ a sequence such that $y_n \to t \in B$. Since the pair (A, B) has weak P-property, it follows from the weak P-property that

$$d(y_n, y_m) \to 0 \Rightarrow d(x_n, x_m) \to 0,$$

as $m, n \to \infty$, and $x_n, x_m \in A_0$ and $d(x_n, y_n) = d(x_m, y_m) = d(A, B)$.
Thus $\{x_n\}$ is a Cauchy sequence and converges strongly to a point $s \in A$. By the continuity of the metric d, we have $d(s, t) = d(A, B)$, that is $t \in B_0$ and hence B_0 is closed.
Let $\overline{A_0}$ be the closure of A_0; now we have to prove that $F(\overline{A_0}) \subseteq B_0$. If we take $x \in \overline{A_0} \setminus A_0$, then there exists a sequence $\{x_n\} \subseteq A_0$ such that $x_n \to x$. By the continuity of F and the closeness of B_0, we get as $Fx = \lim_{n\to\infty} Fx_n \in B_0$. That is, $F(\overline{A_0}) \subseteq B_0$.
Since F is $\mathcal{Z}$-contraction,which implies that

$$\begin{aligned} 0 &\leq \zeta(d(Fx_1, Fx_2), d(x_1, x_2)) \\ &< d(x_1, x_2) - d(Fx_1, Fx_2), \end{aligned}$$

implies that

$$d(Fx_1, Fx_2) < d(x_1, x_2). \tag{3.3}$$

Define an operator $P_{A_0} : F(\overline{A_0}) \to A_0$, by $P_{\overline{A_0}} y = \{x \in A_0 : d(x, y) = d(A, B)\}$. Since the pair (A, B) has weak P-property and using (5), we have

$$\begin{aligned} d(P_{A_0}Fx_1, P_{A_0}Fx_2) &\leq d(Fx_1, Fx_2) \\ &< d(x_1, x_2) \end{aligned}$$

for any $x_1, x_2 \in \overline{A_o}$. Hence $\zeta(d(P_{A_0}Fx_1, P_{A_0}Fx_2), d(x_1, x_2)) \geq 0$. So, $P_{A_0}F : \overline{A_0} \to \overline{A_0}$ is a $\mathcal{Z}$-contraction from complete metric subspace $\overline{A_0}$ into itself. Since by using Remark (2.1), every $\mathcal{Z}$-contraction is a contraction and hence a Banach contraction. Thus by using theorem (2.2), $P_{A_0}F$ has unique fixed point, that is $P_{A_0}Fx^* = x^* \in A_0$, which implies that

$$d(x^*, Fx^*) = d(A, B).$$

Therefore, x^* is unique in A_0 such that $d(x^*, Fx^*) = d(A, B)$. It is easily seen that x^* is unique one in A such that $d(x^*, Fx^*) = d(A, B)$. The Picard Iterative sequence

$$x_{n+1} = P_{A_0}Fx_n, \quad n = 0, 1, 2, ...$$

converges, for every $x_0 \in A_0$, to x^*. The iteration sequence $\{x_{2n}\}$, for $n = 0, 1, 2, ...$ defined by,

$$x_{2n+1} = Fx_{2n}, \quad d(x_{2n+2}, x_{2n+1}) = d(A, B), \quad n = 0, 1, 2, ...$$

is exactly a subsequence of $\{x_n\}$, so that it converges to x^*, for every $x_0 \in A_0$. □

Theorem 3.6. *Let (A, B) be the pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Define a mapping $F : A \to B$ satisfying the following conditions:*

(1) *F is Suzuki type $\mathcal{Z}$-contraction with $F(A_0) \subseteq B_0$;*
(2) *the pair (A, B) has the weak P-property.*

Then there exists unique x^* in A such that $d(x^*, Fx^*) = d(A, B)$ and the iteration sequence $\{x_{2n}\}$ defined by

$$x_{2n+1} = Fx_{2n}, \quad d(x_{2n+2}, x_{2n+1}) = d(A, B), \quad n = 0, 1, 2, ...$$

converges, for every $x_0 \in A_0$ to x^*.

