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Josefa Marín and Jesús Rodríguez-López

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WATS'16

Workshop on Applied Topological Structures WATS'16

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Preface

General Topology has become one of the fundamental parts of mathematics. Nowadays, as a consequence of an intensive research activity, this mathematical branch has been shown to be very useful in modeling several problems which arise in some branches of applied sciences as Economics, Artificial Intelligence and Computer Science. Due to this increasing interaction between applied and topological problems, we have promoted the creation of an annual or biennial workshop to encourage the collaboration between different national and international research groups in the area of General Topology and its Applications. This year it has been given the name of Workshop on Applied Topological Structures (WATS).

This book contains a collection of papers presented by the participants in this workshop which took place in Valencia (Spain) from June 22 to 23, 2016.

All the papers of the book have been strictly refereed.

We would like to thank all participants, the plenary speakers and the regular ones, for their excellent contributions.

We express our gratitude to the Instituto Universitario de Matemática Pura y Aplicada for its financial support without which this workshop would not have been possible.

We are certain of all participants have established fruitful scientific relations during the Workshop.

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Some results on weak fuzzy normed spaces

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Abstract

The notion of weak fuzzy norm appears in the theory of fuzzy normed spaces when dealing with duality in this framework. We present canonical examples of weak fuzzy norms and sumarize some results about the topological structure of the weak fuzzy normed spaces.

1. Introduction

The study of fuzzy normed spaces is relatively recent in the field of fuzzy functional analysis. The first definition of fuzzy norm on a linear space was given by Katsaras [9] in 1984 while studying topological vector spaces. Following this work, Felbin [7] offered in 1992 an alternative definition of a fuzzy norm on a linear space with an associated metric of Kaleva and Seikkala's type [8]. In 1994 Cheng and Mordeson [6] gave another definition of fuzzy norm that corresponds with the notion of a fuzzy metric as defined by Kramosil and Michalek in [10].

[1]This research is supported under grant MTM2015-64373-P (MINECO/FEDER, UE).

The notion of weak fuzzy norm on a real vector space generalizes the notion of fuzzy norm. Weak fuzzy norms appear in the theory of fuzzy normed spaces when dealing with the duality in this context (see [2]). Indeed, if (X, N) is a fuzzy normed space in the sense of Cheng and Mordeson ([6]) then its topological dual X^* can be equipped with a weak fuzzy norm N^* that plays a similar role to that the dual norm on the classical theory of normed spaces ([2]). On the other hand, there is in the last years a growing interest in the theory of extended normed spaces ([3, 4, 5]) and this class of spaces, as we shall see in this paper, provides a natural class of examples of weak fuzzy normed spaces. These facts motivate a fully exploration of the weak fuzzy normed spaces. In this direction we here study some aspects of the topological structure of these spaces and its relation with the classical topological vector spaces.

Let X be a linear space and let a function $\|\cdot\| : X \to \mathbb{R}^+ \cup \{\infty\}$. If $\|\cdot\|$ satisfies the conditions of a norm we say that $\|\cdot\|$ is an extended norm. The pair $(X, \|\cdot\|)$ is called an extended normed space (see [3, 5]).

According to [13] a binary operation $* : [0,1]\times[0,1] \to [0,1]$ is a continuous t-norm if $*$ satisfies the following conditions: (i) $*$ is associative and commutative; (ii) $*$ is continuous; (iii) $a * 1 = a$ for every $a \in [0,1]$; (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0,1]$.

Three paradigmatic examples of continuous t-norms are $\wedge$, $\cdot$ and $*_L$ (the Lukasiewicz t-norm), which are defined by $a \wedge b = \min\{a, b\}$, $a \cdot b = ab$ and $a *_L b = \max\{a + b - 1, 0\}$, respectively. Recall that $*_L \leq \cdot \leq \wedge$. In fact, $* \leq \wedge$ for every continuous t-norm $*$.

2. Weak fuzzy normed spaces

Definition 1 ([2]). If X be a real vector space, a weak fuzzy norm on X is a pair $(N, *)$ such that, $*$ is a continuous t-norm and N is a fuzzy set in $X \times [0, \infty)$ satisfying the following conditions for every $x, y \in X$, and $t, s \geq 0$:

(FN1) $N(x, 0) = 0$.

(FN2) $N(x, t) = 1$ for all $t > 0 \Leftrightarrow x = \mathbf{0}$.

(FN3) $N(cx,t) = N(x, t/ \mid c \mid)$ for every $c \in \mathbb{R}\backslash\{0\}$.

(FN4) $N(x+y, t+s) \geq N(x,t) * N(y,s)$.

(FN5) $N(x,\cdot) : [0,\infty) \to [0,1]$ is left continuous.

The triple $(X, N, *)$ is called a weak fuzzy normed space.

If $(N, *)$ is a weak fuzzy norm on X satisfying:

$(FN6)$ $lim_{t\to\infty} N(x,t) = 1$ for all $x \in X$

then $(N, *)$ is a fuzzy norm on X.

If, in addition, $* = \wedge$, then one has the notion of a fuzzy norm as given by Cheng and Morderson [6].

Example 2. Let $(X, \|\cdot\|)$ be an extended normed space.

(a) Let $N : X \times [0,\infty) \to [0,1]$ given by $N(x,0) = 0$ for all $x \in X$ and

$$N(x,t) = \frac{t}{t + \|x\|},$$

for all $x \in X$ and $t > 0$. Then $(N, *)$ is a weak fuzzy norm on X, where $*$ is any continuous t-norm. Note that if there exists $x \in X$ such that $\|x\| = \infty$, then $(N, *)$ is not a fuzzy norm because $\lim_{t\to\infty} N(x,t) = 0$.

(b) Let $N : X \times [0,\infty) \to [0,1]$ given by $N(x,t) = 0$ if $t \leq \|x\|$ and $N(x,t) = 1$ if $t > \|x\|$. Then $(N, *)$ is a fuzzy norm on X, where $*$ is any continuous t-norm. As above, if $\|x\| = \infty$, then $\lim_{t\to\infty} N(x,t) = 0$.

If $(X, N, *)$ is a weak fuzzy normed space, the open ball $B_N(x,r,t)$ with center x, radius r, $0 < r < 1$, and $t > 0$ is defined as follows:

$$B_N(x,r,t) = \{y \in X : N(y-x,t) > 1-r\}.$$

We note that $B_N(x,r,t) = x + B_N(\mathbf{0},r,t)$, for all $x \in X$ and $0 < r < 1,\ t > 0$. The closed ball $\bar{B}_N(x,r,t)$ with center x, radius r, $0 < r < 1$, and $t > 0$ is defined as follows:

$$\bar{B}_N(x,r,t) = \{y \in X : N(y-x,t) \geq 1-r\}.$$

It is clear that if $(X, N, *)$ is a weak fuzzy normed space, the fuzzy set M_N in $X \times X \times [0,\infty)$ given by $M_N(x,y,t) = N(y-x,t)$ is a fuzzy metric on X in the sense of Kramosil and Michalek [10]. This fuzzy metric induces a topology τ_N on X, which has as a base the collection $\{B_N(x,r,t) : x \in X, 0 < r < 1, t > 0\}$. Moreover τ_N is metrizable and the countable collection of balls $\{B_N(x, 1/n, 1/n) : n = 2, 3, ...\}$ forms a fundamental system of neighborhoods of x, for all $x \in X$.

It is easy to see that if $(X, \|\cdot\|)$ is an extended normed space, then the topology τ_N agrees with the topology induced by the extended norm $\|\cdot\|$ where $(N, *)$ is one of the weak fuzzy norms of Example 1. Therefore, the extended normed spaces are included in the class of weak fuzzy normed spaces.

In the same way that the subspace of an extended norm space consisting of all vectors with finite norm is a normed space, the subspace of a weak fuzzy normed space consisting of all vectors that satisfy condition (FN6) is a fuzzy normed space.

If $(X, N, *)$ is a weak fuzzy normed space, from Proposition 1 and 3 of [1], we obtain the following properties of the open balls with center in the origin.

Proposition 3. *Let $(X, N, *)$ be a weak fuzzy normed space and let $\mathcal{B}$ the family of open balls with center in the origin. Then*

(a) *$B_N(\mathbf{0}, r, t)$ is balanced for all $t > 0$ and $0 < r < 1$.*
(b) *$\lambda B_N(\mathbf{0}, r, t) = B_N(\mathbf{0}, r, \lambda t)$, for every $\lambda > 0$, $t > 0$ and $0 < r < 1$.*
(c) *If $U \in \mathcal{B}$ there is $V \in \mathcal{B}$ such that $V + V \subset U$.*
(d) *If $U, V \in \mathcal{B}$ there is $W \in \mathcal{B}$. such that $W \subset U \cap V$.*
e) *If $* = \wedge$, then $B_N(\mathbf{0}, r, t)$ is convex for all $t > 0$ and $0 < r < 1$.*

In the following proposition we show the closed relationship between the absorbency of the open balls and condition (FN6).

Proposition 4. *A weak fuzzy normed space $(X, N, *)$ is a fuzzy normed space if and only if $B(\mathbf{0}, r, t)$ is an absorbent set for all $t > 0$ and $0 < r < 1$.*

Proof. The 'only if' part follows from [1, Proposition 1]. For the converse, suppose that there exists $x_0 \in X$ such that $\lim_{t\to\infty} N(x_0, t) \neq 1$. Then there exists $0 < \varepsilon < 1$ such that $N(x_0, t) < 1 - \epsilon$ for all $t > 0$. So that $N(x_0, \lambda t) < 1 - \epsilon$, for all $\lambda > 0$, i.e., $\frac{x_0}{\lambda} \notin B(\mathbf{0}, \varepsilon, t)$. Therefore $B(\mathbf{0}, \varepsilon, t)$ is not an absorbent set. □

It is well known that if $(X, N, *)$ is a fuzzy normed space, then (X, τ_N) is a topological vector space. This is not the case in general if $(N, *)$ a weak fuzzy norm on X. Indeed, by Proposition 4, if there exists $x \in X$ such that $\lim_{t\to\infty} N(x, t) \neq 1$, there exist neighborhoods of $\mathbf{0}$ that are not absorbent sets. Consequently, (X, τ_N) is not a topological vector space.

If (X, τ) is a topological vector space and $(N, *)$ is a weak fuzzy norm on X, we say that $(N, *)$ is compatible with τ if $\tau_N = \tau$.

Proposition 5. *If (X, τ) is a topological vector space and $(N, *)$ is a weak fuzzy norm on X compatible with τ, then $(N, *)$ is a fuzzy norm.*

Proof. If $\tau_N = \tau$, then $B_N(\mathbf{0}, r, t)$ is neighborhood of $\mathbf{0}$ in (X, τ) for all $t > 0$ and $0 < r < 1$. Since (X, τ) is a topological vector space, we have that $B_N(\mathbf{0}, r, t)$ is an absorbent set for all $t > 0$ and $0 < r < 1$ and so, by Proposition 4, $(N, *)$ is a fuzzy norm on X. □

Then, we can obtain the following characterizations of metrizable topological vector spaces in terms of weak fuzzy norms.

Theorem 6. *For a topological vector space (X, τ) the following conditions are equivalent:*

(1) *(X, τ) is metrizable;*
(2) *there is a fuzzy norm $(N, *)$ on X compatible with τ;*
(3) *there is a weak fuzzy norm $(N, *)$ on X compatible with τ.*

Proof. (1) ⇒ (2) If (X, τ) is metrizable then there is a fuzzy norm $(N, *_L)$ on X compatible with τ (see [11] or Theorem 2 of [1]).

(2) ⇒ (1) If $(N, *)$ is a fuzzy norm on X then (X, τ_N) is a metrizable topological vector space (see [12] or Theorem 1 (A) of [1]). Since $\tau_N = \tau$, (X, τ) is metrizable.

(2) ⇔ (3) This follows from Proposition 5. □

Theorem 7. *For a topological vector space (X, τ) the following conditions are equivalent:*

(1) *(X, τ) is metrizable and locally convex;*
(2) *there is a fuzzy norm $(N, \wedge)$ on X compatible with τ;*
(3) *there is a weak fuzzy norm $(N, \wedge)$ on X compatible with τ.*

Proof. (1) ⇔ (2) This is Theorem 4 of [1].

(2) ⇔ (3) This follows from Proposition 5.

□

From Proposition 5 and Theorems 5 and 7 of [1], we can also obtain characterizations of those topological vector spaces that are locally bounded and normable in terms of weak fuzzy norms.

Theorem 8. *For a topological vector space (X, τ) the following conditions are equivalent:*

(1) *(X, τ) is locally bounded;*
(2) *there is a fuzzy norm $(N, *)$ on X compatible with τ such that $\lim_{t\to\infty} N(x,t) = 1$ uniformly on an open ball centered at origin;*
(3) *there is a weak fuzzy norm $(N, *)$ on X compatible with τ such that $\lim_{t\to\infty} N(x,t) = 1$ uniformly on an open ball centered at origin.*

Theorem 9. *For a topological vector space (X, τ) the following conditions are equivalent:*

(1) *(X, τ) is normable;*
(2) *there is a fuzzy norm $(N, \wedge)$ on X compatible with τ such that $\lim_{t\to\infty} N(x,t) = 1$ uniformly on an open ball centered at origin;*
(3) *there is a weak fuzzy norm $(N, \wedge)$ on X compatible with τ such that $\lim_{t\to\infty} N(x,t) = 1$ uniformly on an open ball centered at origin.*

3. The dual of a fuzzy normed space

As mentioned at the beginning of this paper the notion of weak fuzzy norm appears in the study of fuzzy normed spaces when it comes to constructing the dual space of a fuzzy normed spaces. In Section 4 of [2], after an extensive research on structural properties of the fuzzy normed spaces, the authors constructed an appropriate weak fuzzy norm on the topological dual of a fuzzy normed space $(X, N, \wedge)$ and then they proved a theorem of Hahn-Banach type in the frame of fuzzy normed spaces which generalizes the classical one for normed spaces.

Let $(X, N, \wedge)$ a fuzzy normed space and let $(N_s, \wedge)$ be the standard fuzzy norm on $\mathbb{R}$, i.e., $N_s(x, 0) = 0$ for $x \in \mathbb{R}$ and $N_s(x, t) = t/(t + |x|)$ for all $x \in \mathbb{R}$ and $t > 0$.

Denote by X^* the set of all continuous linear mappings from (X, τ_N) to $(\mathbb{R}, \tau_{N_s})$. (Note that τ_{N_s} is the usual topology of $\mathbb{R}$.)

The weak fuzzy norm of X^* is defined by $N^*(f, 0) = 0$ for all $f \in X^*$, and

$$N^*(f, t) = \sup\{\alpha \in [0, 1) : \|f\|^*_\alpha < t\},$$

for all $f \in X^*$, where

$$\|f\|^*_\alpha = \sup\{|f(x)| : \|x\|_{1-\alpha} \leq 1\},$$

and

$$\|x\|_{1-\alpha} = \inf\{t > 0 : N(x, t) \geq 1 - \alpha\}.$$

The following example shows that $(N^*, \wedge)$ is not in general a fuzzy norm on X^*.

Example 10. (See Example 19 of [2]) Let X be the linear space of all sequences $x := (x_n)_n$ of real scalars and let $(N, \wedge)$ be the fuzzy norm induced on X by the ascending family of separating seminorms $\{\| \cdot \|_\alpha : \alpha \in (0, 1)\}$ given by $\|x\|_\alpha = q_n(x)$ if $\alpha \in (\frac{n-1}{n}, \frac{n}{n+1}]$, for all $n \in \mathbb{N}$, where $q_n(x) = \max\{|x_1|, \ldots, |x_n|\}$.

Let $f : X \to \mathbb{R}$ be the linear function given by $f(x) = x_1 + x_2 + x_3$. Then, $f \in X^*$ and $\|f\|^*_{1/2} = \infty$. Since $\|f\|^*_\alpha \leq \|f\|^*_\beta$ whenever $\alpha \leq \beta$, we have that

$$N^*(f, t) = \sup\{\alpha \in [0, 1) : \|f\|^*_\alpha < t\} < 1/2,$$

for all $t > 0$, therefore $\lim_{t\to\infty} N^*(f, t) \leq 1/2$.

References

[1] C. Alegre, S. Romaguera, Characterization of metrizable topological vector spaces and their asymmetric generalization in terms of fuzzy (quasi-)norms, Fuzzy Sets Syst. 161 (2010), 2181–2192.

[2] C. Alegre, S. Romaguera, The Hahn Banach theorem for fuzzy normed spaces revisited, Abstract Appl. Anal. 2014 (2014), 7 pages.

[3] G. Beer, Norms with infinite values, J. Convex. Anal. 22 (2015), 35–58.

[4] G. Beer, The structure of extended-real valued metric spaces, Set-Valued Var. Anal. 21 (2013), 591–602.

[5] G. Beer, J. Vanderwerff, Estructural properties of extended normed spaces, Set-Valued Var. Anal. (DOI 10.1007/S1228-015-0331).

[6] S. C. Cheng, J. N. Mordeson, Fuzzy linear operator and fuzzy normed linear spaces, Bull. Calcutta Math. Soc. 86 (1994), 429–436.

[7] C. Felbin, Finite dimensional fuzzy normed linear spaces, Fuzzy Sets Syst. 48 (1992), 239–248.

[8] O. Kaleva, S. Seikkala, On fuzzy metric spaces, Fuzzy Sets Syst. 12 (1984), 215–229.

[9] A. K. Katsaras, Fuzzy topological vector spaces II, Fuzzy Sets Syst. 12 (1984), 143–154.

[10] I. Kramosil, J. Michalek, Fuzzy metrics and statistical metric spaces, Kybernetika 11 (1975), 326–334.

[11] D. H. Muštari, On the linearity of isometric mapping on random normed spaces, Kazan Gos. Univ. Uchen. Zap. 128 (1968), 86–90.

[12] V. Radu, On the relationship between locally (K)-convex spaces and random normed spaces over valued fields, Seminarul de Teoria Probabilitatilor STPA, West University of Timisoara, Vol. 37, 1978.

[13] B. Schweizer, A. Sklar, Statistical metric spaces, Pacific J. Math. 10 (1960), 314–334.

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Kannan mappings vs. Caristi mappings: An easy example

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Abstract

We give an easy example of a Kannan mapping T on a complete metric space (X, d) for which the function $x \to d(x, Tx)$ is not lower semicontinuous on X.

MSC: *54H25; 54E50; 47H10.*

In 1922, Banach published his famous fixed point theorem which is stated as follows.

Theorem 1 (Banach [1])**.** *Let (X, d) be a complete metric space. If T is a self-mapping of X such that there is a constant $c \in [0, 1)$ satisfying*

$$d(Tx, Ty) \leq cd(x, y), \tag{1}$$

for all $x, y \in X$, then T has a unique fixed point.

[1]This research is supported under grant MTM2015-64373-P (MINECO/FEDER, UE).

In [4], Kannan proved the following fixed point theorem which is independent from Banach's fixed point theorem.

Theorem 2 (Kannan [4]). *Let (X, d) be a complete metric space. If T is a self-mapping of X such that there is a constant $c \in [0, 1/2)$ satisfying*

$$d(Tx, Ty) \leq c(d(x, Tx) + d(y, Ty)), \tag{2}$$

for all $x, y \in X$, then T has a unique fixed point.