Proof. First of all, we have to show that B_0 is closed. For this, let us take $\{y_n\} \subseteq B_0$ a sequence such that $y_n \to g \in B$. Since the pair (A, B) has weak P-property, it follows from weak P-property that

$$d(y_n, y_m) \to 0 \Rightarrow d(x_n, x_m) \to 0,$$

as $m, n \to \infty$, and $x_n, x_m \in A_0$ and $d(x_n, y_n) = d(x_m, y_m) = d(A, B)$.
Thus $\{x_n\}$ is a Cauchy sequence and converges strongly to a point $f \in A$. By the continuity of the metric d, we have $d(f, g) = d(A, B)$, that is $g \in B_0$ and hence B_0 is closed.
Let $\overline{A_0}$ be the closure of A_0; now we have to prove that $F(\overline{A_0}) \subseteq B_0$. If we take $x \in \overline{A_0} \setminus A_0$, then there exists a sequence $\{x_n\} \subseteq A_0$ such that $x_n \to x$. By the continuity of F and the closeness of B_0, we get as $Fx = \lim_{n\to\infty} Fx_n \in B_0$. That is, $F(\overline{A_0}) \subseteq B_0$.
Define an operator $P_{A_0} : F(\overline{A_0}) \to A_0$, by $P_{\overline{A_0}}y = \{x \in A_0 : d(x, y) = d(A, B)\}$. Since F is Suzuki type $\mathcal{Z}$-contraction, such that for $\frac{1}{2}d(x_1, Fx_1) < d(x_1, y_1)$, we have

$$\zeta(d(Fx_1, Fy_1), d(x_1, y_1)) \geq 0.$$

Now, we claim that $P_{A_0}F$ is Suzuki type $\mathcal{Z}$-contraction. For this, we have to prove that $\frac{1}{2}d(x_1, P_{A_0}Fx_1) < d(x_1, y_1)$, for all $x, y \in A$. Since F is Suzuki type $\mathcal{Z}$-contraction, that is $d(Fx, Fy) < d(x, y)$. By using P-property, $P_{\overline{A_0}}y = \{x \in A_0 : d(x, y) = d(A, B)\}$ and triangular inequality, we obtain

$$\begin{aligned} \frac{1}{2}d(x_1, P_{A_0}Fx_1) &\leq \frac{1}{2}[d(x_1, y_1) + d(y_1, P_{A_0}Fx_1)] \\ &= \frac{1}{2}[d(x_1, y_1) + d(y_1, x_1)] \\ &= d(x_1, y_1) \\ &= d(Fx_1, Fy_1) \\ &< d(x_1, y_1) \end{aligned}$$

Hence,

$$\frac{1}{2}d(x_1, P_{A_0}Fx_1) < d(x_1, y_1). \tag{3.4}$$

for any $x_1, y_1 \in \overline{A_0}$. Which shows that $\zeta(d(P_{A_0}Fx_1, P_{A_0}Fy_1), d(x_1, y_1)) \geq 0$, where $x_1, y_1 \in \overline{A_0}$. Thus, $P_{A_0}F : \overline{A_0} \to \overline{A_0}$ is a Suzuki type $\mathcal{Z}$-contraction from complete metric subspace $\overline{A_0}$ into itself. Consequently, one may write by using the fact that $P_{A_0}F$ is a Suzuki type $\mathcal{Z}$-contraction and remark (3.1) as

$$\Rightarrow d(P_{A_0}Fx_1, P_{A_0}Fy_1) \quad < \quad d(x_1, y_1).$$

Then by using Theorem (2.1), $P_{A_0}F$ has unique fixed point, that is $P_{A_0}Fx^* = x^* \in A_0$, which implies that

$$d(x^*, Fx^*) = d(A, B).$$

Therefore, x^* is unique in A_0 such that $d(x^*, Fx^*) = d(A, B)$. It is easily seen that x^* is unique one in A such that $d(x^*, Fx^*) = d(A, B)$. The Picard Iterative sequence

$$x_{n+1} = P_{A_0}Fx_n, \quad n = 0, 1, 2, ...$$

converges, for every $x_0 \in A_0$, to x^*. The iteration sequence $\{x_{2n}\}$, for $n = 0, 1, 2, ...$ defined by,

$$x_{2n+1} = Fx_{2n}, \quad d(x_{2n+2}, x_{2n+1}) = d(A, B), \quad n = 0, 1, 2, ...$$

is exactly a subsequence of $\{x_n\}$, so that it converges to x^*, for every $x_0 \in A_0$. □

Corollary 3.7. *Let (X, d) be a complete metric space. Define a mapping $F : X \to X$ satisfying the following conditions:*

(1) *F is $\mathcal{Z}$-contraction.*

Then there exists unique fixed point in X and the iteration sequence $\{x_{2n}\}$ defined by

$$x_{2n+1} = Fx_{2n}, \quad d(x_{2n+2}, x_{2n+1}) = d(A, B), \quad n = 0, 1, 2, ...$$

converges to x^*, for every $x_0 \in A_0$.

Proof. Taking self mapping $A = B = X$ in Theorem (3.1), then we get desired result. □

Remark 3.8. By Taking self mapping in Theorem (3.2), we obtain Theorem (2.1).

There is an example to justify our results and remarks.

Example 3.9. Consider $X = \mathbb{R}^2$, with the usual metric d. Define the sets $A = \{(x, 1) : x \geq 0\}$ and $B = \{(x, 0) : x \geq 0\}$. Let $A_0 = A$ and $B_0 = B$ and clearly, the pair (A, B) has the P-property, also satisfies weak P-property. Also define $f : A \to B$ as:

$$f(x, 1) = (\frac{x^2}{x+1}, 0),$$

we take $A_0 = A \neq \varnothing, B_0 = B, f(A_0) \subseteq B_0$. Then,

$$\begin{aligned} d(f(x_1,1), f(x_2,1)) &= |\frac{x_1^2}{x_1+1} - \frac{x_2^2}{x_2+1}| \\ &= \frac{|x_1^2(x_2+1) - x_2^2(x_1+1)|}{(x_1+1)(x_2+1)} = \frac{|(x_1x_2+x_1+x_2)(x_2-x_1)|}{|(x_1+1)(x_2+1)|} \\ &= \frac{x_1x_2+x_1+x_2}{(x_1+1)(x_2+1)}|x_1-x_2| \\ &= \frac{x_1x_2+x_1+x_2}{x_1x_2+x_1+x_2+1}|x_1-x_2| \\ &< |x_1-x_2| = d((x_1,1),(x_2,1)). \end{aligned}$$

i.e. $d(f(x_1,1), f(x_2,1)) < d((x_1,1),(x_2,1))$,
which implies that $\zeta(d(f(x_1,1), f(x_2,1)), d((x_1,1),(x_2,1))) \geq 0$, i.e. f is ζ-contraction. Thus, all the conditions of the Theorem (3.1) are satisfied, and the conclusion of that theorem is also correct, that is, f has a unique best proximity point $z^* = (0,1) \in A_0$ such that $d(z^*, fz^*) = d((0,1),(0,0)) = d(A,B) = 1$.
On the other hand, it is clear that the iteration sequence $\{z_{2k}\}, \quad k = 0,1,2,...$ defined by

$$z_{2k+1} = f\{z_{2k}\}, \quad d(z_{2k+2}, z_{2k+1}) = d(A,B) = 1, \quad k = 0,1,2,...,$$

converges for every $z_0 \in A_0$, to z^*, since

$$z_{2(k+1)} = (x_{2(k+1)}, 1) = (\frac{x_{2k}^2}{x_{2k}+1}, 1) \to (0,1).$$

In fact, from $x_{2(k+1)} = \frac{x_{2k}^2}{x_{2k}+1}$, we know that $x_{2k+1} \leq x_{2k}$, so there exists a number x^* such that $x_{2k} \to x^*$. Furthermore, $x^* = \frac{(x^*)^2}{x^*+1}$ and hence $x^* = 0$.