Later on, Chatterjea [3] obtained the following variant of Kannan's fixed point theorem.

Theorem 3 (Chatterjea [3]). *Let (X, d) be a complete metric space. If T is a self-mapping of X such that there is a constant $c \in [0, 1/2)$ satisfying*

$$d(Tx, Ty) \leq c(d(x, Ty) + d(y, Tx)), \tag{3}$$

for all $x, y \in X$, then T has a unique fixed point.

The above results suggest the following well-established notion.

Definition 4. Let T be a self-map of a metric space (X, d). Then T is called a Banach contraction (resp. a Kannan mapping, a Chatterjea mapping) if T satisfies condition (1) (resp. condition (2), condition (3)) for all $x, y \in X$.

Contrarily to the Banach contractions, not every Kannan mapping is a continuous mapping and not every Chatterjea mapping is a continuous mapping.

On the other hand, Banach's fixed point theorem does not characterize metric completeness. Indeed, there exist examples of non complete metric spaces for which every Banach contraction has a fixed point (see e.g. [9, 10]). However, both Kannan's fixed point theorem and Chatterjea's fixed point theorem characterize metric completeness, which was showed by Subrahmanyam in [9] as follows.

Theorem 5 (Subrahmanyam [9]). *For a metric space (X, d) the following conditions are equivalent.*

(1) (X, d) is complete.

(2) Every Kannan mapping on X has a fixed point.

(3) Every Chatterjea mapping on X has a fixed point.

In his well-known paper [2], Caristi proved the following important fixed point theorem that also allows to characterize metric completeness and is "equivalent" to the Ekeland Variational Principle.

Theorem 6 (Caristi [2]). *Let (X, d) be a complete metric space. If T is a self-mapping of X such that there is a lower semicontinuous function $\varphi : X \to [0, \infty)$ satisfying*

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx), \tag{4}$$

for all $x \in X$, then T has a fixed point.

A self-mapping T on a metric space (X, d) for which there is a lower semicontinuous function $\varphi : X \to [0, \infty)$ satisfying condition (4) for all $x \in X$ is called a Caristi mapping.

In [5] Kirk proved the "if" part of the following characterization.

Theorem 7 (Kirk [5]). *A metric space (X, d) is complete if and only if every Caristi mapping on X has a fixed point.*

The relationship between Banach mappings and Caristi mappins, as well as between Kannan mappings (resp. Chatterjea mappings) and Caristi mappings has been considered by several authors. Thus, following a construction suggested by Weston (see [11, p. 188]), Park asserted in [6, p. 24] that if T is a Chatterjea mapping on a metric space (X, d), with constant $c \in [0, 1/2)$, then the function $\varphi : X \to [0, \infty)$ defined as

$$\varphi(x) = \frac{1 - c}{1 - 2c} d(x, Tx), \tag{5}$$

for all $x \in X$, is a Caristi mapping, and, hence, every Chatterjea mapping is a Caristi mapping.

In this direction, Shioji, Suzuki and Takahashi [8, p. 3118], and [7, p. 118] claim that every Banach contraction and every Kannan mapping on a metric space is a Caristi mapping. In fact (see e.g. [7, p. 118]) if T is a self-mapping on a metric space (X, d) the following facts are asserted (actually, they are obvious consequences of a more general claimed facts):

(A) If T is a Banach contraction, with constant $c \in [0, 1)$, then the function $\varphi : X \to [0, \infty)$ defined as

$$\varphi(x) = \frac{1}{1-c} d(x, Tx), \tag{6}$$

for all $x \in X$, is a Caristi mapping on X.

(B) If T is a Kannan mapping, with constant $c \in [0, 1/2)$, then the function $\varphi : X \to [0, \infty)$ defined as in (5) for all $x \in X$, is a Caristi mapping on X.

It is easy to check that assertion (A) is correct. However, in the case that T is a Kannan mapping, or a Chatterjea mapping, on a metric space (X, d), the function φ given by (5) indeed satisfies $d(x, Tx) \leq \varphi(x) - \varphi(Tx)$ for all $x \in X$, but, unfortunately, the function $x \to d(x, Tx)$ is not lower semicontinuous in general, as Example 8 below shows. Therefore, it seems that the following question still is open: Is every Kannan mapping on a metric space X a Caristi mapping on X?

Example 8. Let $X = [0, \infty)$ and let d be the usual metric on X. Fix $\delta \in (0, 1)$ and define $T : X \to X$ as

$Tx = 0$ if $x \in [0, 1 - \delta)$;

$Tx = x/4$ if $x \in [1 - \delta, 1)$, and

$Tx = (1 - \delta)/4$ if $x \geq 1$.

We show that T is both a Kannan mapping and a Chatterjea mapping on (X, d).

Indeed, let $x, y \in X$.

If $x, y \in [0, 1 - \delta)$ or $x, y \geq 1$, then $d(Tx, Ty) = 0$.

If $x, y \in [1-\delta, 1)$, then

$$\begin{aligned} d(Tx, Ty) &= d\left(\frac{x}{4}, \frac{y}{4}\right) = \frac{|x-y|}{4} \leq \frac{1}{4}(x+y) = \frac{1}{3}\left(x - \frac{x}{4} + y - \frac{y}{4}\right) \\ &= \frac{1}{3}[d(x, Tx) + d(y, Ty)] = \frac{1}{3}[d(x, Ty) + d(y, Tx)]. \end{aligned}$$

If $x \in [0, 1-\delta)$ and $y \in [1-\delta, 1)$, then

$$\begin{aligned} d(Tx, Ty) &= d\left(0, \frac{y}{4}\right) = \frac{y}{4} \leq \frac{1}{3}\left(x + y - \frac{y}{4}\right) \\ &= \frac{1}{3}[d(x, Tx) + d(y, Ty)] = \frac{1}{3}[d(x, Ty) + d(y, Tx)]. \end{aligned}$$

If $x \in [0, 1-\delta)$ and $y \geq 1$, then

$$\begin{aligned} d(Tx, Ty) &= d\left(0, \frac{1-\delta}{4}\right) = \frac{1-\delta}{4} \leq \frac{1}{3}\left(x + y - \frac{1-\delta}{4}\right) \\ &= \frac{1}{3}[d(x, Tx) + d(y, Ty)] \leq \frac{1}{3}[d(x, Ty) + d(y, Tx)]. \end{aligned}$$

If $x \in [1-\delta, 1)$ and $y \geq 1$, then

$$\begin{aligned} d(Tx, Ty) &= d\left(\frac{x}{4}, \frac{1-\delta}{4}\right) = \frac{x-(1-\delta)}{4} < \frac{\delta}{4} < \frac{1}{3}\left(x - \frac{x}{4} + y - \frac{1-\delta}{4}\right) \\ &= \frac{1}{3}[d(x, Tx) + d(y, Ty)] = \frac{1}{3}[d(x, Ty) + d(y, Tx)]. \end{aligned}$$

We have shown that T is both a Kannan mapping and a Chatterjea mapping on (X, d) with constant $c = 1/3$.

Finally, define $f : X \to [0, \infty)$ as $f(x) = d(x, Tx)$. We show that f is not lower semicontinuous at $x = 1$.

Indeed, choose a sequence $(x_n)_n$ in X such that $x_n \in (1-\delta, 1)$ and $d(1, x_n) \to 0$. We have

$$\begin{aligned} f(1) - f(x_n) &= d\left(1, \frac{1-\delta}{4}\right) - d\left(x_n, \frac{x_n}{4}\right) = \frac{3+\delta}{4} - \frac{3x_n}{4} \\ &> \frac{3+\delta}{4} - \frac{3}{4} = \frac{\delta}{4}, \end{aligned}$$

for all n, so f is not lower semicontinuous at $x = 1$. We conclude that the function φ as defined by (5) is not a Caristi mapping on (X, d).

References

[1] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fundamenta Mathematicae 3 (1922), 133-181.

[2] J. Caristi, Fixed point theorems for mappings satisfying inwardness conditions, Transactions of the American Mathematical Society 215 (1976), 241–251.

[3] S. K. Chatterjea, Fixed-point theorems, Comptes Rendus de l'Académie Bulgare des Sciences 25 (972), 727–730.

[4] R. Kannan, Some results on fixed points, Bulletin of the Calcutta Mathematical Society 60 (1968), 71–76.

[5] W. A. Kirk, Caristi's fixed point theorem and metric convexity, Colloquium Mathematicum 36 (1976), 81–86.

[6] S. Park, Characterizations of metric completeness, Colloquium Mathematicum 49 (1984), 21–26.

[7] A. Petrusel, Caristi type operators and applications, Studia Universitatis Babeş-Bolyai Mathematica 48 (2003), 115–123.

[8] N. Shioji, T. Suzuki, W. Takahashi, Contractive mappings, Kannan mappings and metric completeness, Proceedings of the American Mathematical Society 126 (1998), 3117–3124.

[9] P. V. Subrahmanyam, Completeness and fixed-points, Monatshefte für Mathematik 80 (1975), 325–330.

[10] T. Suzuki, W. Takahashi, Fixed point theorems and characterizations of metric completeness, Topological Methods in Nonlinear Analysis 8 (1996), 371–382.

[11] J. D. Weston, A characterization of metric completeness, Proceedings of the American Mathematical Society 64 (1977), 186–188.

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Visualization of multi-objective optimization processes and asymmetric norms

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Abstract

Asymmetric norms can be used in the mathematical development of specific tools for visualization of multi-objective optimization problems. The canonical asymmetric norm on a finite dimensional Banach lattice provides a topology that make compatible the arguments based on Euclidean distances and the notion of domination, that is often used in the study of Pareto sets in optimization problems. We show how this tool can be used for helping the decision maker to understand the information provided by a multi-objective optimization program.

Keywords: *asymmetric normed space; visualization; level diagram; optimization.*
MSC: *Primary 46N10; 57N17; Secondary 52A07; 52A20.*

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1. Introduction

Visualization of the set of solutions of a multi-objective problem provided by computer calculations is sometimes the key step for finding the correct answer to the original problem. In the recent paper [3], a new mathematical tool based on asymmetric distances has been introduced. It provides a new way of measuring the (non-symmetric) distance in the finite dimensional space to which the set of solution belongs. These ideas start from the simple assumption that the distance with respect to which the optimal solution is searched is not symmetric, in the sense that the distance from x to y does not coincide with the distance from y to x. From this point of view, two points in the Pareto set of suitable solutions to the problem can be compared using a non classical criterion, mixing the key concept of *domination* and the Euclidean distance in the definition of the topology. In the context of mathematical optimization, a point in a lattice is said to *dominate* other point if the first one is less or equal than the second one in the lattice order. This represent the idea that the first one is closer to the optimal point 0 than the second one.

The proposed mathematical structured can be used for analyzing the approximation of a set of solutions of the problem to a particular "ideal solution point" with respect to a (what is called) "lattice asymmetric norm". If the associated non-symmetric distance d given by the asymmetric norm is performed in a particular way, this problem is equivalent to the one of finding the best approximation with respect to an Euclidean norm from a set of points to a convex cone.

The theory of asymmetric normed spaces is the foundation of the technique explained here. It started to be systematically developed at the beginning of the nineties of the XX century (see for example [2, 8, 10, 11]). The reader can find a complete presentation of this theory in [4] and the references therein.

We recall that an *asymmetric norm* q is a real function $q : X \to [0, \infty)$ satisfying

(1) $q(tx) = tq(x)$ for $t \geq 0$, $x \in X$,
(2) $q(x + y) \leq q(x) + q(y)$,
(3) $q(x) = 0 = q(-x)$ if and only if $x = -x = 0$.

A couple (X, q) is called an asymmetric normed linear space; it can be considered as a topological space with a non-symmetric topology on X that is generated by the (non symmetric) open balls $B_q(x, \varepsilon) = \{y \in X \mid q(y-x) < \varepsilon\}$. This topology is not Hausdorff in the general case (see [4, §1, §2]). However, it is always T_0. The non-Hausdorff case is the one that is used in the present paper, when the asymmetric norm is given by a lattice norm. That is, the mathematical structure is given by what is called an asymmetric Euclidean lattice, that are Banach spaces with a norm $\| \cdot \|$ endowed with an ordering $\leq$ such that if $0 \leq x \leq y$, then $\|x\| \leq \|y\|$. The asymmetric topology in the space is defined then by the asymmetric norm given by

$$q(x) := \|x \vee 0\|, \quad x \in \mathbb{R}^n.$$

The reader can find a lot of information on this particular class of asymmetric norms in [2, 5, 6, 7, 8, 14].

Concretely, we are concerned with the finite dimensional case. Suppose that we have the canonical Euclidean lattice norm on $\mathbb{R}^n$ given by

$$\|(x_1, ..., x_n)\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}, \quad (x_1, ..., x_n) \in \mathbb{R}^n.$$

Consider the canonical lattice order $\leq$ that is provided by the pointwise order. The norm $\| \cdot \|$ is a lattice norm, since if we have two ordered positive elements,

$$(0, ..., 0) \leq (x_1, ..., x_n) \leq (y_1, ..., y_n),$$

we also have that $\|(x_1, ..., x_n)\|_2 \leq \|(y_1, ..., y_n)\|_2$. The asymmetric normed linear lattice that is then considered in our model is

$$q(x) := \|(x_1, ..., x_n) \vee (0, ..., 0)\|_2 = \sqrt{\sum_{i=1}^{n} |\max\{x_i, 0\}|^2}, \quad (x_1, ..., x_n) \in \mathbb{R}^n.$$

In the next section we will show that this can be slightly improved by better adapting our geometric arguments to a broader class of domination relations.

From the point of view of the convergence of the optimization algorithms, compactness and convexity of the Pareto sets are fundamental properties to be studied. These properties for asymmetric normed spaces and asymmetric lattices have been analyzed in recent years. The reader can find the corresponding information in [1, 5, 6, 7, 9, 12, 13] and the references therein; for a complete survey on the fundamental questions in this topic, see also [4, §1.2, §2.5] and [14].

2. Domination relations, preference directions and asymmetric norms

Let us fix now the abstract framework of the multi-objective optimization problems. Suppose that there is a function $\Phi : \Omega \to \mathbb{R}^n$ with a set of values $A \subset \mathbb{R}^n$, that is, $\Phi(\Omega) = A$. Suppose that the set A is already giving a set of solutions of a problem and we are interested in finding a "better" subset of A adding some new optimization requirement. Our technique is given by the following scheme.

- Choose first a point x_0 not belonging to A that would give "an ideal solution to the problem" because of its good properties.
- Choose a set of directions D in $\mathbb{R}^n$ satisfying that, if $a \in A$ is an optimal solution to the problem and $v \in D$, $a+v$ then it is still an optimal solution to the problem if it belongs to A. That is, if we move the optimal solutions in the directions defined by the set D, there is no loss of accuracy in the solution obtained.
- In this setting, the optimization argument is based on the *domination* relation: a point x dominates other point y whenever $y \in x + C^+$, where C^+ is the positive cone of a lattice that must coincide with the set of directions D.

For the aim of this work, we assume that the set of directions is given by the canonical basic vectors of $\mathbb{R}^n$. However, this can be modified in order to define a different sets of preference directions by giving a different basis $\mathcal{B} := \{b_1, ..., b_n\}$ of $\mathbb{R}^n$ and constructing the suitable norm in order to be compatible with the order generated by the cone

$$C^+ := \{\sum_{i=1}^{n} \lambda_i b_i : \lambda_1, ..., \lambda_n \geq 0\}.$$

A classical geometric argument provides the formula

$$\|(x_1,...,x_n)\| := \sqrt{(x_1,...,x_n)(M^{-1})^T \cdot (M^{-1}) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}} = \sqrt{\sum_{i=1}^{n} \alpha_i^2},$$

where $(x_1,...,x_n) \in \mathbb{R}^n$. The matrix M is defined by the coordinates of the vectors $b_1,...,b_n$ as columns, and so $(\alpha_1,...,\alpha_n)$ are the coordinates of the vector $(x_1,...,x_n)$ with respect to the basis $\mathcal{B}$. Figure 2 shows the Euclidean unit ball of a non-canonical Euclidean norm.

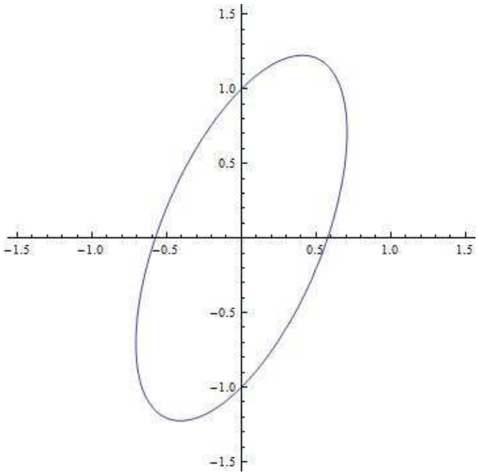

FIGURE 1. Euclidean unit ball of an Euclidean norm defined according to a non-canonical basis $\mathcal{B}$.

The corresponding asymmetric norm is then given by

$$q(((x_1,...,x_n)) = \sqrt{\sum_{i=1}^{n} (\max\{\alpha_i, 0\})^2}. \quad (1)$$

3. Visualizing the domination relations and decision making

Using these tools we obtain the next mathematical structure that may help the decision maker to find his best solution to the problem by providing adequate visualization tools.

- First, the decision maker must define a new domination order in the space given by the convex cone D of preference directions.
- This is the positive cone of a new lattice Euclidean norm generated by the new basis $\mathcal{B}$. This cone —that may be the canonical positive cone— defines the asymmetric norm given in (1).
- Consider the quasi-distance $d(w, v) := q(v-w)$, $v, w \in \mathbb{R}^n$. It is compatible with the ordering and has the meaning that, if $\leq x$ $d(x_0, x)$, it gives an Euclidean distance, and if $x \leq x_0$, then $d(x_0, x) = 0$.
- The optimization with respect to this new distance implies that *there is a gain in minimizing the distance d from x_0 whenever the point $x \in A$ is as close as possible of satisfying $x \leq x_0$.*

Figure 3 illustrates the algorithm when the cone considered is the one generated by a basis $\mathcal{B} := \{b_1, b_2\}$. The vectors $v_1 = -b_1$ and $v_2 = -b_2$ show the (opposite to the) preference directions. All the vectors in the cone E generated by v_1 and v_2 are "better" than x_0 with respect to the optimization criterion fixed by the preference directions. Therefore, when this cone is translated to the points of the set A, all the points that satisfy that $x_0 \in a + E$ has an asymmetric distance to x_0 that coincides with the corresponding Euclidean distance. However, the points of A that do not satisfy this property has an asymmetric distance that measures "how much the point must be moved in order to dominate the point x_0". This gives the interpretation key to the decision maker.

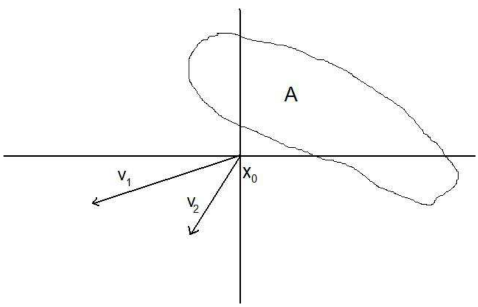

FIGURE 2. Optimal point x_0 and the opposite vectors v_1 and v_2

References

[1] C. Alegre, I. Ferrando, L. M. García-Raffi and E. A. Sánchez-Pérez, Compactness in asymmetric normed spaces, Topology Appl. 155 (2008), 527–539.

[2] C. Alegre, J. Ferrer and V. Gregori, On the Hahn-Banach theorem in certain linear quasi-uniform structures, Acta Math. Hungar. 82 (1999), 315–320.

[3] X. Blasco, G. Reynoso-Meza, E. A. Sánchez-Pérez and J. V. Sánchez Pérez, Asymmetric distances to improve n-dimensional Pareto fronts graphical analysis, Information Sc. 340 (2016), 228–249.

[4] S. Cobzaş, Functional Analysis in Asymmetric Normed spaces, Birkhäuser, Basel, 2013.

[5] J. Conradie, Asymmetric norms, cones and partial orders, Topology and its Applications 193 (2015), 100–115.

[6] J. J. Conradie and M. D. Mabula, Completeness, precompactness and compactness in finite-dimensional asymmetrically normed lattices, Topology and its Applications 160 (2013), 2012–2024.

[7] J. J. Conradie and M. D. Mabula, Convergence and completeness in asymmetrically normed sequence lattices, Quaestiones Mathematicae 38 (2015), 73–81.

[8] J. Ferrer, V. Gregori and C. Alegre, Quasi-uniform structures in linear lattices, Rocky Mountain J. Math. 23 (1993), 877–884.

[9] L. M. García-Raffi, Compactness and finite dimension in asymmetric normed linear spaces, Topology Appl. 153 (2005), 844–853.

[10] L. M. García-Raffi, S. Romaguera and E. A. Sánchez-Pérez, The supremum asymmetric norm on sequence algebras: a general framework to measure complexity distances, Electronic Notes in Theoretical Computer Science 74 (2003), 39–50.

[11] L. M. García Raffi, S. Romaguera and E. A. Sánchez Pérez, Weak topologies on asymmetric normed linear spaces and non-asymptotic criteria in the theory of Complexity Analysis of algorithms, J. Anal. Appl. 2, no. 3 (2004), 125–138.

[12] N. Jonard-Pérez and E. A. Sánchez-Pérez, Compact convex sets in 2-dimensional asymmetric normed lattices, Quaestiones Mathematicae 39 (2016), 73–82.

[13] N. Jonard-Pérez and E. A. Sánchez-Pérez, Extreme points and geometric aspects of convex compact sets in asymmetric normed spaces, Topology and its Applications 203 (2016), 12–21.

[14] M. D. Mabula, Compactness in asymmetrically normed lattices, Doctoral Dissertation, University of Cape Town, Cape Town, 2012.

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Extending products to compactifications

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Abstract

We describe how Arens products can be used to introduce an algebraic structure on compactifications of semigroups. We then survey known results around Arens regularity, and especially around the extreme non-Arens regularity of the group algebra $L_1(G)$, with particular emphasis on tools from Topological Algebra.

1. Densely defined multiplications

Enlarging a topological space to get a compact one, *compactifying a topological space*, is one of the best known procedures in General Topology. In the context of Topological Algebra, topological spaces often carry an algebraic structure and it is only natural to seek for a way to extend this structure to the compactification.

1.1. **Compactifications of semigroups.** We start with a semigroup $\mathbf{S}$ equipped with a topology and a compactification $\mathbf{S}^{\mathcal{X}}$ induced by a unital C^*-algebra of

[1]This extended abstract is based on work carried out jointly with Mahmoud Filali over the last years. I would like to thank him for letting me reproduce here some samples of this work.

continuous and bounded functions $\mathcal{X} \subset \mathcal{CB}(\mathbf{S})$. The compactification $\mathbf{S}^{\mathcal{X}}$ is then the *spectrum* of $\mathcal{X}$ (the set of nonzero multiplicative elements of the dual space $\mathcal{X}^*$) and evaluations define a continuous map with dense range $\epsilon_{\mathcal{X}}\colon G \to G^{\mathcal{X}}$.

The best known example of such a compactification is probably the Stone-Čech compactification $\beta\mathbf{S}$ of $\mathbf{S}$ with $\mathbf{S}$ discrete. It is well known that addition in $\mathbb{N}$ and $\mathbb{Z}$ may be extended to a binary operation in $\beta\mathbb{N}$ or $\beta\mathbb{Z}$ and that subtle combinatorial properties are encoded in the algebraic properties of this operation, see [13] for a wide assortment of examples.

Other examples of compactifications of this sort are the Bohr (or almost periodic) compactification, the weakly almost periodic compactification or the LUC-compactification induced, respectively, by the algebras $\mathcal{AP}(G)$, $\mathcal{WAP}(G)$ and $LUC(G)(G)$, see [3] for more information on these algebras.

As a matter of fact, the first extension of semigroup multiplications to compactifications appeared in [1] in the context of Banach algebras. The unit ball $\mathfrak{A}_1$ of a Banach algebra $\mathfrak{A}$ is a multiplicative semigroup. The unit ball $\mathfrak{A}_1^{**}$ of the bidual of $\mathfrak{A}$ with the weak*-topology is then a compactification of $\mathfrak{A}_1$. The embedding given by the usual evaluation map: $\epsilon_{\mathfrak{A}}\colon \mathfrak{A} \to \mathfrak{A}^{**}$.

1.2. **Extending the semigroup operation. Naïve approach.** Let $(\mathbf{S}, \cdot)$ be a semigroup with a topology and let $\mathbf{S}^{\mathcal{X}}$ be a compactification of $\mathbf{S}$. We would like to extend the operation $\cdot$ to $\mathbf{S}^{\mathcal{X}}$. If $\Phi, \Psi \in \mathbf{S}^{\mathcal{X}}$ and $\Phi = \lim_\alpha \epsilon_{\mathcal{X}}(a_\alpha)$, $\Psi = \lim_\beta \epsilon_{\mathcal{X}}(b_\beta)$, with $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ nets in S, we could define:

$$\Phi \Box \Psi = \lim_\alpha \lim_\beta \epsilon_{\mathcal{X}}(a_\alpha)\epsilon_{\mathcal{X}}(b_\beta). \tag{1}$$

This definition is clearly problematic, it is not even clear whether $\Phi\Box\Psi$ is uniquely determined. There is also an arbitrary choice of sides that should be clarified.

When $S = \mathfrak{A}$ is a Banach algebra and $T = \mathfrak{A}^{**}$, Arens [1] proved that it is possible to define a product on $\mathfrak{A}^{**}$ that satisfies (1), makes $\mathfrak{A}^{**}$ into a Banach algebra and the morphism $\mathfrak{A} \to \mathfrak{A}^{**}$ into a Banach algebra homomorphism.

If $\mathbf{S}$ is a discrete semigroup and $\mathcal{X} = LUC(G)(\mathbf{S}) = \ell_\infty(\mathbf{S})$, then $\mathbf{S}^{\mathcal{X}} = \beta\mathbf{S}$, the Stone-Čech compactification of $\mathbf{S}$. In this case $\mathbf{S}^{\mathcal{X}}$ is a subsemigroup of $\ell_1(\mathbf{S})^{**} =$

$M(\beta\mathbf{S})$ under $\Box$. Hence, $\Box$ defines an operation on $\mathbf{S}^{\mathcal{X}}$. The same idea of Arens can be used to extend the group operation to other compactifications in such a way that $\epsilon_{\mathcal{X}}$ is a semigroup homomorphism. The relation between the resulting algebraic structure and the topology varies depending on the algebra and the kind of group.

2. Defining the Arens product

Arens rigorous definition of product applied to compactifications of semigroups follows.

Definition 1. Let $\mathbf{S}$ be a semigroup with a topology and let $\mathcal{X} \subseteq \mathcal{CB}(\mathbf{S})$ be a C^***-algebra**. Let $p, q \in \mathbf{S}^{\mathcal{X}}$ and $\phi \in \mathcal{X}$:

(1) First, define $T_{q,\phi}\colon \mathbf{S} \to \mathbb{C}$, by $T_{q,\phi}(g) = q(L_g\phi)$
(2) **If** $\mathbf{T}_{\mathbf{q},\phi} \in \mathcal{X}$, define $\rho_q\colon \mathbf{S}^{\mathcal{X}} \to \mathbf{S}^{\mathcal{X}}$ by $\rho_q(p)(\phi) = p\,(T_{q,\phi})$.
(3) Finally, $\mathbf{p}\Box\mathbf{q} = \rho_{\mathbf{q}}(\mathbf{p})$.

This product is not defined for every C*-subalgebra of $\mathcal{CB}(\mathbf{S})$, it works only for those satisfying the condition in (2). When G is, e.g., a nondiscrete locally compact group and $\mathcal{X} = \mathcal{CB}(G)$, that condition is not satisfied. When $\mathcal{X}$ is, in the terminology of [3, Definition 2.10] m-admissible, then $T_{q,\phi} \in \mathcal{X}$ and $(\mathbf{S}^{\mathcal{X}}, \Box)$ is a semigroup; it is actually a right topological semigroup.

With $\Box$ thus defined, $\mathbf{S}^{\mathcal{X}}$ satisfies the limit property in (1).

Banach algebras. The Arens product was originally introduced in [1] as a way to extend a *bilinear operation* to the bidual of a Banach space. The definition of $\Box$ in Definition 1 reduces to the original Arens product when $\mathbf{S} = \mathfrak{A}_1$, the unit ball of a Banach algebra $\mathfrak{A}$, $\mathcal{X} = \mathfrak{A}^*$ (restricted to $\mathfrak{A}_1$) and the bilinear operation is multiplication on $\mathfrak{A}$. In that case, given $\Phi, \Psi \in \mathfrak{A}^{**}$, $f \in \mathfrak{A}^*$ and $a, b \in \mathfrak{A}$ the above product can be read as follows.

(1) First, a right action of $\mathfrak{A}$ on $\mathfrak{A}^*$ is defined by $\langle f \cdot a, b\rangle = \langle f, a\,b\rangle$.
(2) Then a left action of $\mathfrak{A}^{**}$ on $\mathfrak{A}^*$ by $\langle \Psi \cdot f, a\rangle = \langle \Psi, f \cdot a\rangle$ $(\Psi \in \mathfrak{A}^{**})$.
(3) Finally: $\langle \mathbf{\Phi}\Box\mathbf{\Psi}, \mathbf{f}\rangle = \langle \mathbf{\Phi}, \mathbf{\Psi} \cdot \mathbf{f}\rangle$.

Observe that, in this case, $T_{\psi,f} = f \cdot a$. It is easily checked that $f \cdot a \in \mathfrak{A}^*$, hence the Condition in (2) of Definition 1 holds.

3. Properties of $\square$

We summarize here some of the properties of $\square$:

Theorem 2. *Let* $\mathbf{S}$ *be a semitopological semigroup and let* $\mathfrak{A}$ *be a Banach algebra*

(1) $(\mathfrak{A}^{**}, \square)$ *is a Banach algebra and the map* $\Psi \mapsto \Psi\square\Phi$ *is weak∗-continuous for every* $\Phi \in \mathfrak{A}^{**}$.
(2) *If* $\mathbf{S}$ *is discrete,* $(\beta\mathbf{S}, \square)$ *is a right topological semigroup.*
(3) *If* $\mathcal{X} \subset \mathcal{CB}(\mathbf{S})$, $(\mathbf{S}^{\mathcal{X}}, \square)$ *is a semitopological semigroup if and only if* $\mathcal{X} \subset \mathcal{WAP}(\mathbf{S})$, *where* $\mathcal{WAP}(\mathbf{S})$ *is the algebra of weakly almost periodic functions.* $(\mathbf{S}^{\mathcal{X}}, \square)$ *is a topological group if and only if* $\mathcal{X} \subset \mathcal{AP}(\mathbf{S})$, *where* $\mathcal{AP}(\mathbf{S})$ *is the algebra of almost periodic functions.*

4. Taking sides

Arens noticed that the order in the limits in (1) is important. We could actually have swapped the order in the definitions of Section 2 and obtain another product on $\mathfrak{A}^{**}$ or $G^{\mathcal{X}}$. If we denote this product as $\Diamond$, then

$$\Phi\Diamond\Psi = \lim_{\beta}\lim_{\alpha} \epsilon_{\mathcal{X}}(a_\alpha)\epsilon_{\mathcal{X}}(b_\beta). \tag{2}$$

We see next that $\square$ and $\Diamond$ are sometimes different, and sometimes equal, and that this does not depend only on the Banach space structure.

Example 3. Consider the Banach algebras $\mathfrak{A}_1 = (\ell_1(\mathbb{Z}), \cdot)$ and $\mathfrak{A}_2 = (\ell_1(\mathbb{Z}), *)$ where $\cdot$ and $*$ denote, respectively, the pointwise and convolution products. Then:

(1) $\Phi\Diamond\Psi = \Phi\square\Psi$, for every Φ, $\Psi \in \mathfrak{A}_1^{**}$
(2) There are $p, q \in \beta\mathbb{Z} \subset \mathfrak{A}_2^{**}$ such that $p\Diamond q \neq p\square q$.

Proof. **(1) Pointwise Product.** Consider the usual duality $c_0(\mathbb{Z})^* \cong \ell_1(\mathbb{Z})$, $\ell_1(\mathbb{Z})^* \cong \ell_\infty(\mathbb{Z})$. If $\mathfrak{R} = c_0(\mathbb{Z})^\perp \subset \ell_1(\mathbb{Z})^{**}$, then it is not difficult to see that $\mathfrak{A}_1^{**} = \mathfrak{A}_1 \oplus \mathfrak{R}$.

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One can deduce then from (1) and (2) that

$$\Psi\Box\Phi = \Psi\Diamond\Phi = 0, \qquad \text{if either } \Psi \text{ or } \Phi \text{ belong to } \mathfrak{R}. \tag{3}$$

This, together with the equality $\mathfrak{A}_1^{**} = \mathfrak{A}_1 \oplus \mathfrak{R}$, shows that $\Psi\Box\Phi = \Psi\Diamond\Phi$ for every $\Psi, \Phi \in \ell_1(\mathbb{Z})^{**}$ and that, actually, all the action takes place in $\mathfrak{A}_1$. **(2) Convolution Product.** Consider for each $z \in \mathbb{Z}$, the element $\delta_z \in \ell_1(\mathbb{Z}) = \mathfrak{A}_2$ taking the value 1 at z and 0 for the remaining integers. Then $\delta_{z_1} * \delta_{z_2} = \delta_{z_1+z_2}$. Identifying δ_z with a point-mass measure, the map $z \mapsto \delta_z$ sends $\mathbb{Z}$ into $M(\beta\mathbb{Z})$, the measure algebra of $\beta\mathbb{Z}$. Observe that $\beta\mathbb{Z} \subset M(\beta\mathbb{Z}) = \mathfrak{A}_2^{**}$.

Let $p \in \beta\mathbb{Z} \setminus \mathbb{Z}$ be an accumulation point of $\mathbb{N}$, that is $p \in \mathfrak{A}_2^{**}$ is an accumulation point of $\{\delta_n \colon n \in \mathbb{N}\} \subset \mathfrak{A}_2$, and let $q \in \beta\mathbb{Z} \setminus \mathbb{Z}$ be an accumulation point of $-\mathbb{N}$. Then $p = \lim_\alpha \delta_{n_\alpha}$ and $q = \lim_\beta \delta_{-m_\beta}$, with $n_\alpha, m_\beta \in \mathbb{N}$. Then $p\Box q = \lim_\alpha \lim_\beta \delta_{n_\alpha - m_\beta} \in \overline{-\mathbb{N}}$, while $p\Diamond q \in \overline{N}$. Denoting by $\chi_{\mathbb{N}} \in \mathfrak{A}_2^*$ the characteristic function of $\mathbb{N}$, we see that $\langle p\Box q, \chi_{\mathbb{N}}\rangle \neq \langle p\Diamond q, \chi_{\mathbb{N}}\rangle$. $\Box$

Theorem 4. *Let* $\mathbf{S}$ *be a commutative semigroup and let* $\mathbf{S}^{\mathfrak{X}}$ *be a compactification of* S*. Let* $\Box$ *and* $\Diamond$ *be operations on* $\mathbf{S}^{\mathfrak{X}}$ *that, respectively, satisfy conditions* (1) *and* (2)*, then* $(\mathbf{S}^{\mathfrak{X}}, \Box)$ *is commutative if and only if* $\Box = \Diamond$*.*

We see as a consequence of Example 3 that $\beta\mathbb{Z}$ (and hence $\ell_1(\mathbb{Z})^{**}$ with convolution) are not commutative.

5. Arens regularity and the center

We start by naming those Banach algebras for which both Arens products coincide.

Definition 5. We say that a semigroup $\mathbf{S}$ with compactification $\mathbf{S}^{\mathfrak{X}}$ is *Arens-regular* if $p\Box q = p\Diamond q$ for any $p, q \in \mathbf{S}^{\mathfrak{X}}$. In the case of a Banach algebra $\mathfrak{A}$ the term is applied to every $\Psi, \Phi \in \mathfrak{A}^{**}$.

Example 3 already shows that $\ell_1(\mathbb{Z})$ is Arens regular under pointwise product, and that, under convolution it is not.

Once we have seen that even for algebras as simple as $\ell_1(\mathbb{Z})$, the two Arens products may be different, we wonder for what kind of elements this may happen. We then define:

Definition 6. Let $(\mathbf{S},\cdot)$ be a semigroup with compactification $\mathbf{S}^{\mathcal{X}}$. We define the *left* and *right topological center* of $\mathbf{S}^{\mathcal{X}}$ as:

$$Z^{(\ell)}(\mathbf{S}^{\mathcal{X}}) = \left\{\Phi \in \mathbf{S}^{\mathcal{X}} \colon \Phi\Box\Psi = \Phi\Diamond\Psi,\ \text{for all } \Psi \in \mathbf{S}^{\mathcal{X}}\right\} \tag{4}$$

$$Z^{(r)}(\mathbf{S}^{\mathcal{X}}) = \left\{\Phi \in \mathbf{S}^{\mathcal{X}} \colon \Psi\Box\Phi = \Psi\Diamond\Phi,\ \text{for all } \Psi \in \mathbf{S}^{\mathcal{X}}\right\} \tag{5}$$

Particularizing to the case of Banach algebras, it is not difficult to see that:

Proposition 7. *Let $\mathfrak{A}$ be a Banach algebra. Then:*

(1) $\mathfrak{A} \subset Z^{(\ell)}(\mathfrak{A}) \cap Z^{(r)}(\mathfrak{A})$.
(2) $\Phi \in Z^{(\ell)}(\mathfrak{A})$ *if and only if the map* $\Psi \mapsto \Phi\Box\Psi$ *is weak*-continuous.*

Theorem 8 (Some basic facts around regularity)**.** *We collect here a few general results on Arens-regularity that constitute the core of what is known on the subject.*

(1) *(Takeda, 1954; Civin and Yood, 1961) C^*-algebras are Arens-regular.*
(2) *(Pym, 1965) A Banach algebra $\mathfrak{A}$ is Arens-regular if and only if $\mathfrak{A}^* = \mathcal{WAP}(\mathfrak{A})$, where*

$$\mathcal{WAP}(\mathfrak{A}) = \left\{f \in \mathfrak{A}^* \colon \lim_n \lim_m f(a_n \cdot b_m) = \lim_m \lim_n f(a_n \cdot b_m),\ \forall\, (a_n)_n, (b_m)_m \subset \mathfrak{A}\right\}.$$

(3) *(Ülger, 1986) $\mathcal{WAP}(L_1(G)) = \mathcal{WAP}(G)$.*
(4) *(Young 1973) $L_1(G)$ is Arens-regular if and only if G is finite.*

6. Irregularity

Since the main algebras of Harmonic Analysis are not regular it comes into question how irregular they may be. Two ways of measuring when irregularity is as strong as possible can be found in the literature: when the center is as small as possible and when the quotienty $\mathfrak{A}/\mathcal{WAP}(\mathcal{A})$ is as large as possible.