Example 3.10. Consider $X = \mathbb{R}^2$, with the usual metric d. Define the sets $A = \{(x,1) : x \geq 0\}$ and $B = \{(x,0) : x \geq 0\}$. Let $A_0 = A$ and $B_0 = B$ and clearly, the pair (A,B) has the P-property, also satisfies weak P-property. Also define $f : A \to B$ as:

$$f(x,1) = (\frac{x^2}{x+1}, 0),$$

we take $A_0 = A \neq \varnothing, B_0 = B, f(A_0) \subseteq B_0$. Then,

$$\begin{aligned}
\frac{1}{2}d((x_1,1), f(x_1,1)) &= \frac{1}{2}d((x_1,1),(\frac{x_1^2}{x_1+1},0)) \\
&= \frac{1}{2}|1+(x_1-\frac{x_1^2}{x_1+1})| \\
&= \frac{1}{2}|1+\frac{1}{1+x_1}| \\
&= \frac{1}{2}\frac{|1+\frac{1}{1+x_1}|}{|x_1-x_2|}|x_1-x_2| \\
&= \frac{1}{2}\frac{|x_1+2|}{|(x_1-x_2)(x_1+1)|}|x_1-x_2| \\
&< |x_1-x_2| = d((x_1,1),(x_2,1)).
\end{aligned}$$

Thus, $d((x_1,1), f(x_1,1)) < d((x_1,1),(x_2,1))$, which implies that $\zeta(d(f(x_1,1), f(x_2,1)), d((x_1,1),(x_2,1))) \geq 0$, and f is Suzuki type $\mathcal{Z}$-contraction with respect to ζ. Thus, all the conditions of the Theorem (3.2) are satisfied, and the conclusion of that theorem is also correct, that is, f has a unique best proximity point $z^* = (0,1) \in A_0$ such that $d(z^*, fz^*) = d((0,1),(0,0)) = d(A,B) = 1$ On the other hand, it is clear that the iteration sequence $\{z_{2k}\}$, $k = 0,1,2,...$ defined by

$$z_{2k+1} = f\{z_{2k}\} \quad d(z_{2k+2}, z_{2k+1}) = d(A,B) = 1, \quad k = 0,1,2,...,$$

converges for every $z_0 \in A_0$, to z^*, since

$$z_{2(k+1)} = (x_{2(k+1)},1) = (\frac{x_{2k}^2}{x_{2k}+1},1) \to (0,1).$$

In fact, from $x_{2(k+1)} = \frac{x_{2k}^2}{x_{2k}+1}$, we know that $x_{2k+1} \leq x_{2k}$, so there exists a number x^* such that $x_{2k} \to x^*$. Furthermore, $x^* = \frac{(x^*)^2}{x^*+1}$ and hence $x^* = 0$.

Example 3.11. If we change the defined mapping on same conditions of above example and on little change on given sets like for $A = \{(1,y) : y \geq 0\}$ and $B = \{(0,y) : y \geq 0\}$ and $A_0 = A$ and $B_0 = B$. Define $f : A \to B$ as:

$$f(1,y) = (0, \frac{y^2}{y+1}),$$

as given in [25], then also with this defined mapping there exists a best proximity point for both $\mathcal{Z}$ and Suzuki type $\mathcal{Z}$-contractions, also after such change in the conditions, examples (3.1) and (3.2), theorems (3.1) and (3.2) verified, and that best proximity point is $(1,0)$ for both, that is, $d(x, fx) = d((1,0),(0,0)) = d(A,B) = 1$. If there are two best proximity points for same sets, $(1,0)$ and $(0,1)$, then their uniqueness can be proved easily as $d((1,0),(0,0)) = d(A,B)$ and $d((0,1),(0,0)) = d(A,B) = 1$, then one can write as: $d((1,0),(0,0)) = d((0,1),(0,0)) = d(A,B)$, this implies that $(1,0) = (0,1)$. Hence, existence and uniqueness of best proximity point in the metric space has proved.