Definition 9. Let $\mathfrak{A}$ be a Banach algebra.

(1) (Dales and Lau [6]) $\mathfrak{A}$ is *strongly Arens irregular (SAI)* if $Z^{(\ell)}(\mathfrak{A}) = Z^{(r)}(\mathfrak{A}) = \mathfrak{A}$.
(2) (Granirer [11]) $\mathfrak{A}$ is *extremely non Arens-regular (ENAR)* if there is a closed linear subspace $J \subset \dfrac{\mathfrak{A}^*}{\mathcal{WAP}(A)}$ that has $\mathfrak{A}^*$ as linear quotient.

Before the term had been coined Lau and Losert had already proved that $L_1(G)$ is SAI for every locally compact group.

Theorem 10 (Lau and Losert, [15]). *Let G be a locally compact group. $L_1(G)$ is SAI if and only if G is finite.*

7. $L_1(G)$ is ENAR

We outline in this section how to prove that $L_\infty(G)/\mathcal{CB}(G)$ contains an isometric copy of $L_\infty(G)$ for every locally compact group G. This means that $L_1(G)$ is ENAR. We start with a useful observation.

Proposition 11 ([10, 7]). *If $\kappa = \max\{\kappa(G), \chi(G)\}$, then there is a linear isometry $\Psi\colon L_\infty(G) \to \ell_\infty(\kappa)$.*

7.1. **Quotients.** The focus is now on finding copies of $\ell_\infty(\kappa)$, with κ as large as possible, in the quotients $L_\infty(G)/\mathcal{WAP}(G)$.

The concepts of $\mathcal{X}$-interpolation set and approximable $\mathcal{X}$-interpolation set proved to be very helpful to find copies of $\ell_\infty(\kappa)$ in quotients. We address the reader to [9] for the definitions of these terms and an in-length study of these concepts. In [8] we found a fairly general method to find such quotients, based on a technique first introduced by C. Chou:

Theorem 12 (Filali and Galindo, [8]). *Let G be a locally compact group and let $\mathcal{X} \subset \mathcal{Y} \subset LUC(G)(G)$ be two C^*-subalgebras of $\mathcal{CB}(G)$. Let U be a compact neighborhood of e such that T is right U-uniformly discrete. If G contains a family of pairwise disjoint sets $\{\mathbf{T}_\eta \colon \eta < \kappa\}$ with:*

1. *T_η is not an $\mathcal{X}$-interpolation set for any $\eta < \kappa$.*
2. *$\bigcup_{\eta<\kappa} T_\eta$ is an approximable $\mathcal{Y}$-interpolation set.*

Then, there is a linear isometry $\Psi\colon \ell_\infty(\kappa) \to \mathcal{Y}/\mathcal{X}$.

In a more or less explicit way, this approach is present in all the following results:

Theorem 13. *Let G be a locally compact group. In the following cases there exists a linear isometry Ξ.*

(1) ([8]) $\Xi\colon \ell_\infty\big(\kappa(\mathbf{G})\big) \to \dfrac{\mathcal{CB}(\mathbf{G})}{LUC(\mathbf{G})(\mathbf{G})}$, *if G is noncompact and nondiscrete.*

(2) ([8]) $\Xi\colon \ell_\infty\big(\kappa(\mathbf{G})\big) \to \dfrac{\mathcal{WAP}(\mathbf{G})}{\mathcal{B}(\mathbf{G})}$, *if G has small invariant neighbourhood.*

(3) ([4]) $\Xi\colon \ell_\infty\big(\kappa(\mathbf{G})\big) \to \dfrac{LUC(\mathbf{G})(\mathbf{G})}{\mathcal{WAP}(\mathbf{G})}$, *if G is not compact.*

It follows from (3) above and Proposition 11 that groups that are far from being compact are ENAR. Let $\kappa(G)$ denote the compact covering number of G and $\chi(G)$ the local character:

Theorem 14 (Fong and Neufang, [10]; Bouziad and Filali [4])**.** *If $\kappa(G) \geq \chi(G)$, then $L_1(G)$ is ENAR.*

7.2. **The compact case.** After Theorem 14, it is clear that the case of compact groups is the obviously outstanding one. For small groups this was solved by Bouziad and Filali:

Theorem 15 (Bouziad and Filali, [4])**.** *If G is compact, then there is a linear isometry $\boldsymbol{\psi}\colon \ell_\infty \to \dfrac{\mathbf{L}_\infty(\mathbf{G})}{\mathcal{CB}(\mathbf{G})} = \dfrac{L_\infty(G)}{\mathcal{WAP}(G)}$. Therefore $L_1(G)$ is ENAR if G is compact and metrizable.*

The proof of Theorem 15 is based on constructing an infinite collection of open disjoint subsets of G to embed ℓ_∞ in the quotient $L_\infty(G)/\mathcal{CB}(G)$ and then apply Proposition 11. For nonmetrizable G we would need an uncountable family of open disjoint subsets of G to be able to replace ℓ_∞ by $\ell_\infty(\kappa)$. But compact groups have countable cellularity, see [2, Corollary 4.18], and no such family exists.

To work with larger compact groups it was useful to understand that compact groups are much like products.

Theorem 16 (Adapted from Grekas and Merkourakis, [12])**.** *If G is a compact group, there are two metrizable groups M_1 and M_2, two compact groups K_1 and K_2 and two* Haar measure preserving *quotient maps:*

$$\phi_1\colon M_1 \times K_1 \to G \quad \text{and} \quad \phi_2\colon G \to M_2 \times K_2.$$

Theorem 16 reduces the compact case to the case of a product $G = M \times K$ with M metrizable. We can then adapt the construction of Theorem 12 to this situation:

Theorem 17 (Filali and Galindo [7]). *Let H and M be locally compact* σ-compact *groups with M* nondiscrete and metrizable. *Then there exists a linear isometry* $\Xi_{\mathbf{0}} \colon \ell_\infty(\mathbf{L}_\infty(\mathbf{H})) \longrightarrow \dfrac{\boldsymbol{L_\infty(M \times H)}}{\boldsymbol{\mathcal{CB}(M \times H)}}$.

With this theorem providing the isometry Ξ_0, it is possible to put together a family of linear isometries like the following one that ultimately leads to Theorem 18.

$$\begin{array}{ccccc}
\dfrac{\mathbf{L}_\infty(\mathbf{G})}{\mathcal{CB}(\mathbf{G})} & \xleftarrow{\;\Xi_3\;} & \dfrac{L_\infty(M_1 \times K_1)}{\mathcal{CB}(M_1 \times K_1)} & \xleftarrow{\;\Xi_0\;} & \ell_\infty\left(L_\infty(K_1)\right) \\
 & & & & \Big\uparrow{\scriptstyle \Xi_4} \\
\mathbf{L}_\infty(\mathbf{G}) & \xrightarrow{\;\Xi_2\;} & L_\infty\left(M_2 \times K_2\right) & \xrightarrow{\;\Xi_1\;} & \ell_\infty\left(L_\infty(K_2)\right).
\end{array}$$

Theorem 18 (Filali and Galindo [7]). *If G is a compact group, there is a linear isometry* $\boldsymbol{\psi} \colon L_\infty(G) \to \dfrac{L_\infty(G)}{\mathcal{CB}(G)}$. *$L_1(G)$ is therefore ENAR.*

A classical theorem of Davis to the effect that any locally compact group G contains an open subgroup that is Haar-homeomorphic (i.e. admits a homeomorpism that preserves Haar measure) to $\mathbb{R}^n \times K$ with K compact together with Theorem 18 and Theorem 15 finally gives:

Theorem 19 (Filali and Galindo [7]). *If G is any locally compact group, then $L_1(G)$ is ENAR.*

8. Concluding Remarks

I would like to add that the study of Arens regularity in the algebras of Fourier Analysis is by no means closed. While it is known that $L_1(G)$ is SAI and ENAR for every infinite locally compact group, it is not known whether the Fourier algebra is irregular for every such group. If $\kappa(G) \leq \chi(G)$, Hu [14] proved that $A(G)$ is ENAR. Observe that, for Abelian G, $A(G)$ is isometrically isomorphic to $L_1(\widehat{G})$ and that $\kappa(G) = \chi(\widehat{G})$, so in the Abelian case, this would follow from Theorem 15. Losert [16] recently posted a preprint showing that $A(F(a,b))$ is not SAI, proving therefore that the analog of Theorem 10 is not true for the Fourier algebra of a nonAbelian group. On the other hand the algebra $M(G)$ was recently shown to be SAI for every locally compact group [17].

References

[1] R. Arens, The adjoint of a bilinear operation, Proc. Amer. Math. Soc. 2 (1951), 839–848.

[2] A. Arhangel′skii and M. Tkachenko, Topological groups and related structures, Atlantis Studies in Mathematics, vol. 1, Atlantis Press, Paris, 2008.

[3] J. F. Berglund, H. D. Junghenn, and Paul Milnes, Analysis on semigroups, John Wiley & Sons Inc., New York, 1989.

[4] A. Bouziad and M. Filali, On the size of quotients of function spaces on a topological group, Studia Math. 202 (2011), no. 3, 243–259.

[5] C. Chou, Weakly almost periodic functions and thin sets in discrete groups, Trans. Amer. Math. Soc. 321 (1990), no. 1, 333–346.

[6] H. G. Dales and A. T.-M. Lau, The second duals of Beurling algebras, Mem. Amer. Math. Soc. 177 (2005), no. 836, vi+191.

[7] M. Filali and J. Galindo, Extreme non-Arens regularity of the group algebra, preprint (`http://arxiv.org/abs/1307.1000`).

[8] M. Filali and J. Galindo, Interpolation sets and the size of quotients of function spaces on a locally compact group, Trans. Amer. Math. Soc., to appear.

[9] M. Filali and J. Galindo, Approximable $\mathcal{WAP}$- and $\mathcal{LUC}$-interpolation sets, Adv. Math. 233 (2013), 87–114.

[10] C. K. Fong and M. Neufang, On the quotient space and extreme non-arens regularity of UC(G)/WAP(G), preprint (2006).

[11] E. E. Granirer, Day points for quotients of the Fourier algebra $A(G)$, extreme nonergodicity of their duals and extreme non-Arens regularity, Illinois J. Math. 40 (1996), no. 3, 402–419.

[12] S. Grekas and S. Mercourakis, On the measure-theoretic structure of compact groups, Trans. Amer. Math. Soc. 350 (1998), no. 7, 2779–2796.

[13] N. Hindman and D. Strauss, Algebra in the Stone-Čech compactification, de Gruyter Expositions in Mathematics, vol. 27, Walter de Gruyter & Co., Berlin, 1998, Theory and applications.

[14] Z. Hu, Inductive extreme non-Arens regularity of the Fourier algebra $A(G)$, Studia Math. 151 (2002), no. 3, 247–264.

[15] A. T. Lau and V. Losert, On the second conjugate algebra of $L_1(G)$ of a locally compact group, J. London Math. Soc. (2) 37 (1988), no. 3, 464–470.

[16] V. Losert, The centre of the bidual of fourier algebras (discrete groups), preprint (`https://arxiv.org/abs/1605.04523`).

[17] V. Losert, M. Neufang, J. Pachl and J. Steprāns, Proof of the Ghahramani-Lau conjecture, Adv. Math. 290 (2016), 709–738.

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Completeness type properties on $C_p(X, Y)$ spaces

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Abstract

In this paper we improve several results presented in [2] *and in* [4] *related to characterize several kind of pseudocompleteness and compactness properties in spaces of continuous functions of the form* $C_p(X, Y)$. *In particular, we prove that for every space* X *and every separable metrizable topological group* G *for which* $C_p(X, G)$ *is dense in* G^X, $C_p(X, G)$ *is weakly* α*-favorable if and only if* X *is* u_G*-discrete. This result helps us to obtain two generalizations of a theorem due to V.V. Tkachuk in* [14]. *Besides, we obtain several applications to weakly pseudocompact spaces.*

1. Notations, basic definitions and introduction

Throughout this article all topological spaces are considered Tychonoff and with more than one point if the contrary is not specified.

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We are going to denote by $C_p(X,Y)$ the space of continuous functions from X to Y with the topology inherited by the product topology in Y^X; that is, the topology in $C_p(X,Y)$ is the topology generated by all the sets of the form:

$$[x_1,\ldots,x_n;B_1,\ldots,B_n] = \{f \in C_p(X,Y) : f(x_i) \in B_i, i = 1,\ldots,n\}$$

where $n \in \omega$, $\{x_1,\ldots,x_n\} \subseteq X$, and $B_1,\ldots,B_n$ are open subsets of Y. When Y is the real line with its usual topology, we write $C_p(X)$ instead of $C_p(X,\mathbb{R})$. For every space of the form $C_p(X,Y)$ considered in this article, the spaces X and Y are such that $C_p(X,Y)$ is dense in Y^X.

A compactification of a space X is a compact space K containing X as a dense subspace. The statement *"$X \subseteq Y$ is G_δ-dense in Y"* means that each nonempty G_δ-set in Y contains at least one point in X.

Pseudocompactness and the Baire property are outstanding classes of spaces, but they are not productive. Efforts have been made to define classes of spaces which contain all pseudocompact spaces, satisfy the Baire Category Theorem and are closed under arbitrary topological products. One of the first successful achievement in this direction was the introduction of the class of pseudocomplete spaces defined by J.C. Oxtoby in [11]. This paper was followed by that of A.R. Todd [15] where he modified the definition given by Oxtoby and proved that his new class of pseudocomplete spaces also satisfies the required conditions. We are going to call these spaces *Oxtoby complete* and *Todd complete*, respectively, following the terminology used in [2]. Whether the two properties are equivalent remains an open question.

D.J. Lutzer and R.A. McCoy in [10] and V.V. Tkachuk in [13] analyzed Oxtoby pseudocompleteness in spaces of real-valued continuous functions with the pointwise covergence topology. They proved:

Theorem 1 ([10, Theorem 8.3]). *Let X be a pseudonormal space. Then the following are equivalent: (1) $C_p(X)$ is Oxtoby complete; (2) $C_p(X)$ is weakly α-favorable; (3) Player II has a winning strategy in the game $\Gamma(X)$; (4) X is ω-discrete; (5) $C_p(X)$ is G_δ-dense in $\mathbb{R}^X$.*

Theorem 2 ([13, Theorem 4.1]). *The following conditions are equivalent for every space X: (1) X is a C_ω-discrete space; (2) $C_p(X)$ is Oxtoby complete; (3) $\upsilon C_p(X) = \mathbb{R}^X$.*

Another class of spaces which remains between pseudocompact spaces and Baire spaces and is productive is the class of the so called *weakly pseudocompact* spaces introduced in [7]. A space X is *weakly pseudocompact* if it is G_δ-dense in one of its compactifications. F.W. Eckertson asked in [5] if the spaces $\mathbb{R}^{\omega_1}$ or $\mathbb{Z}^{\omega_1}$ are weakly pseudocompact. A more general problem is to characterize spaces X and Y for which the space of continuous functions from X to Y with the pointwise convergence topology, $C_p(X,Y)$, is weakly pseudocompact. Some contributions in the direction of solving this problem are made in [2], [3] and [4].

In this article, we are going to continue the study of when a space $C_p(X,Y)$ satisfies one of the three pseudocompleteness properties described above for Tychonoff spaces X and Y such that $C_p(X,Y)$ is dense in Y^X. Throughout this paper, we improve some of the results obtained in [2] and [4], and we generalize a classic result by V.V. Tkachuk.

2. Weakly α-favorable spaces

Definition 3. (1) A family $\mathcal{B}$ of sets in a topological space X is called π-*base* (respectively, π-*pseudobase*) if every element of $\mathcal{B}$ is open (respectively, has a nonempty interior) and every nonempty open set in X contains an element of $\mathcal{B}$.

(2) A space is *Oxtoby complete* (respectively, *Todd complete, Telgarsky complete*) if there is a sequence $\{\mathcal{B}_n : n < \omega\}$ of π-bases, (respectively, π-pseudobases, bases) in X such that for any sequence $\{U_n : n < \omega\}$ where $U_n \in \mathcal{B}_n$ and $cl_X U_{n+1} \subseteq int U_n$ for all n, then $\bigcap_{n<\omega} U_n \neq \emptyset$. A sequence $\{\mathcal{B}_n : n < \omega\}$ of π-bases, (respectively, π-pseudobases, bases) in X which testifies that X is Oxtoby complete (resp., Todd complete, Telgarsky complete) is called *Oxtoby sequence* (resp., *Todd sequence, Telgarsky sequence*).

J.C. Oxtoby in [11] proved that all Oxtoby complete spaces have the Baire property, and arbitrary products of Oxtoby complete spaces are Oxtoby complete. A.R. Todd in [15] came up with a modification of the Oxtoby completeness property: the Todd completeness. Of course, every Oxtoby complete space is Todd complete. In [15], A.R. Todd observed that using similar argumentations to those given by Oxtoby, it happens that every completely metrizable topological space and every locally compact T_2 space is Oxtoby complete, every Todd complete space is a Baire space, and the arbitrary product of Todd complete spaces is Todd complete. Moreover, he proved that every pseudocompact space is Oxtoby complete (Proposition 2.4 in [15]). If X contains a dense Oxtoby complete subspace, then X is Oxtoby complete [1]. For Telgarsky and Todd completeness:

Proposition 4. *If X is dense in Y and X is Todd (resp., Telgarsky) complete, then Y is Todd (resp., Telgarsky) complete.*

The classical *Banach-Mazur game* $\mathrm{BM}(X)$ in a topological space X is defined as follows: There are two players I and II. Player I starts the game by choosing a nonempty open set $B_0 \subseteq X$, and II responds by choosing a nonempty open set $B_1 \subseteq B_0$. In the $(n+1)^{st}$ inning I chooses a nonempty open set $B_{2n+2} \subseteq B_{2n+1}$, and II responds with a nonempty open set $B_{2n+3} \subseteq B_{2n+2}$, and so on. In this way the two players produce a *play*

$$B_0, B_1, \ldots, B_{2n}, B_{2n+1}, \ldots (n < \omega).$$

Player II wins this play if $\bigcap_{n<\omega} B_n \neq \emptyset$, and I wins otherwise. We will say that a space X is *weakly α-favorable* if Player II has a winning strategy in the Banach-Mazur game on X. (Choquet terminology: the Players I and II are called β and α respectively). A space X is *α-favorable* if Player II has a stationary winning strategy (a strategy which depends on the opponent's last move only) in $\mathrm{BM}(X)$. In [4] it was proved that every Todd complete space is weakly α-favorable; but that proof shows, indeed, that every Todd complete space is α-favorable.

As it is well known, the open continuous image of a Baire space is a Baire space. Whether such mappings also preserve Oxtoby completeness is an open problem. In [1] the authors prove that the metrizable continuous image of an Oxtoby complete space is Oxtoby complete. The following theorem is a corollary of item (11) of the Theorem in [16] where X is asked to be T_0 and with a countable order base. However, we would like to include a proof of this result with some modifications to the original one given in [16] .

Theorem 5. *Every weakly α-favorable metrizable space X contains a dense zero-dimensional G_δ completely metrizable subspace.*

Corollary 6. *If X is weakly α-favorable, Y is metrizable and $f : X \to Y$ is an onto open and continuous function, then Y contains a dense subset which is completely metrizable.*

Lemma 7. *Every completely metrizable space is Telgarsky complete.*

Corollary 8. *A metrizable space Y is weakly α-favorable if and only if Y contains a dense completely metrizable subspace.*

Let $\mathcal{P}$ be the set of the following properties considered and defined in [2]: Oxtoby countable compactness, Todd countable compactness, strong Sánchez-Okunev completeness, strong Oxtoby completeness, strong Todd completeness, Sánchez-Okunev countable compactness and Sánchez-Okunev completeness. And let $\mathcal{Q}$ be the properties considered in [4]: α-favorability, weak α-favorability, having a Markov winning strategy in the Choquet game. Diagram 2 in [2] shows the relations of properties in $\mathcal{P}$. This diagram can be completed adding the properties in $\mathcal{Q}$ by taking into account that every Todd complete space is α-favorable [4], and in every Telgarsky complete space Z Player II has a Markov winning strategy in the Choquet game $\mathrm{Ch}(Z)$ [12, Theorem 2.2]. And Player II having a Markov winning strategy in the Choquet game $\mathrm{Ch}(Z)$ implies that Player II has a Markov winning strategy in the Banach-Mazur game $\mathrm{BM}(Z)$. Finally, Galvin and Telgarsky in [8], Corollary 9, proved that a Markov winning strategy made by Player II in $\mathrm{BM}(Z)$ can be reduced to a stationary winning strategy; that is, when Player II has a Markov winning strategy in the Choquet game $\mathrm{Ch}(Z)$ it implies that Z is α-favorable. Next theorem adds properties of $\mathcal{Q}$ to Theorem 4.1 in [2].

Theorem 9. *If X is metrizable, then the following claims are equivalent: (1) X contains a dense completely metrizable subspace; (2) X has property $P \in \mathcal{P}$; (3)*

X has property $Q \in \mathcal{Q}$; (4) X is Telgarsky complete; (5) X is Oxtoby complete; (6) X is Todd complete.

3. Weakly pseudocompact spaces

A well known result by E. Hewitt [9] states that a space X is pseudocompact if and only if X is G_δ-dense in βX. Then, every pseudocompact space is weakly pseudocompact. In [7], S. García-Ferreira and A. García-Máynez introduced the concept of weakly pseudocompact space, and proved the following theorems:

Theorem 10. *(1) Every weakly pseudocompact space is Baire. (2) If X is weakly pseudocompact and Lindelöf, then X is compact. (3) Weakly pseudocompactness is productive.*

Theorem 11 ([5])**.** *(1) Weak pseudocompactness is not an inverse invariant of perfect maps. (2) Weak pseudocompactness is not a direct invariant of open continuous maps with compact fibers. (3) Every zero set in a pseudocompact space is weakly pseudocompact.*

The following spaces are weakly pseudocompact: (1) Every locally compact non Lindelöf space; (2) The completely metrizable hedgehog $J(\kappa)$ with $\kappa \neq \omega$ spines.

Problem 12 ([5])**.** *Are the uncountable products $\mathbb{Z}^{\omega_1}, \mathbb{R}^{\omega_1}$ and $\mathbb{S}^{\omega_1}$, where $\mathbb{S}$ is the Sorgenfrey line, weakly pseudocompact spaces?*

We have the following result which says that weak pseudocompactness can be reflected from a "big" power to certain "small" power of a separable space.

Theorem 13. *Let X be a separable space. If X^κ is weakly pseudocompact for some cardinal κ, then $X^\mathfrak{c}$ is weakly pseudocompact.*

Proof. If $\kappa \leq \mathfrak{c}$, there is nothing to prove since weak pseudocompactness is a productive property. Assume that $\mathfrak{c} < \kappa$, and choose a compactification K of X^κ such that X^κ is G_δ-dense in K. Fix $x_0 \in X$. Let $Z = \Sigma_{x_0} X^\kappa$ and $Z_A = \{f \in X^\kappa : f \restriction_{(\kappa \setminus A)} \equiv x_0\}$, for each $A \subseteq \kappa$. We will recursively construct a sequence $\{A_\alpha : \alpha < \mathfrak{c}\}$ of subsets of κ of cardinality at most $\mathfrak{c}$ such that $A = \bigcup_{\alpha<\mathfrak{c}} A_\alpha$

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satisfies that Z_A is G_δ-dense in $cl_K Z_A$. Let A_0 be an arbitrary subset of κ of cardinality $\mathfrak{c}$. Assume that $\alpha < \mathfrak{c}$ and that we have constructed A_β for each $\beta < \alpha$. If α is a limit, we choose $A_\alpha = \bigcup\{A_\beta : \beta < \alpha\}$. Assume now that $\alpha = \beta + 1$. By the Hewitt-Marczewski-Pondiczery Theorem, the space Z_{A_β} is separable. Since any Tychonof separable space has weight at most $\mathfrak{c}$, we can choose a base $\mathcal{B}_\beta$ for the space $cl_K Z_{A_\beta}$ of cardinality at most $\mathfrak{c}$. Let $\mathcal{G}_\beta = \{\bigcap \mathcal{B} : \mathcal{B} \subseteq \mathcal{B}_\beta$ and $|\mathcal{B}| \leq \omega\}$. Observe that $|\mathcal{G}_\beta| \leq \mathfrak{c}$. Since Z is G_δ-dense in X^κ and X^κ is G_δ-dense in K, the set Z is G_δ-dense in K. For each nonempty set $G \in \mathcal{G}_\beta$, fix a map $f_G \in G \cap Z$. Finally, let $A_\alpha = A_\beta \cup \bigcup\{supp(f_G) : G \in \mathcal{G}_\beta\}$, where $supp(f_G) = \{x \in \kappa : f_G(x) \neq x_0\}$. It is clear that the cardinality of A_α is at most $\mathfrak{c}$. We have finished our construction. Now, let $A = \bigcup\{A_\alpha : \alpha < \mathfrak{c}\}$ and observe that $|A| \leq \mathfrak{c}$.

Claim. The set Z_A is G_δ-dense in $cl_K Z_A$.

Proof of the Claim. Let G be a nonempty closed G_δ-set in $cl_K Z_A$. We may assume that there exists a sequence $\{U_n : n \in \omega\}$ of open sets in $cl_K Z_A$ such that $G = \bigcap\{U_n : n \in \omega\} = \bigcap\{clU_n : n \in \omega\}$. The compactness of G and $cl_K Z_A$ implies that $\chi(G, cl_K Z_A) = \psi(G, cl_K Z_A) \leq \omega$. Then we may assume that, in addition, $\{U_n : n \in \omega\}$ is a base for the set G in $cl_K Z_A$. We will prove that $G \cap cl_K Z_{A_\alpha} \neq \emptyset$ for some $\alpha < \mathfrak{c}$. Assume the contrary, that $G \cap cl_K Z_{A_\alpha} = \emptyset$ for each $\alpha < \kappa$. Given $\alpha < \kappa$, fix $n_\alpha \in \omega$ such that $G \subseteq U_{n_\alpha}$ and $U_{n_\alpha} \cap cl_K Z_\alpha = \emptyset$. Since the cofinality of $\mathfrak{c}$ is uncountable, there exists $k < \omega$ and a cofinal subset B of $\mathfrak{c}$ such that $n_\alpha = k$ for each $\alpha \in B$. Since the family $\{Z_{A_\alpha} : \alpha < \mathfrak{c}\}$ is increasing, we must have that $U_k \cap \bigcup\{cl_K Z_{A_\alpha} : \alpha < \mathfrak{c}\} = \emptyset$. The fact that $\bigcup\{Z_{A_\alpha} : \alpha < \mathfrak{c}\}$ is dense in Z_A implies that $U_k \cap Z_A = \emptyset$. It follows from $G \subseteq U_k$ that $G = G \cap cl_K Z_A = \emptyset$; which is a contradiction. Then we can fix $\alpha < \mathfrak{c}$ such that $G \cap cl_K Z_{A_\alpha} \neq \emptyset$. Fix $f \in G \cap cl_K Z_{A_\alpha}$. Since $\mathcal{B}_\alpha$ is a base for $cl_K Z_{A_\alpha}$, for each $n < \omega$ we can choose $B_n \in \mathcal{B}_\alpha$ such that $f \in B_n \subseteq U_n$. Set $H = \bigcap\{B_n : n \in \omega\}$. Note that $H \in \mathcal{G}_\alpha$ and $f \in H \subseteq G$. We know that $f_H \in H \subseteq G$. On the other hand, $\mathrm{supp}(f_H) \in A_{\alpha+1} \subseteq A$ and so $f_H \in Z_A$. Therefore, $f_H \in G \cap Z_A$.

Since X^A is homeomorphic to Z_A, the claim implies that X^A is weakly pseudocompact. Finally, since $|A| \leq \mathfrak{c}$ and weak pseudocompactness is productive, $X^\mathfrak{c}$ is weakly pseudocompact. □

4. Completeness type properties on $C_p(X)$

Now, we give some notions which will relate the completeness properties in function spaces $C_p(X,Y)$ with properties on X and Y.

Definition 14. (1) A space X is *ω-discrete* if all countable subspaces of X are discrete (or equivalently, are closed) in X.

(2) A subspace N of a space X is *C_Y-embedded* (resp., *C-embedded*) in X if every continuous function $f : N \to Y$ (resp., every continuous function $f : N \to \mathbb{R}$) has a continuous extension to all of X.

(3) A space X is *u_Y-discrete* (resp., *u-discrete*, *b-discrete*) if every countable subset of X is discrete and C_Y-embedded (resp., discrete and C-embedded) in X.

(4) A space X is *C^Y_ω-discrete* if each one of its countable subsets is C_Y-embedded.

Lemma 15. *For every topological group and its dense subgroup $H \subseteq G$ we have $H = G$ in case that H has a dense Čech-complete subspace.*

Theorem 16. *If $C_p(X)$ is weakly α-favorable, then X is u-discrete. In particular if $C_p(X)$ is weakly pseudocompact, then X is u-discrete.*

Proof. If $C_p(X)$ is weakly α-favorable, then X is ω-discrete (see the proof of (b) $\Rightarrow$ (c) $\Rightarrow$ (d) of Theorem 8.4 in [10], in which the extra hypothesis "*X is pseudonormal*" required in the initial conditions of that theorem was not used). Let A be a countable subset of X. Since X is ω-discrete, A is closed. Then, $\pi_A : C_p(X) \to \mathbb{R}^A$ is open and continuous. The set $\pi_A[C_p(X)]$ is a dense topological subgroup of $\mathbb{R}^A$. Because of Corollary 6 and Lemma 15, $\pi_A[C_p(X)] = C_p(A) = \mathbb{R}^A$. This means that A is C-embedded in X; that is, X is u-discrete. □

The following result includes the properties in $\mathcal{Q}$ as giving before Theorem 9, and the list of equivalences of Theorem 5.2 in [2] for $G = \mathbb{R}$.

Theorem 17. *The following conditions are equivalent. (1) X is u-discrete; (2) $C_p(X)$ is Todd complete; (3) $C_p(X)$ is Oxtoby complete; (4) $C_p(X)$ is Telgarsky* *complete; (5) $C_p(X)$ satisfies $P \in \mathcal{P}$; (6) $C_p(X)$ satisfies $Q \in \mathcal{Q}$; (7) $C_p(X)$ is G_δ-dense in $\mathbb{R}^X$; (8) $\upsilon C_p(X) = \mathbb{R}^X$.*

Proof. By Proposition 16, the weaker property in $\mathcal{Q}$, weak α-favorability, implies that (1) holds. The equivalence of the conditions in (3), (4), (5) and (7) was proved in [2, Th. 5.2]. Telgarsky completeness implies having a Markov winning strategy in the Choquet game, and this last property implies α-favorability; so (4) implies one of the properties in (6) which any one of them implies (1). Oxtoby completeness implies Todd completeness and this implies α-favorability, hence (3) $\Rightarrow$ (2) $\Rightarrow$ one of the properties in (6). Finally, (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (8) is the Tkachuk Theorem ([13, Theorem 4.1]; see Theorem 2, above). □

With respect to weak pseudocompactness and Problem 12, we have:

Corollary 18. *The space $\mathbb{R}^\kappa$ is weakly pseudocompact if and only if $C_p(X)$ is weakly pseudocompact for every u-discrete space X with $min\{\mathfrak{c}, \kappa\} \leq |X|$.*

Proof. If $\mathbb{R}^\kappa$ is weakly pseudocompact, then $\mathbb{R}^\mathfrak{c}$ is weakly pseudocompact (Theorem 13). Then $\mathbb{R}^X$ is weakly pseudocompact because this property is productive. Let K be a compactification of $\mathbb{R}^X$ in which $\mathbb{R}^X$ is G_δ-dense. If X is u-discrete, then $C_p(X)$ is G_δ-dense in $\mathbb{R}^X$. Then K is a compactification of $C_p(X)$ in which $C_p(X)$ is G_δ-dense. □

5. An application to separable metrizable topological groups

Lemma 19. *Let G be a separable metrizable topological group and let D be a dense subspace of G^X. If $\phi : D \to G^Y$ is a continuous map, then there exists a subset A of X and a continuous map $\psi : \pi_A(D) \to G^Y$ such that $|A| \leq |Y| \cdot \omega$ and $\phi = \psi \circ \pi_A \restriction_D$, where π_A is the canonical projection from G^X onto G^A.*

Lemma 20. *Let G be a separable metrizable topological group and let $\phi : G^X \to G^X$ be an embedding with dense image. Then for every $A \subseteq X$ there exists $E \subseteq X$ and an embedding $\phi_E : G^E \to G^E$ such that $A \subseteq E$, $|E| \leq |A| \cdot \omega$ and $\phi_E \circ \pi_E = \pi_E \circ \phi$.*

Lemma 21. *Let G be a separable metrizable topological group and let $\phi : G^X \to G^X$ be an embedding. Then, (1) If G is Čech complete and $\phi(G^X)$ is a dense subgroup of G^X, then $\phi(G^X)$ is G_δ-dense in G^X; and (2) If $\phi(G^X)$ is G_δ-dense in G^X, then $\phi(G^X) = G^X$.*

Lemma 22. *Let G be a space. If H is a dense subset of G^X, then $|X| \leq \chi(H)$.*

The following theorem is a generalization of the Theorem in [14].

Theorem 23. *Let G be a separable completely metrizable topological group. If H is a dense subgroup of G^X and H is homeomorphic to G^Y for some set Y, then $H = G^X$.*

Proof. By Lemma 22 we know that $|X| \leq \chi(H)$. Let $\phi : G^Y \to H$ be a homeomorphism. Note that $|Y| = w(G^Y) = w(H) \leq w(G^X) = |X| \leq \chi(H) = \chi(G^Y) = |Y|$ and hence $|Y| = |X|$. As a consequence, we can assume that $Y = X$. In virtue of Lemma 21 we must have that $H = \phi(G^X) = G^X$. □

References

[1] J. M. Aarts and D. J. Lutzer, Pseudocompactness and the product of Baire Spaces, Pacific Journal of Math. 48 1 (1973), 1–10.

[2] A. Dorantes-Aldama and D. Shakhmatov, Completeness and compactness properties in metric spaces, topological groups and function spaces, manuscript.

[3] A. Dorantes-Aldama and Á. Tamariz-Mascarúa, Some results on weakly pseudocompact spaces, Houston J. Math, to appear.

[4] A. Dorantes-Aldama, R. Rojas-Hernández and Á. Tamariz-Mascarúa, Weak pseudocompactness of spaces of continuous functions, Topology Appl. 196 (2015), 72–91.

[5] F. W. Eckertson, Sums, products, and mappings of weakly pseudocompact spaces, Topology Appl. 72 (1996), 149–157.

[6] R. Engelking, General Topology, Sigma Series in Pure Mathematics, 1989.

[7] S. García-Ferreira and A. García- Maynez, On weakly-pseudocompact spaces, Houston J. Math. 20 (1) (1994), 145–159.

[8] F. Galvin and R. Telgársky, Stationary strategies in topological games, Topology Appl. 22 (1986), 51–69.

[9] E. Hewitt, Rings of continuous functions I, Trans. Amer. Math. Soc. 48 (1948), 45–99.

[10] D. J. Lutzer and R. A. McCoy, Category in Function Spaces, I, Pacific J. Math. 90, no. 1 (1980), 145–168.

[11] J.C. Oxtoby, Cartesian products of Baire spaces, Fundam. Math. 49 (1961), 157–166.

[12] R. Telgárski, Remarks on a game of Choquet, Colloq. Math. 51 (1987), 365–372.

[13] V. V. Tkachuk, The spaces $C_p(X)$: Descomposition into a countable union of bounded subspaces and completeness properties, Topology Appl. 22 (1986), 241–253.

[14] V. V. Tkachuk, Cardinal invariants of the Suslin number type, Soviet Math. Dokl. 27 (1987) 135-136. (Translated from Russian).

[15] A. R. Todd, Quasiregular products of Baire spaces, Pac. J. Math. 95, no. 1 (1981), 233–250.

[16] H. White, Topological spaces which are α-favorable for a player with perfect information, Proc. Amer. Math. Soc. 50 (1975), 477–482.

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Some comments to cone metric spaces

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Abstract

In this paper, we do some observations about the order defined on a real Banach space, in order to revise the concept of cone metric space.

Keywords: *Banach space; regular (normal) cone.*
MSC: *54A40; 54D35; 54E50.*

1. Introduction

The aim of this paper and its continuation (Some remarks on cone metric spaces, in this same issue) is to revise some concepts and results appeared in the notion of cone metric space given in [1] by L. Huang and X. Zhang. Here we will do some observations on the order that formally is given in a real Banach space.

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2. Order on a real Banach space

Regarding to the appearance of cone metric spaces, at the end of [2] the authors suggest to replace Banach spaces by certain topological groups in the construction of new spaces. So, we begin stating, for groups and for topological vector spaces, a similar order to the given for Banach spaces.

In this section $(E, +)$ is an additive group and θ denotes the zero of E.

Definition 1. Let H be a non-trivial monoid of the additive group $(E, +)$. We will say that H is strict if $H \cap (-H) = \{\theta\}$ (i.e. $h, -h \in H$ implies $h = \theta$), where $-H = \{e \in E : -e \in H\}$.

Obviously, if H is a strict monoid then $-H$ is also a strict monoid.