4. Application

In this section, we present an application of our fixed point results derived in previous section to establish the existence of solution of fractional order functional differential equation.
Consider the following initial value problem (IVP for short) of the form

$$D^\alpha y(t) = f(t, y_t), \text{ for each } t \in J = [0, b], 0 < \alpha < 1, \tag{4.1}$$

$$y(t) = \phi(t), t \in (-\infty, 0] \tag{4.2}$$

where D^α is the standard Riemann-Liouville fractional derivative, $f : J \times B \to \mathbb{R}$, $\phi \in B, \phi(0) = 0$ and B is called a phase space or state space satisfying some fundamental axioms (H-1, H-2, H-3) given below which were introduced by Hale and Kato in [14].
For any function y defined on $(-\infty, b]$ and any $t \in J$, we denote by y_t the element of B defined by

$$y_t(\theta) = y(t + \theta), \theta \in (-\infty, 0].$$

Here $y_t(\cdot)$ represents the history of the state from $-\infty$ up to present time t.

By $C(J, \mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$||y||_\infty := \sup\{|y(t)| : t \in J\}$$

where $| \cdot |$ denotes a suitable complete norm on $\mathbb{R}$.

- (H-1) If $y : (-\infty, b] \to \mathbb{R}$, and $y_0 \in B$, then for every $t \in [0, b]$ the following conditions hold:
  - (i) y_t is in B,
  - (ii) $||y_t||_B \leq K(t) \sup\{|y(s)| : 0 \leq s \leq t\} + M(t)||y_0||_B$,
  - (iii) $|y(t)| \leq H||y_t||_B$,
    where $H \geq 0$ is a constant, $K : [0, b] \to [0, \infty)$ is continuous, $M : [0, \infty) \to [0, \infty)$ is locally bounded and H, K, M are independent of $y(\cdot)$.
- (H-2) For the function $y(\cdot)$ in (H-1), y_t is a B-valued continuous function on $[0, b]$.
- (H-3) The space B is complete.

By a solution of problem (4.1)-(4.2), we mean a space $\Omega = \{y : (-\infty, b] \to \mathbb{R} : y|_{(-\infty,0]} \in B$ and $y|_{[0,b]}$ is continuous$\}$. Thus a function $y \in \Omega$ is said to be a solution of (4.1)-(4.2) if y satisfies the equation $D^\alpha y(t) = f(t, y_t)$ on J, and the condition $y(t) = \phi(t)$ on $(-\infty, 0]$.

The following lemma is crucial to prove our existence theorem for the problem (4.1)-(4.2).

Lemma 4.1 (see [11]). *Let $0 < \alpha < 1$ and let $h : (0, b] \to \mathbb{R}$ be continuous and $\lim_{t\to 0^+} h(t) = h(0^+) \in \mathbb{R}$. Then y is a solution of the fractional integral equation*

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds,$$

if and only if y is a solution of the initial value problem for the fractional differential equation

$$D^\alpha y(t) = h(t), t \in (0, b],$$
$$y(0) = 0.$$

Now we are ready to prove following existence theorem.

Theorem 4.2. *Let $f : J \times B \to \mathbb{R}$. Assume*
(H) there exists $q > 0$ such that

$$|f(t, u) - f(t, v)| \leq q||u - v||_B, \quad \text{for } t \in J \quad \text{and every } u, v \in B.$$

If $\frac{b^\alpha K_b q}{\Gamma(\alpha+1)} = \lambda < 1$ where

$$K_b = \sup\{|K(t)| : t \in [0, b]\},$$

then there exists a unique solution for the IVP (4.1)-(4.2) on the interval $(-\infty, b]$.