We can define a partial order $\leq_H$ in E as follows:

For all $x, y \in E$ we define $x \leq_H y$ if and only if $y - x \in H$. As usual $x <_H y$ denotes $x \leq_H y$ with $x \neq y$.

For simplicity, if confusion is not possible, the order $\leq_H$ defied by H on E will be denoted by $\lesssim$. The notation $\leq$ will be used for the usual order of $\mathbb{R}$.

Example 2. (a) Take $E = \mathbb{R}$ with the usual addition $+$. Consider the strict monoid $H = [0, \infty[$. The order $\lesssim$ defined by H on H is the following.

For all $x, y \in E$ we have that $x \lesssim$ if and only if $y - x \in H$ if and only if $y - x \geq 0$ if and only if $y \geq x$. So, in this case, $\lesssim$ is the usual order on $\mathbb{R}$. So $(E, \lesssim)$ is a totally ordered set. If we take the strict monoid $-H =]-\infty, 0]$ then we obtain the converse order, that is $x \lesssim y$ if and only if $x \geq y$.

(b) Take $E = \mathbb{Z}$ with the usual addition $+$. Consider the strict monoid $H = \{0, 2, 4, ...\}$. The order $\lesssim$ defined by H on E is the following.

For all $x, y \in E$ we have that $x \lesssim y$ if and only if $x \leq y$ and both x and y are even or both are odd. In this case $(E, \lesssim)$ is not totally ordered; nevertheless $(H, \lesssim)$ is well-ordered.

(c) Take $E = \mathbb{R}^2$ with the usual addition of vectors. Consider the strict monoid $H = \{(x, y) : x \geq 0, y \geq 0\}$. The order $\lesssim$ defined by H on E is the following.

For each $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ we have that $(x_1, y_1) \lesssim (x_2, y_2)$ if and only $x_1 \leq x_2$ and $y_1 \leq y_2$. Clearly $(E, \lesssim)$ is not totally ordered, but it is a lattice. Indeed, denote by $M = \max\{x_1, x_2\}, N = \max\{y_1, y_2\}, m = \min\{x_1, x_2\}, n = \min\{y_1, y_2\}$. Then $(x_1, y_1) \vee (x_2, y_2) = (M, N)$ and $(x_1, y_1) \wedge (x_2, y_2) = (m, n)$, where $\vee$ and $\wedge$ denotes supremum and infimum, respectively.

Also, $(H, \lesssim)$ is a lattice and not totally ordered.

(d) Take $E = \mathbb{R}^2$ with the usual addition of vectors. Consider the strict monoid $H = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\} \cup \{(0, 0)\}$. The order $\lesssim$ defined by H on $E$is the following.

For each $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ we have that $(x_1, y_1) \lesssim (x_2, y_2)$ if and only $x_1 < x_2$ and $y_1 < y_2$, whenever $(x_1, y_1) \neq (x_2, y_2)$. Clearly $(E, \lesssim)$ is not totally ordered. Further, $(E, \lesssim)$ is not a lattice. Indeed, for instance, the set of upper bounds of $\{(-3, 5), (4, 2)\}$ is $U = \{(m, n) : m > 4, n > 5\}$ and U has not a least element.

Also, $(H, \lesssim)$ is not totally ordered and it is not a lattice neither.

For achieving the cone metric space's concept and some of its properties we will need some richer structures on E and on H. So, in the rest of this section $(E, +)$ will be a real vector space.

Definition 3. A subset $H \neq \{\theta\}$ of the real vector space $(E, +)$ will be called a hemispace if $\alpha x + \beta y \in H$ whenever $\alpha, \beta \geq 0, x, y \in H$, and $H \cap (-H) = \{\theta\}$.

Clearly, a hemispace is a strict monoid.

Let $B = \{v_1, \ldots, v_r\}$ a linearly independent system of vectors in E. We define the following subset generated by B:

$$< B >= \{v \in E : v = \sum_{i=1}^{r} \lambda_i v_i, \lambda_i \geq 0, i = 1, \ldots, r\}.$$

The scalars λ_i, $i = 1, \ldots, r$, are called coordinates of v (with respect to B), and obviously they are unique.

Proposition 4. *$< B >$ is a hemispace of E.*

Proof. If $x, y \in < B >$ and $\alpha, \beta \geq 0$ it is obvious that $\alpha x + \beta y \in < B >$.

Now, suppose that $h \in < B >$ and $h \in - < B >$. Note that it means that $-h \in < B >$ and so there exist $\lambda_i, \mu_i \geq 0$, $i = 1, \ldots, r$, such that $h = \sum_{i=1}^{r} \lambda_i v_i$ and $-h = \sum_{i=1}^{r} \mu_i v_i$. Then, $\theta = h - h = \sum_{i=1}^{r} \lambda_i v_i + \sum_{i=1}^{r} \mu_i v_i = \sum_{i=1}^{r} (\lambda_i + \mu_i) v_i$, thus $\lambda_i + \mu_i = 0$ for each $i = 1, \ldots, r$ and consequently $\lambda_i = \mu_i$ for each $i = 1, \ldots, r$. Hence, $h = \theta$. □

We will say that B is a (finite) base for $< B >$ and it is easy to observe that if B is not a linearly independent system of vectors then $< B >$ could not be a hemispace.

Example 5. (a) The strict monoid H of (c) in Example 2, is a hemispace of $(\mathbb{R}^2, +)$. Clearly, $\{(1,0), (0,1)\}$ is a base for H.

(b) The strict monoid H of (d) in Example 2 is a hemispace of $(\mathbb{R}^2, +)$, but there not exists any base that generates that hemispace. Indeed, suppose that $\{(x_1, y_1), (x_2, y_2)\}$ is a base for $H = \{(x, y \in \mathbb{R}^2 : x > 0, y > 0\} \cup \{(0,0)\}$. Necessarily it is satisfied $x_i > 0$ and $y_i > 0$, $i = 1, 2$. Suppose, without lose of generality, that $\frac{y_2}{x_2} > \frac{y_1}{x_1}$.

Take $(x, y) \in H$, with $x > 0, y > 0$. Suppose that (x, y) cannot be written as $\alpha(x_1, y_1)$ with $\alpha \geq 0$, neither $\beta(x_2, y_2)$ with $\beta \geq 0$. Then, $(x, y) = \alpha(x_1, y_1) + \beta(x_2, y_2)$.

Now, from $\frac{\beta y_2}{\beta x_2} > \frac{\alpha y_1}{\alpha x_1}$ we have that

$$\frac{y}{x} = \frac{\alpha y_1 + \beta y_2}{\alpha x_1 + \beta x_2} > \frac{\alpha y_1}{\alpha x_1} = \frac{y_1}{x_1}.$$

In particular, $(x_1, \frac{1}{2} y_1) \in H$, should satisfy the last inequality but it is not fulfilled since $\frac{\frac{1}{2} y_1}{x_1} = \frac{y_1}{2x_1} < \frac{y_1}{x_1}$. So, $(x_1, \frac{1}{2} y_1)$ cannot be written as a linear expression of $\{(x_1, y_1), (x_2, y_2)\}$

At the light of the last example it raises the following question.

Question 6. *Let $(E, +)$ be a Hausdorff topological vector space. If the hemispace H of E has a finite base then, is H closed?*

Next we recall the concept of cone in a real Banach space.

Definition 7 (Huang and Zhang [1])**.** Let $(E, +)$ be a real Banach space and let P a non-empty subset of E. Then P is called a cone of E if it satisfies the following

(i) P is closed and $P \neq \{\theta\}$.
(ii) $x, y \in P$, $\alpha, \beta \geq 0$ implies $\alpha x + \beta y \in P$.
(iii) $P \cap (-P) = \{\theta\}$.

So, P is a cone of E if P is a closed (non-trivial) hemispace of $(E, +)$.

Then, for a given cone $P \subset E$ we can define the above partial order $\lesssim$ on E, with respect ot P, given by

$$x, y \in E \text{ then } x \lesssim y \text{ if and only if } y - x \in P.$$

Notice that $\theta \lesssim x$, for each $x \in E$, is equivalent to write $x \in P$.

For $x, y \in E$ it is denoted $x \ll y$ whenever $y - x \in intP$, where $intP$ denotes the interior of P. Then $\theta \ll x$ is equivalent to say that $x \in intP$.

Next we give an example of a cone of $\mathbb{R}^2$, which is related with Example 5 and Question 6.

Example 8. Let $E = \mathbb{R}^2$ with the Euclidean norm. Then $P = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ is closed in E and in addition it is a cone of $\mathbb{R}^2$, but $H = \{(x, y)\mathbb{R}^2 : x > 0, y > 0\} \cup \{(0, 0)\}$ is not a cone since H is not closed.

3. Regular and normal cones

In this section $(E, +)$ is a real Banach space.

Definition 9 (Huang and Zhang [1]). A cone P of E is called normal if there exists $M > 0$ such that for all $x, y \in P$ it is satisfied

$$x \lesssim y \text{ implies } ||x|| \leq M \cdot ||y||. \tag{1}$$

In [4] the authors have proved that there are not normal cones with $M < 1$ in (1).

Example 10. Let $E = \mathbb{R}$ with the usual norm $|\cdot|$. Then, $P = [0, \infty[$ is a cone of E and it is normal. Indeed, for every $x, y \in P$ with $x \lesssim y$ we have that $y - x \in P$, that is $y - x \geq 0$ or equivalently $x \lesssim y$ and so $|x| \leq |y|$, since $y \geq x$.

Then (1) is fulfilled by $M = 1$, and by the result mentioned above of [4], 1 is the least number satisfying (1).

The following definition appeared in [1] (and it was reproduced in [4] and [3]).

Definition 11. The least positive number satisfying (1) in a normal cone P is called the normal constant of P.

We have just seen that 1 is the normal constant in Example 10. Nevertheless, the following is an immediate question.

Question 12. *There exists a normal constant for each cone? (That is, there exists a least value M satisfying condition (1)?).*

In Proposition 2.2 of [4] the authors write: "For each $K > 1$ there is a normal cone with normal constant $M > K$". In the proof the authors constructed a family of normed spaces $\{E_K : K > 1\}$ and for each one they find a normal cone P_K such that (1) is fulfilled for some $M > K$. Now, the authors did not prove the existence of the least value M satisfying condition (1).

Remark 13. We notice that the authors of [5] call normal constant of P simply to any number $M \geq 1$ satisfying condition (1).

Definition 14. A cone P of E is called regular if every increasing sequence which is bounded from above is convergent, that is if $\{x_n\}$ is a sequence in P such that

$$x_1 \lesssim x_2 \lesssim \cdots \lesssim x_n \lesssim \cdots \lesssim y,$$

for some $y \in E$, then there is $x \in E$ such that $||x_n - x|| \to 0$ as $n \to \infty$.

Equivalently, P is regular if and only if every decreasing sequence which is bounded from below is convergent.

Remark 15. The concept of regular cone appeared in [1] is reproduced in [4, 5]; nevertheless this concept is not used in [1] neither in [5]. In these three cases one should suppose that, in general, the sequence $\{x_n\}$ of Definition 14 is in E. In fact, it is in [2] where it is explicitly written that $x_n \in E$, for each $n \in \mathbb{N}$.

Now, Proposition 2.2 of [4], in which this concept is used, sheds no light on the belonging of x_n, and on the other hand, in the proof, by contradiction, that regular implies normal (Lemma 1.1 [4]) it shows (implicitly) that the following weaker version also implies normality:

A cone P of E is regular if for every sequence $\{x_n\} \in P$ satisfying $\theta \lesssim x_1 \lesssim x_2 \lesssim \cdots \lesssim x_n \lesssim \cdots \lesssim y$, for some $y \in P$, there exits $x \in P$ such that $\lim_n ||x_n - x|| = 0$.

The authors have observed that both definitions are equivalent.

References

[1] L. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, Journal of Mathematical Analysis and Applications 332 (2007), 1468–1476.

[2] M. Khani, M. Pourmahdian, On the metrizability of cone metric spaces, Topology and its Applications 158 (2011), 2258–2261.

[3] T. Öner, M. B. Kandemir, Fuzzy cone metric spaces, Journal of Nonlinear Analysis and Applications 8 (2015), 610–616.

[4] Sh. Rezapour, R. Halmbarami, Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings", Journal of Mathematical Analysis and Applications 345 (2008), 719–724.

[5] D. Turkoglu, M. Abuloha, Cone metric spaces and fixed point theorems in diametrically contractive mappings, Acta Mathematica Sinica 26, no. 3 (2010), 489–496.

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Some remarks on cone metric spaces

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Abstract

In this paper we continue our "Some comments to cone metric spaces". We revise the concept of cone metric space and some results related to it. We also observe some controversies appeared on that concept.

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MSC: *54A40; 54D35; 54E50.*

1. Introduction

As we have said in the first part (Some comments to cone metric spaces, in this same issue), recently, Huang and Zhang [3] have introduced the concept of cone metric space (Definition 1). Basically, a cone metric space is a structure regarding the theory of metric spaces but replacing the set of non-negative real numbers by an ordered Banach space, and so, obviously, the class of cone metric spaces contains the class of metric spaces. The aim of the authors were to obtain fixed point theorems that generalize the Banach Contraction Principle and fixed point

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theorems given for other notions of contractive mappings in metric spaces. The above study was continued in [7] (where some results on fixed point theory were generalized) and in [3] (where the authors proved that a Hausdorff topology is deduced from a cone metric). Later in [5] the authors have proved that cone metric spaces are metrizable. Recently in [6] the authors have extended the theory to fuzzy setting.

In this paper, continuing our first part, we do some comments and analyse some ideas which arise in the reading of the mentioned papers. At the end, a section devoted to some controversies appeared on that concept, is included

2. Cone metric spaces

In this section $(E,+)$ is a real Banach space, P is a cone in E with $intP \neq \emptyset$ and $\lesssim$ the partial order on E with respect to P.

Definition 1 (Huang and Zhang [3])**.** A cone metric space is an ordered quadruple (X,d,E,P), where X is non-empty set, E is a real Banach space, P is a cone of E with $intP \neq \emptyset$ and $d : X \times X \to P$ is a mapping satisfying

- (d1) $d(x,y) = \theta$ if and only if $x = y$;
- (d2) $d(x,y) = d(y,x)$ for all $x,y \in X$;
- (d3) $d(x,z) \lesssim d(x,y) + d(y,z)$ for all $x,y,z \in X$.

In this case, we will say that d is a cone metric on (X,E,P).

If confusion is not possible, we will say that (X,d) is a cone metric space or d is a cone metric on X.

Remark 2. Clearly, a classical metric space (X,d) is a cone metric space. (Indeed, notice that $P = [0,\infty[$ is a cone with $intP \neq \emptyset$ in the real Banach space $(\mathbb{R},|\cdot|)$ of Example 10 in the first part.

Definition 3 (Huang and Zhang [3])**.** Let $\{x_n\}$ be a sequence in the cone metric space (X,d) and let $x \in X$. Then:

(i) $\{x_n\}$ is called convergent to x if for any $c \in P$ with $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n \geq n_0$.

(ii) $\{x_n\}$ is called Cauchy if for any $c \in P$ with $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m \geq n_0$.

(iii) a cone metric (X, d) is called complete if every Cauchy sequence is convergent.

Remark 4. As the reader can observe, the last two definitions regard the corresponding concepts in classical metric space, and formally, they only differ from the classical concepts on slight changes in the notations.

It is easy to observe the possibility of continuing this theory generalizing classical concepts or extending this theory to other fields, for instance, to fuzzy setting. In fact, the concept of fuzzy cone metric space, that we could say in the sense of George and Veeramani, has recently appeared [6].

But, what about the proofs? As the reader has been noted the role of the positive $\epsilon \in \mathbb{R}^+$, is taken by $c \in intP$. In the case that for a given positive $\epsilon \in \mathbb{R}^+$ one should need $c \gg \theta$ with $||c|| < \epsilon$, a way for obtaining a such c is the following. Since $intP \neq \emptyset$ we can take $e \in intP$ with $e \neq \theta$, and we also can choose $\lambda > 0$ such that $\lambda||e|| < \epsilon$. If we call $c = \lambda e$, clearly $c \in intP$ and $||c|| = \lambda||e|| < \epsilon$.

In the case that P is a normal cone satisfying (1) in the first part for some $M > 0$, and if one is interested in obtaining $c \in intP$ with $M \cdot ||c|| < \epsilon$, then, attending to the last paragraph it is enough to choose $\lambda > 0$ such that $\frac{\lambda}{M}||e|| < \epsilon$.

In the next lemma we have compiled some results used in [3, 6, 7, 9], for obtaining the proofs of their results.

Lemma 5. *Let P be a cone of E. Then:*

(i) $intP + intP \subset intP$, [7].

(ii) $\lambda \cdot intP \subset intP$, [7].

(iii) *For each $c \gg \theta$ there exists $\delta > 0$ such that $(c - x) \in intP$ for each $x \in X$ satisfying $||x|| < \delta$ (i.e. $x \ll c$ whenever $||x|| < \delta$, $x \in E$),* [3, 9].

(iv) *For each $c_1, c_2 \gg \theta$ there exists $\theta \ll c$ such that $c \ll c_1$ and $c \ll c_2$,* [9].

For proving the metrizability of cone metric spaces the authors proved (Lemma 3.1, 3.2, 3.3 of [5]):

Lemma 6. *Let (X, d) be a cone metric space. Then:*

(i) *For $x \in P$ and $y \in intP$ there exists $n \in \mathbb{N}$ such that $x \ll ny$.*
(ii) *If $y \in intP$ then $x \gtrsim y$ implies $x \in intP$.*
(iii) *$x \lesssim y \ll z$ implies $x \ll z$.*

Definition 7 ([9])**.** A cone P of E is called minihedral if for each $x, y \in E$ there exists $\sup\{x, y\}$, and strongly minihedral if every subset of E which is bounded from above has a supremum.

In [3] the fixed point theorems are stated for normal cones. For instance the Banach Contraction Principle is written as follows.

Theorem 8. *Let (X, d, E, P) be a complete cone metric space, where P is a normal cone with normal constant $M \geq 1$. Suppose that the mapping $T : X \to X$ satisfies the contractive condition*

$$d(Tx, Ty) \lesssim K \cdot d(x, y), \text{ for each } x, y \in X,$$

where $K \in]0, 1[$. Then, T has a unique fixed point in X and for each $x \in X$ the iterative sequence $\{T^n x\}$ converges to the fixed point.

This theorem is a generalization of the Banach Contraction Principle since, attending to Remark 2, every metric space is a cone metric space, when considering the cone $P = [0, \infty[$ of $(\mathbb{R}, |\cdot|)$. For this cone we have just seen in Example 10 in the first part that 1 is the normal constant of P.

Other three fixed point theorems are given in [3] for complete cone metric space assuming the existence of the normal constant. Now, it is easy to observe in the proofs of these theorems that it is only required the existence of a value M satisfying condition (1) in the first part. Further, in [7] the authors have generalized these theorems removing in all of them the condition of normality of P.

3. Some controversies on cone metric spaces

After the authors in [3] defined the concept of cone metric space, many authors are working on this concept. But, some of them wonder if cone metric spaces are really a generalization of metric spaces. Other authors ensure that some fixed point generalizations given for cone metric spaces are not so.