Proof. To prove the existence of solution for the IVP (4.1)-(4.2), we transform it into a fixed point problem. For this, consider the operator $N : \Omega \to \Omega$ defined by

$$N(y)(t) = \begin{cases} \phi(t) & t \in (-\infty, 0], \\ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s)ds & t \in [0, b]. \end{cases}$$

Let $x(\cdot) : (-\infty, b] \to \mathbb{R}$ be the function defined by

$$x(t) = \begin{cases} \phi(t) & t \in (-\infty, 0], \\ 0 & t \in [0, b]. \end{cases}$$

Then $x_0 = \phi$. For each $z \in C([0, b], \mathbb{R})$ with $z(0) = 0$, we denote by $\bar{z}$ the function defined by

$$\bar{z}(t) = \begin{cases} 0 & \text{if } t \in (-\infty, 0], \\ z(t) & \text{if } t \in [0, b]. \end{cases}$$

If $y(\cdot)$ satisfies the integral equation

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s)ds,$$

we can decompose $y(\cdot)$ as $y(t) = \bar{z}(t) + x(t), 0 \leq t \leq b$, which implies $y_t = \bar{z}_t + x_t$, for every $0 \leq t \leq b$, and the function $z(\cdot)$ satisfies

$$z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \bar{z}_s + x_s)ds$$

Set

$$C_0 = \{z \in C([0, b], \mathbb{R}) : z_0 = 0\},$$

and let $||\cdot||_b$ be the seminorm in C_0 defined by

$$||z||_b = ||z_0||_B + \sup\{|z(t)|; 0 \le t \le b\} = \sup\{|z(t)|; 0 \le t \le b\},\ z \in C_0.$$

C_0 is a Banach space with norm $||\cdot||_b$. Let the operator $P : C_0 \to C_0$ be defined by

$$(Pz)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \bar{z}_s + x_s) ds,\ t \in [0, b]. \tag{4.3}$$

That the operator N has a fixed point is equivalent to P has a fixed point, and so we turn to proving that P has a fixed point. Indeed, consider $z, z^* \in C_0$. Then we have for each $t \in [0, b]$

$$\begin{aligned}
|P(z)(t) - P(z^*)(t)| &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, \bar{z}_s + x_s) - f(s, \bar{z}^*_s + x_s)|\ ds \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} q||\bar{z}_s - \bar{z}^*_s||_B\ ds \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} q K_b \sup_{s\in[o,t]} ||z(s) - z^*(s)||\ ds \\
&\le \frac{K_b}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} q\ ds\ ||z - z^*||_b.
\end{aligned}$$

Therefore

$$||P(z) - P(z^*)||_b \le \frac{q b^\alpha K_b}{\Gamma(\alpha+1)} ||z - z^*||_b,$$

i.e.

$$d(P(z), P(z^*)) \le \lambda d(z, z^*).$$

Now we observe that the function $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ defined by $\zeta(t, s) = \lambda s - t$ for all $t, s \in [0, \infty)$, is in $\mathcal{Z}$ and so we deduce that the operator P satisfies all the hypothesis of corollary (3.7). Thus P has unique fixed point. □

Acknowledgements. *The authors thank Editor-in-Chief and Referee(s) for their valuable comments and suggestions, which were very useful to improve the paper significantly. The authors would like to thank the Petchra Pra Jom Klao Ph.d. Research Scholarship for financial support. Also, Somayya Komal was supported by the Petchra Pra Jom Klao Doctoral Scholarship Academic for Ph.D. Program at KMUTT. This work was completed while the third author (Dr. Gopal) was visiting Theoretical and Computational Science Center (TaCS), Science Laboratory Building, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bangkok, Thailand. He thanks Professor Poom Kumam and the University for their hospitality and support. Moreover, this project was supported by the Theoretical and Computational Science (TaCS) Center under Computational and Applied Science for Smart Innovation Research Cluster (CLASSIC), Faculty of Science, KMUTT.*

References

[1] M. Abbas, A. Hussain and P. Kumam, A coincidence best proximity point problem in G-metric spaces, Abst. and Appl. Anal. 2015 (2015), Article ID 243753, 12 pages.

[2] A. Akbar and M. Gabeleh, Generalized cyclic contractions in partially ordered metric spaces, Optim. Lett. 6 (2012), 1819–1830.

[3] A. Akbar and M. Gabeleh, Global optimal solutions of noncyclic mappings in metric spaces. J. Optim. Theory Appl. 153 (2012), 298–305.