In this sense, in [1] the authors assert that by renorming an ordered Banach Space, every cone P can be converted to a normal cone with normal constant $M = 1$ and consequently due to this approach every cone metric space is really a metric space and every theorem given in metric spaces is valid for cone metric spaces, automatically.

Now, the authors in [8] show that content of the last paper is not true. We note that the quote is not exactly [1], since they referenced it as [M. Asadi, S. M. Vaezpour, H. Soleimani, *Metrizability of cone metric spaces via renorming teh BAnach space*, arXiv:1102.2040v1[math.FA] 10 Feb 2011.].

In [4] the author discusses the concept of cone metric space and he shows that many of the new results are merely copies of the classical ones. He basis his assertion proving that cone metric spaces have a metric type structure [4, Theorem 2.6].

Finally, in the abstract of [2] (See the tittle of the paper) the author writes: "In this short note we present a different proof of the known fact that the notion of a cone metric space is not more general than that of a metric space". Its assertion is based on his next result:

Theorem 9 (Z. Ercan [2, Theorem 1.4])**.** *Let X be a non-empty set, E be an ordered vector space with cone P and $e \in int(P)$. For a function $d : X \times X \to P$, define $\bar{d} : X \times X \to \mathbb{R}^+$ by $\bar{d}(x, y) = \inf\{\lambda \in \mathbb{R}^+ : d(x, y) \leq \lambda e\}$.*

(i) *If d is a cone metric, then $\bar{d}$ is a metric.*
(ii) *If $\rho : X \times X \to \mathbb{R}^+$ is a metric, then there exists a cone metric $p : X \times X \to P$ such that $\rho = \bar{p}$.*

References

[1] M. Asadi, S. M. Vaezpour, B. E. Rhoades, H. Soleimani, Metrizability of cone metric spaces via renorming teh BAnach space, Journal of Nonlinear Analysis and Application 2012 (2012), Article ID jnaa-00160, doi: 10.5899/2012/jnaa-00160

[2] Z. Ercan, On the end of cone metric spaces, Topology and its Applications 166 (2014), 10–14.

[3] L. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, Journal of Mathematical Analysis and Applications 332 (2007), 1468–1476.

[4] M. A. Khamsi, Remarks on cone metric spaces and fixed point theorems of contractive mappings, Fixed Point Theory and Applications 2010 Article ID 315398, 7 pages, doi:10.1155/2010/315398.

[5] M. Khani, M. Pourmahdian, On the metrizability of cone metric spaces, Topology and its Applications 158 (2011), 2258–2261.

[6] T. Öner, M. B. Kandemir, Fuzzy cone metric spaces, Journal of Nonlinear Analysis and Applications 8 (2015), 610–616.

[7] Sh. Rezapour, R. Halmbarami, Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings", Journal of Mathematical Analysis and Applications 345 (2008), 719–724.

[8] K. P. R. Sastry, Ch. Srinivasa Rao, A. Chandra Sekhar, M. Balaiah, On non metrizability of cone metric spaces, International Journal of Science and Applications 1, no. 3 (2011), 1533–1535.

[9] D. Turkoglu, M. Abuloha, Cone metric spaces and fixed point theorems in diametrically contractive mappings, Acta Mathematica Sinica 26, no. 3 (2010), 489–496.

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Ultracomplete spaces

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Abstract

A space is called ultracomplete (cofinally Čech-complete) if its remainder in some compactification is hemicompact. A space is called almost locally compact if its remainder in some compactification is locally compact and Lindelof. In this work we summarize several known results about ultracomplete and almost locally compact spaces.

1. Introduction and preliminaries

Romaguera introduced in [14] the class of cofinally Čech-complete spaces in order to characterize metrizable spaces which admits cofinally complete metric, he proved that a metrizable space has a cofinally complete metric if and only if it is cofinally Čech-complete. In [13] Ponomarev and Tkachuk defined a space X to be strongly complete, if $\chi(X, \beta X) \leq \omega$. Buhagiar and Yoshioka proved in [4] that cofinal Čech-completeness is equivalent to strong completeness and renamed it as ultracompleteness. In [8] a space X is said to be almost locally compact if

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there is a compact $K \subset X$ such that $\chi(K, X) \leq \omega$ and $nlc(X) = \{x \in X : X$ is not locally compact at $x\} \subset K$. In this work we summarize several properties of ultracomplete and almost locally compact spaces obtained in [2]–[16].

All spaces under consideration are assumed to be Tychonoff. The space $\mathbb{R}$ is the set of real numbers with its natural topology. For any space X let $C_p(X)$ be the space of continuous functions from X to $\mathbb{R}$ endowed with the topology of pointwise convergence. The Stone-Čech compactification of a space X is denoted by βX. The character of X at its subspace $A \subset X$, denoted by $\chi(A, X)$, is the minimal of the cardinalities of all outer bases of A in X. A space X is Čech-complete if it is a G_δ-set in some compactification cX (equivalently in any compactification kX). A space X is called hemicompact if there is a countable family $\{K_n : n \in \mathbb{N}\}$ of compact subsets of X such that for any compact $K \subset X$ there exists $n \in \mathbb{N}$ for which $K \subset K_n$. A space X is of (pointwise) countable type if for any compact $F \subset X$ ($x \in X$) there exists a compact $K \subset X$ such that $F \subset K$ ($x \in K$) and $\chi(K, X) \leq \omega$. Every Čech-complete space if of countable type. A sequence $\{x_n : n \in \mathbb{N}\}$ in a metric space (X, d) is called cofinally Cauchy if $\forall \epsilon > 0$ there exists an infinite set $\mathbb{N}_\epsilon \subset \mathbb{N}$ such that for any $i, j \in \mathbb{N}_\epsilon$ we have $d(x_i, x_j) < \epsilon$. A metric space is called cofinally complete if any cofinally Cauchy sequence has a cluster point.

2. Ultracomplete spaces and their properties.

Definition 1. A space X is called cofinally Čech-complete ([14]) if there is a countable collection $\{\mathcal{G}_n : n \in \mathbb{N}\}$ of open covers of X satisfying the property that whenever $\mathcal{F}$ is filter on X such that for each $n \in \mathbb{N}$ there is some $G_n \in \mathcal{G}_n$ which meets all the members of $\mathcal{F}$, then $\mathcal{F}$ has a cluster point.

Romaguera introduced the previous definition in order to characterize those metrizable spaces which admit cofinally complete metric.

Theorem 2 ([14], Theorem 2)**.** *A metrizable space admits a cofinally complete metric if and only if it is cofinally Čech-complete.*

Theorem 3. *For any Tychonoff space X, the following conditions are equivalent:*

i) $\chi(X, cX) \leq \omega$ for some compactification cX of X;

ii) $\chi(X, kX) \leq \omega$ for every compactification kX of X;

iii) $cX \setminus X$ is hemicompact for some compactification cX of X;

iv) $kX \setminus X$ is hemicompact for every compactification kX of X.

Definition 4 ([13])**.** A space X is called ultracomplete if it satisfies one of the conditions of the previous proposition.

Buhagiar and Yoshioka proved in Theorem 2.2 of [5] that a space is ultracomplete if and only if it is cofinally Čech-complete. From definition it follows that any locally compact space is ultracomplete and that any ultracomplete space is Čech-complete. For a given space X, denote by $nlc(X)$ the set of all the points of non local compactness of X. In [6] and [13] it was proved the next fact about $nlc(X)$.

Proposition 5. *If X is an ultracomplete space, then $nlc(X)$ is bounded in X.*

From the previous fact it follows that an ultracomplete space without points of local compactness is pseudocompact.

Example 6. i) The set of all irrationals numbers with its natural topology induced from $\mathbb{R}$ is a non-ultracomplete Čech-complete space.

ii) The space $X = [0, 1] \setminus \{\frac{1}{n} : n \in \mathbb{N}\}$ is ultracomplete and it is not locally compact.

It was proved independently in [6] and [13] that a paracompact space X is ultracomplete if and only if the subspace $nlc(X)$ is contained in a compact subset of countable outer character. The following definition was motivated by the last fact.

Definition 7 ([8], Definition 2.2)**.** Call a space X almost locally compact if there is a compact $K \subset X$ such that $\chi(K, X) \leq \omega$ and $nlc(X) \subset K$. It is worth mentioning that in [16] an almost locally compact space is called c-ultracomplete.

Thus a paracompact space is ultracomplete if and only if it is almost locally compact. If X is a Čech-complete space and $nlc(X)$ is compact, then X is almost locally compact, because it is of countable type. From definitions we have the implications:

$$\text{locally compact} \Rightarrow \text{almost locally compact} \Rightarrow \text{ultracomplete} \Rightarrow \check{\text{C}}\text{ech-complete}$$

Proposition 8 ([16], Theorem 1.8)**.** *For any space X, the following conditions are equivalent:*

i) X is almost locally compact;

ii) $cX \setminus X$ is locally compact and Lindelöf for some compactification cX of X;

iii) $kX \setminus X$ is locally compact and Lindelöf for every compactification kX of X.

Buhagiar and Yoshioka proved in [4] that ω_1^ω is an ultracomplete space without points of local compactness, hence it is not almost locally compact. The following proposition provides a family of ultracomplete spaces without points of local compactness, these spaces are not almost locally compact.

Proposition 9 ([8], Theorem 2.10)**.** *Let X be an infinite compact space. If $F(X)$ is the Markov free topological group of X, then $Y = \beta(F(X)) \setminus F(X)$ is an ultracomplete space without points of local compactness.*

The examples given above, of ultracomplete spaces without points of local compactes, are countably compact. In contrast we have the following

Example 10. In Theorem 3.15 of [9] it was stablished the existence a dense countable subspace of $\{0,1\}^{\mathfrak{c}}$ without nontrivial convergent sequences. If D is such subspace, then it was proved in Example 3.16 of [9] that $X = \{0,1\}^{\mathfrak{c}} \setminus D$ is ultracomplete non-countably compact and has no points of local compactness.

Buhagiar and Yoshioka asked, in Problem 5.3 of [4], whether a countably compact Čech-complete space is ultracomplete. In [10] we can find the answer to this question.

Example 11. In Lemma 3.10 of [10] it was proved the existence of an open set $U \subset \mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ such that U is countably compact and $\mathbb{N}^* \setminus \overline{U}$ is not σ-compact. The authors proved in Theorem 3.11 of [10] that $Y = (\mathbb{N}^* \setminus \overline{A}) \cup U$, where A

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is a countably infinite discrete subspace of $\mathbb{N}^*$, is a non-ultracomplete countably compact Čech-complete space.

From definitions it easily follows that if X is ultracomplete (almost locally compact) and $Y \subset X$ is closed, then Y is ultracomplete (almost locally compact). It is well known that a G_δ subset of a Čech-complete space is Čech-complete, in contrast we have the following

Example 12 ([4], Example 2.3)**.** If $A = \{x_n : n \in \mathbb{N}\} \subset \beta\mathbb{N} \setminus \mathbb{N}$, let $X \subset \beta\mathbb{N}$ be defined by $X = \beta\mathbb{N} \setminus A$. Then X is a countably compact, ultracomplete, non-locally compact space. The space $X \times \beta\mathbb{N}$ is ultracomplete, $X \times \mathbb{N}$ is open in $X \times \beta\mathbb{N}$ and it is not ultracomplete.

Example 13 ([4], Example 3.3)**.** Let $p : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the natural quotient function. The space $\mathbb{R}$ is ultracomplete and p is a closed function, because $\mathbb{Z}$ is closed in $\mathbb{R}$. The space $\mathbb{R}/\mathbb{Z}$ is not of pointwise countable type and so it cannot be Čech-complete.

An open continuous image of a Čech-complete space is not necessarily Čech-complete. Perfect images and perfect preimages of Čech-complete spaces are Čech-complete.

Proposition 14. *i)* ([6], Proposition 2) *The open continuous image of an ultracomplete space is ultracomplete.*

ii) ([6], Theorem 1) *Let f be a perfect function from a space X onto a space Y. Then X is ultracomplete if and only if Y is ultracomplete.*

Proposition 15. *i)* ([8], Proposition 2.3) *An open continuous image of an almost locally compact space is also almost locally compact.*

ii) ([16], Theorem 2.3) *Let f be a perfect function from a space X onto a space Y. Then X is almost locally compact if and only if Y is almost locally compact.*

It was proved the equivalence of ultracompleteness and almost local compactness in some classes of topological spaces. Yoshioka proved in [16] the equivalence of these concepts in the classes of normal γ-spaces and ks-spaces. Buhagiar proved in [3] the same result in the class of GO spaces.

3. Products and hyperspaces of ultracomplete spaces.

Proposition 16 ([4], Theorem 4.12). *Let X and Y be two ultracomplete spaces. Then $X \times Y$ is ultracomplete if, and only if, one of the following conditions holds:*

i) X and Y are locally compact,

ii) either X or Y is countably compact, locally compact,

iii) both X and Y are countably compact.

Proposition 17 ([4], Theorem 4.16). *Let X_n be an ultracomplete, countably compact space for every $n \in \mathbb{N}$, then $X = \prod_{n\in\mathbb{N}} X_n$ is ultracomplete countably compact.*

The space $X = ([0,1] \setminus \{\frac{1}{n} : n \in \mathbb{N}\}) \times \mathbb{N} = (\{0\} \times \mathbb{N}) \cup ((0,1] \setminus \{\frac{1}{n} : n \in \mathbb{N}\}) \times \mathbb{N})$ is the union of two locally compact spaces, and it is not almost locally compact, because $nlc(X) = \{0\} \times \mathbb{N}$ is not compact.

Proposition 18 ([12], Theorem 2.7). *If X is a metrizable space, $n \in \mathbb{N}$ and $X^n = X_1 \cup X_2 \cup \cdots \cup X_n$, where every X_k is ultracomplete then X is also ultracomplete.*

Proposition 19 ([12], Theorem 2.8). *If X^ω is a union of countably many of its ultracomplete subspaces, then the space X^ω is ultracomplete.*

Proposition 20 ([10], Theorem 3.15). *For any space X and $n \in \mathbb{N}$, if $X^n = X_1 \cup X_2 \cup \cdots \cup X_n$ and every X_k is almost locally compact, then X is almost locally compact.*

For any space X let $\mathcal{K}(X)$ ($\mathcal{K}_C(X)$) be the hyperspace of all nonempty compact (compact connected) subsets of X endowed with the Vietoris topology. For any $n \in \mathbb{N}$ we denote by $\mathcal{F}_n(X)$ the hyperspace of all non empty subsets of cardinality $\leq n$ endowed with the topology induced by $\mathcal{K}(X)$. It is well known that X is Čech-complete (locally compact) if and only if $\mathcal{K}(X)$ is Čech-complete (locally compact). If (Y,d) is a metric space, let d_H denote the Hausdorff metric induced by d in the hyperspace $\mathcal{K}(Y)$.

Example 21 ([7], Example 2.12). If $X = [0,1] \setminus \{\frac{1}{n} : n \in \mathbb{N}\}$, then:

i) X and $\mathcal{K}_C(X)$ are almost locally compact,

ii) $\mathcal{K}(X)$ and $\mathcal{F}_n(X)$ are not almost locally compact for all $n > 1$,

Proposition 22 ([7], Proposition 2.11)**.** *The space $\mathcal{K}(X)$ is almost locally compact if and only if X is locally compact.*

From propositions 14 and 16 it follows the next proposition.

Proposition 23. *If the hyperspace $\mathcal{K}(X)$ is ultracomplete, then X is ultracomplete and either countably compact or locally compact. If in addition $\mathcal{K}(X)$ has no points of local compactness, then X is countably compact and has no points of local compactness.*

Example 24. The space X defined in Example 10 is ultracomplete non-countably compact and has no points of local compactness. From Proposition 23 it follows that $\mathcal{K}(X)$ is not ultracomplete.

Proposition 25 ([7], Proposition 3.7)**.** *If X is a countably compact ultracomplete space, then $\mathcal{F}_n(X)$ is ultracomplete countably compact for any $n \in \mathbb{N}$.*

Künzi and Romaguera proved the next

Proposition 26 ([11], Corollary 3.5)**.** *A metric space (X, d) is uniformly locally compact if and only if $(\mathcal{K}(X), d_H)$ is cofinally complete.*

Beer has also worked on cofinal completeness in [2]. He obtained, among other things, the next two propositions.

Proposition 27 ([2], Theorem 4.1)**.** *If X is a metrizable space, then X is ultracomplete if and only if whenever A is a closed subset of $nlc(X)$ we have $\chi(A, X) \leq \omega$.*

If (X, d) is a metric space, $x_0 \in X$ and $\epsilon > 0$, we denote $S_\epsilon(x_0)$ (resp. $S_\epsilon[x_0]$) the open (resp. closed) ϵ-ball with center in x_0. If $x \in X$ has compact neighborhood, set $\nu(x) = \sup\{\epsilon > 0 : S_\epsilon[x_0] \text{ is compact}\}$, otherwise set $\nu(x) = 0$ (see [2]).

Proposition 28 ([2], Theorem 3.2)**.** *If (X, d) is a metric space, then (X, d) is cofinally complete if and only if either X is uniformly locally compact or $nlc(X)$ is nonempty and compact and $\{x \in X : \nu(x) \leq \frac{1}{n}\}$ converges to $nlc(X)$ in the Hausdorff metric d_H.*

4. Ultracompleteness in $C_p(X)$.

Romaguera and Sanchis analized in [15] ultracompleteness of topological groups.

Proposition 29 ([15], Theorem 2.1). *If X is a Hausdorff topological group, then X is ultracomplete if and only if it is locally compact.*

This fact implies that $C_p(X)$ is ultracomplete if and only if X is finite. It is well known that $C_p(X)$ has a dense Čech-complete subspace if and only if X is countable and discrete (see [1], Theorem I.3.1). It is important to observe that ω_1^ω is a non-locally compact homogeneous ultracomplete space.

Proposition 30 ([8], Proposition 2.11). *A function space $C_p(X)$ has a dense ultracomplete subspace if and only if X is finite.*

A space X is called Eberlein-Grothendieck if it is homeomorphic to a subspace of $C_p(Y)$ for some compact space Y. Jardón and Tkachuk proved in [8] that every ultracomplete Eberlein-Grothendieck space is Fréchet-Urysohn and has points of local compactness.

Proposition 31 ([8], Theorem 2.9). *An Eberlein-Grothendieck space is ultracomplete if and only if it is almost locally compact.*

The previous fact was generalized for a wider class of spaces. A compact space X is called Corson compact if $C_p(X)$ is primarily Lindelöf. A space X is called splittable if, for each $f \in \mathbb{R}^X$, there exists a countable set $A \subset C_p(X)$ such that $f \in \overline{A}$ (the closure is taken in $\mathbb{R}^X$). Recall that the topology of pointwise convergence defined on $C_p(X) \subset \mathbb{R}^X$ coincides with the topology induced from the product topology of $\mathbb{R}^X$.

Proposition 32 ([9], Corollary 3.9). *Suppose that the space X is Lindelöf Σ or pseudocompact. Then a subspace $Y \subset C_p(X)$ is ultracomplete if and only if it is almost locally compact.*

Proposition 33 ([9], Corollary 3.10 and Proposition 3.12). *If X is splittable or a subspace of a Corson compact space, then X is ultracomplete if and only if it is almost locally compact.*

If X is a metrizable space and $nlc(X)$ is compact, then X is almost locally compact. This fact cannot be extended to the class of Eberlein-Grothendieck spaces.