[4] H. Argoubi, B. Samet and C. Vetro, Nonlinear contractions involving simulation functions in a metric space with a partial order, J. Nonlinear Sci. Appl. 8 (2015), 1082–1094.

[5] S. Banach, Sur les operations dans les ensembles abstraits et leur applications aux equations integrales, Fundam. Math. 3 (1922), 133–181.

[6] S. S. Basha, Extensions of Banach's contraction principle, Numer. Funct. Anal. Optim. 31 (2010), 569–576.

[7] S. S. Basha, Best proximity points: Global optimal approximate solution, J. Glob. Optim. 49 (2010), 15–21.

[8] S. S. Basha, Best proximity point theorems generalizing the contraction principle, Nonlinear Anal. 74 (2011), 5844–5850.

[9] S. S. Basha, Common best proximity points: Global minimization of multi-objective functions, J. Glob. Optim. 54 (2012), 367–373.

[10] S. S. Basha, N. Shahzad and R. Jeyaraj, Best proximity points: Approximation and optimization, Optim. Lett. 7 (2011), 145–155.

[11] D. Delboso and L. Rodino, Existence and uniqueness for a nonlinear fractional differential equation, J. Math. Anal. Appl. 204 (1996), 609–625.

[12] M. Edestein, On fixed and periodic points under contractive mappings, J. London Math. Soc. 37 (1962), 74–79.

[13] K. Fan, Extensions of two fixed point theorems of F. E. Browder, Mathematische Zeitschrift 112 (1969), 234–240.

[14] J. Hale and J. Kato, Phase space for retarded equations with infinite delay, Funkcial. Ekvac. 21 (1978), 11–41

[15] F. Khojasteh, S. Shukla and S. Redenovi, A new approach to the study fixed point theorems via simulation functions, Filomat 29 (2015), 1188–1194.

[16] P. Kumam, D. Gopal and L. Budhiyi, A new fixed point theorem under Suzuki type Z-contraction mappings, preprint.

[17] A. Nastasi and P. Vetro, Fixed point results on metric and partial metric spaces via simulation functions, J. Nonlinear Sci. Appl. 8 (2015), 1059–1069.

[18] M. Olgun, O. Bier and T. Alyldz, A new aspect to Picard operators with simulation functions, Turk. J. Math. 40 (2016), 832–837.

[19] V. S. Raj, A best proximity point theorem for weakly contractive non-self-mappings, Nonlinear Anal. 74 (2011), 4804-4808.

[20] A. Roldán-López-de-Hierro, E. Karapinar, C. Roldán-López-de-Hierro, J. Martínez-Moreno, Coincidence point theorems on metric spaces via simulation functions, J. Comput. Appl. Math. 275 (2015), 345–355.

[21] A. Roldán-López-de-Hierro and N. Shahzad, Common fixed point theorems under (R,S)-contractivity conditions, Fixed Point Theory Appl. 2016 (2016), 55.

[22] A. Roldán-López-de-Hierro and N. Shahzad, New fixed point theorem under R-contractions, Fixed Point Theory Appl. 2015 (2015), 98.

[23] N. Shahzad, A. Roldán-López-de-Hierro, F. Khojasteh, Some new fixed point theorems under (A,S)-contractivity conditions, RACSAM, to appear (DOI:10.1007/s13398-016-0295-1).

[24] W. Sintunavarat and P. Kumam, Coupled best proximity point theorem in metric spaces, Fixed Point Theory Appl. 2012 (2012), 93.

[25] Y. Sun,Y. Su and J. Zhang, A new method for research of best proximity point theorems for non linear mappings, Fixed Point Theory Appl. 2014 (2014), 116.

[26] T. Suzuki, A generalized Banach contraction principle which characterizes metric completeness, Proc. Am. Math. Soc. 136, no. 5 (2008), 1861–1869.
[27] J. Zhang, Y. Su and Q. Chang, A note on 'A best proximity point theorem for Geraghty-contractions', Fixed Point Theory Appl. 2013 (2013), 99.