Example 34 ([8], Example 2.17)**.** There exists a countable closed subspace $X \subset C_p([0,1])$ without non-trivial convergent sequences which has a unique non-isolated point x_0. The space X is not Fréchet-Urysohn. Thus X is a non-ultracomplete Eberlein-Grothendieck space which is non-locally compact only at the point x_0.

5. Problems

We finish this work with some problems related to ultracomplete spaces.

Problem 35. *Suppose that X is an ultracomplete homogeneous space without points of local compactness. Must X have a dense countably compact subspace?*

Problem 36. *Suppose that the closure of any countably compact subspace of X is compact. If X is ultracomplete, must X have points of local compactness?*

Problem 37. *Let X be a scattered compact space. Is it true that every ultracomplete $Y \subset X$ is almost locally compact?*

Problem 38. *Let X be a scattered compact space of countable tightness. Is it true that every ultracomplete $Y \subset X$ is almost locally compact?*

Problem 39. *Is it true that $\mathcal{K}(\omega_1^\omega)$ is an ultracomplete space? Let X be an infinite compact space. If $F(X)$ is the Markov free topological group of X and define $Y = \beta(F(X)) \setminus F(X)$. Is it true that $\mathcal{K}(Y)$ is an ultracomplete space?*

Problem 40. *Suppose that X is a countably compact ultracomplete space without points of local compactness. Must $\mathcal{K}(X)$ be countably compact, or at least pseudocompact?*

Problem 41. *Suppose that $X^2 = X_1 \cup X_2$ where X_i is ultracomplete for $i = 1, 2$. Must X be ultracomplete?*

References

[1] A.V. Arkhangel'skii, Topological Function Spaces, Kluwer Academic Publishers, Dordretcht (1979).

[2] G. Beer, Between compactness and completeness, Topology Appl. 155 (2008), 503–514.

[3] D. Buhagiar, Non locally compact points in ultracomplete topological spaces, Questions and Answers Gen. Topology 19 (2001), 125–170.

[4] D. Buhagiar, I. Yoshioka, Sums and products of ultracomplete topological spaces, Topology Appl. 122 (2002), 77–86.

[5] D. Buhagiar, I. Yoshioka, Ultracomplete topological spaces, Acta Math. Hungar. 92, no. 1-2 (2001), 19–26.

[6] A. García-Máynez, S. Romaguera, Perfect pre-images of cofinally complete metric spaces, Comment. Math. Univ. Carolin. 40, no. 2 (1999), 335–342.

[7] D. Jardón, Ultracompleteness of hyperspaces of compact sets, Acta Math. Hungar. 137 (2012), 139–152.

[8] D. Jardón, V. V. Tkachuk, Ultracompleteness in Eberlein-Grothendieck spaces, Bol. Soc. Mat. Mexicana 10 (2004), 209–218.

[9] D. Jardón, V. V. Tkachuk, When is an ultracomplete space almost locally compact?, Appl. Gen. Topol. 7 (2006), 191–201.

[10] D. Jardón, V. V. Tkachuk, Ultracomplete metalindelöf spaces are almost locally compact, New Zealand J. Math. 36 (2007), 277–285.

[11] H. P. Künzi, S. Romaguera, Quasi-metric spaces, quasi-metric hyperspaces and uniform local compactness, Rend. Istit. Mat. Univ. Trieste 30 (1999), 133–144.

[12] M. López de Luna, V. V. Tkachuk, Čech-completeness and ultracompleteness in "nice" spaces, Comment. Math. Univ. Carolin. 43, no. 3 (2002), 515–524.

[13] V. I. Ponomarev, V. V. Tkachuk, The countable character of X in βX compared with the countable character of the diagonal in $X \times X$ (in Russian), Vestnik Mosk. Univ. 42, no. 5 (1987), 16–19.

[14] S. Romaguera, On cofinally complete metric spaces, Questions and Answers Gen. Topology 16 (1998), 165–170.

[15] S. Romaguera, M. Sanchis, Locally compact topological groups and cofinal completeness, J. London Math. Soc. 62, no. 2 (2000), 451–460.

[16] I. Yoshioka, On the subsets of non locally compact points of ultracomplete spaces, Comment. Math. Univ. Carolin. 43 (2002), 707–721.

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Extreme points in compact convex sets in asymmetric normed spaces

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Abstract

In this talk we will present some results concerning the existence of extreme points in compact convex subsets of asymmetric normed spaces. We focus our attention in the finite dimensional case, giving a geometric description of all compact convex subsets of a finite dimensional asymmetric normed space.

1. Introduction

An asymmetric normed space is a real vector space X equipped with a so called asymmetric norm q. This means that $q : X \to [0, \infty)$ is a function satisfying

(1) $q(tx) = tq(x)$ for every $t \geq 0$ and $x \in X$,
(2) $q(x + y) \leq q(x) + q(y)$ and
(3) $q(x) = 0 = q(-x)$ if and only if $x = -x = 0$.

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Each asymmetric norm q in an vector space X defines a (symmetric) norm $q^s : X \to [0, \infty)$ defined by the following rule

$$q^s(x) = \max\{q(x), q(-x)\}.$$

Any asymmetric norm induces a non symmetric topology on X that is generated by the asymmetric open balls $B_q(x, \varepsilon) = \{y \in X \mid q(y - x) < \varepsilon\}$.

This topology is a T_0 topology in X for which the vector sum on X is continuous. Furthermore, the topology is T_1 if and only if the set

$$\theta(0) := \{x \mid q(x) = 0\}$$

coincides with the singleton $\{0\}$. However, in most cases this topology is not even Hausdorff and the scalar multiplication is not continuous. Thus (X, q) fails to be a topological vector space. Nevertheless, asymmetric normed spaces are still interesting, mainly because of the many application they have (e.g. in theoretical computer science and particularly in complexity theory).

2. Krein-Milman theorem on asymmetric normed spaces

The well known theorem of Krein-Milman states that every compact convex subset of a locally convex (Hausdorff) space is the closure of the convex hull of its extreme points. If the compact convex set is finite dimensional, then Caratéodory's Theorem says that in that case it is the convex hull of its extreme points. However, in the asymmetric case, neither Krein-Milman nor Caratéodory's Theorem is true, as we can see in the following easy example:

Example 1. Let $u : \mathbb{R} \to [0, \infty)$ be the asymmetric norm defined by the rule

$$u(x) = \max\{x, 0\}.$$

The compact convex set $(0, 1]$ has only one extreme point, the $\{1\}$. However, the convex hull of $\{1\}$ is exactly $\{1\}$ and the closure of $\{1\}$ is not the set $(0, 1]$, but the interval $[1, \infty)$.

However, if the topology of the asymmetric normed space is Hausdorff, then we have the following:

Theorem 2 (Cobzaç [1]). *Let (X, q) be an asymmetric normed space such that the topology τ_q is Hausdorff. Then any nonempty q-compact convex subset of X is the q-closed convex hull of the set of its extreme points.*

In [3] we studied the existence of extreme points in the non-Hausdorff case obtaining the following results:

Theorem 3. *Let (X, q) be an asymmetric normed space. Suppose that $K \subset X$ is a q-compact convex subset of X. Then the set of extreme points of $K + \theta(0)$ is contained in K.*

Theorem 4. *Let K be a q-compact convex subset of an asymmetric normed space (X, q) with the property that $K + \theta(0)$ is q^s-locally compact. Then K has at least one extreme point. In particular, if $K + \theta(0)$ has finite dimension, then K has at least one extreme point.*

In contrast with the normed case, let us observe that Theorem 4 is the best we can say about extreme points in q-compact convex sets. For instance, in any asymmetric normed space (X, q), the set $\theta(x) = x + \theta(0)$ is a q-compact convex set for whom its only extreme point is x itself.

For a compact convex set K let us denote by $S(K)$ the convex hull of its extreme points. Then we have the following

Theorem 5. *Let (X, q) be an asymmetric normed space and K a q-compact convex subset of X such that $K + \theta(0)$ has finite dimension (for example, if X is finite dimensional). Then*

$$S(K) \subset K \subset S(K) + \theta(0) = K + \theta(0).$$

Furthermore, as a corollary we have the following property for the set $S(K)$.

Corollary 6. *Let K be a q-compact convex subset in an asymmetric normed space (X, q) such that $K + \theta(0)$ has finite dimension. If $K_0 \subset X$ is any subset satisfying*

$$S(K) \subset K_0 \subset S(K) + \theta(0)$$

then K_0 is q-compact.

3. Strong compactness in finite dimensional compact convex subsets

Corollary 6 is somehow related with the notion of strong compactness in asymmetric normed spaces.

Let us recall that a subset K in an asymmetric normed space (X, q) is strongly compact iff there exist a q^s-compact subset $S \subset X$ satisfying the following contentions

$$S \subset K \subset S + \theta(0).$$

Is not difficult to prove that every strongly compact set is always compact. If (X, q) is a 2-dimensional asymmetric normed lattice, then for every compact convex subset K, the set $S(K)$ is always q_s-compact and therefore we have the following corollary

Corollary 7 ([2]). *Let q be an asymmetric lattice norm in $\mathbb{R}^2$. Then every q-compact convex set K in $(\mathbb{R}^2, q)$ is strongly q-compact.*

However, the previous result can not be generalized in higher dimensions as we can see in the following example

Example 8. Consider the asymmetric normed lattice $(\mathbb{R}^3, q)$, where $q : \mathbb{R}^3 \to [0, \infty)$ is the asymmetric lattice norm defined by the rule:

$$q(x) = \max\{\max\{x_i, 0\} \mid i = 1, 2, 3\} \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Let $K = \text{co}(A \cup \{(0, 0, 0), (0, 1, 1)\})$ where A is the set defined as

$$A = \{(x_1, 0, x_3) \mid x_1^2 + x_3^2 = 1,\ x_1 \in (0, 1],\ x_3 \geq 0\}.$$

For any q-open cover $\mathcal{U}$ of K, there exists an element $U \in \mathcal{U}$ such that $(0, 1, 1) \in U$. This implies that

$$(0, 1, 1) + \theta_0 \subset U + \theta_0 = U,$$

and therefore the cover $\mathcal{U}$ is a q-open (and q^s-open) cover for $\overline{K}^{q^s}$ which is q^s-compact. Thus, we can extract a finite subcover $\mathcal{V} \subset \mathcal{U}$ for $\overline{K}^{q^s}$. This cover $\mathcal{V}$ is a finite subcover for K too, and then we can conclude that K is q-compact.

To finish this example, let us note that K is not strongly q-compact. For this, simply observe that any set K_0 satisfying $K_0 \subset K \subset K_0 + \theta_0$, must contain the set A. If K_0 is additionally q^s-compact, then it is q^s-closed too and then

$$\overline{A}^{q^s} \subset \overline{K_0}^{q^s} = K_0 \subset K,$$

which is impossible.

References

[1] S. Cobzaş, Functional analysis in asymmetric normed spaces, Birkhäuser. Basel. 2013.

[2] N. Jonard-Pérez and E. A. Sánchez-Pérez, Compact convex sets in 2-dimensional asymmetric normed lattices, Quaestiones Mathematicae (2015), 1–10.

[3] N. Jonard-Pérez and E. A. Sánchez-Pérez, *Extreme points and geometric aspects of compact convex sets in asymmetric normed spaces*, Topology and its Applications 203 (2016) 12–21.

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A study of Takahashi convexity structures in T_0-quasi-metric spaces

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Abstract

We summarize our investigations on convexity structures in T_0-quasi-metric spaces as they were conducted in more detail in [13]. *In particular the reader is referred to* [13] *for proofs of the stated results and detailed references to the literature.*

Keywords: *T_0-quasi-metric; convexity structure, strictly convex; asymmetrically normed vector space; Hausdorff quasi-pseudometric; q-hyperconvex.*
MSC: *54E35; 54E55; 52A01; 47H09.*

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1. Preliminaries

For concepts from asymmetric topology we refer the reader to [8, 11]. In order to fix our terminology we recall the following notions.

Let X be a set and let $d : X \times X \to [0, \infty)$ be a function mapping into the set $[0, \infty)$ of the nonnegative reals. Then d is called a *quasi-pseudometric* on X if

(a) $d(x, x) = 0$ whenever $x \in X$, and

(b) $d(x, z) \leq d(x, y) + d(y, z)$ whenever $x, y, z \in X$. We shall say that d is a T_0-*quasi-metric* provided that d also satisfies the following T_0-condition: For each $x, y \in X$, $d(x, y) = 0 = d(y, x)$ implies that $x = y$.

Remark 1. Let d be a quasi-pseudometric on a set X, then $d^{-1} : X \times X \to [0, \infty)$ defined by $d^{-1}(x, y) = d(y, x)$ whenever $x, y \in X$ is also a quasi-pseudometric, called the *conjugate* or *dual quasi-pseudometric of* d. As usual, a quasi-pseudometric d on X such that $d = d^{-1}$ is called a *pseudometric*. Note that for any quasi-pseudometric d (T_0-quasi-metric), $d^s = \sup\{d, d^{-1}\} = d \vee d^{-1}$ is a pseudometric (metric).

Given a subset A of a quasi-pseudometric space (X, d) we call $\delta(A) = \sup\{d(a, a') : a, a' \in A\}$ the *diameter of* A. The set A is called *bounded* if $\delta(A) < \infty$.

For each $x, y \in \mathbb{R}$ we set $x \dot{-} y = \max\{x - y, 0\}$. Letting $u(x, y) = x \dot{-} y$ whenever $x, y \in \mathbb{R}$ we obtain a natural example of a T_0-quasi-metric space $(\mathbb{R}, u)$. In the following let $I = [0, 1]$ be the set of the real unit interval.

Given a quasi-pseudometric d on a set X, the set of open balls $B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$ with $\epsilon > 0$ and $x \in X$ yields a base of the so-called *quasi-pseudometric topology* $\tau(d)$ on X. Note that for each $x \in X$ and $\epsilon \geq 0$ the "closed" ball $C_d(x, \epsilon) = \{y \in X : d(x, y) \leq \epsilon\}$ is indeed $\tau(d^{-1})$-closed, but may not be $\tau(d)$-closed.

We consider a T_0-quasi-metric space (X, d) equipped with a Takahashi *convexity structure* (briefly TCS). According to our definition, this is a mapping W from $X \times X \times I$ to X (that is, $W(x, y, \lambda)$ defined for all pairs $(x, y) \in X \times X$ and $\lambda \in I$

and valued in X) satisfying the two conditions (called (1) and (2) in the following):

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$$

whenever $u \in X$ (1) and

$$d(W(x, y, \lambda), u) \leq \lambda d(x, u) + (1 - \lambda)d(y, u)$$

whenever $u \in X$ (2).

Evidently the second condition is just the first condition formulated for the dual T_0-quasi-metric. So, by definition, if W is a TCS for (X, d), then it is also a TCS for (X, d^{-1}). Obviously in a metric space our definition yields the concept of a convexity structure in the sense of Takahashi [15]. (We note that in the literature convexity structures are usually studied in the metric setting, see e.g. [4, 14]; the paper [9] seems to be an exception.)

Remark 2. If W is a TCS on a T_0-quasi-metric space (X, d), then W is also a TCS for the metrics $d^+ = d + d^{-1}$ and d^s on X.

Equip $(\mathbb{R}, u)$ with its standard TCS $S(x, y, \lambda) = \lambda x + (1 - \lambda)y$ whenever $x, y \in \mathbb{R}$ and $\lambda \in I$. Fix $\alpha > 0$. For each $x, y \in \mathbb{R}$ and $\lambda \in I$ set $H(x, y, \lambda) = S(x, y, \lambda) - \alpha$. Then H satisfies condition (2) on $(\mathbb{R}, u)$, but it does not satisfy condition (1). Thus H is not a TCS on $(\mathbb{R}, u)$.

The notation $W(x, y, \lambda)$ for a TCS on (X, d) is convenient, although the notation $W(x, y, \lambda, 1 - \lambda)$ would be more appropriate, in particular if one is interested in n-dimensional analogues $W : X^n \times I^n \to X$ (with $n \in \mathbb{N}$ and $n > 2$) of convexity structure functions and their properties. Similarly, as in the metric case, for instance such higher dimensional functions H_n can be obtained from an ordinary TCS W by iterations as follows: For $x_1, \ldots, x_n \in X$ and $\alpha_1, \ldots, \alpha_n \in I$ with $\sum_{i=1}^{n} \alpha_i = 1$ set

$$H_n(x_1, ..., x_n, \alpha_1, \ldots, \alpha_n) = W(H_{n-1}(x_1, \ldots, x_{n-1}, \frac{\alpha_1}{1 - \alpha_n}, \ldots, \frac{\alpha_{n-1}}{1 - \alpha_n}), x_n, 1 - \alpha_n)$$

if $\alpha_n \neq 1$ and $H_n(x_1, \ldots, x_n, 0, \ldots, 0, 1) = x_n$ otherwise.

Many conditions studied in this note have obvious analogues to such higher dimensional functions; for instance we can study the conditions that for any $n \in \mathbb{N}$,

$u \in X$, $x_1, \ldots, x_n \in X$ and $\lambda_1, \ldots, \lambda_n \in I$ $(i = 1, \ldots, n)$ such that $\sum_{i=1}^n \lambda_i = 1$ we have that

$$d(u, T_n(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)) \leq \sum_{i=1}^{n} \lambda_i d(u, x_i)$$

as well as

$$d^{-1}(u, T_n(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)) \leq \sum_{i=1}^{n} \lambda_i d^{-1}(u, x_i).$$

In this note however we shall concentrate on the case $n = 2$.

Remark 3. Let $W(x, y, \lambda)$ be a TCS on a T_0-quasi-metric space (X, d). Then $W^{-1}(x, y, \lambda) := W(y, x, 1 - \lambda)$ whenever $x, y \in X$ and $\lambda \in I$ is a TCS on (X, d).

A TCS W on a T_0-quasi-metric space (X, d) will be called *synchronized* if $W^{-1}(x, y, \lambda) = W(x, y, \lambda)$ whenever $x, y \in X$ and $\lambda \in I$.

Let us mention that the analogous condition is called condition (C) in the metric setting.

Proposition 4. *Let (X, d) be a T_0-quasi-metric space with a TCS W. Then $W(x, x, \lambda) = x$ whenever $x \in X$ and $\lambda \in I$. Furthermore we have $W(y, x, 0) = x$ and $W(y, x, 1) = y$ whenever $x, y \in X$. For $x, y \in X$ and $\lambda \in I$ we have $d(x, W(x, y, \lambda)) = (1 - \lambda)d(x, y)$ and $d(W(x, y, \lambda), y) = \lambda d(x, y)$. Furthermore $d(y, W(x, y, \lambda)) = \lambda d(y, x)$ and $d(W(x, y, \lambda), x) = (1 - \lambda)d(y, x)$.*

Remark 5. Let (X, d) be a T_0-quasi-metric space and $W(x, y, \lambda)$ and $W'(x, y, \lambda)$ be two convexity structures in (X, d).

(a) Suppose that $x, y \in X$ and $\lambda', \lambda \in I$ with $\lambda' \leq \lambda$. Then $(\lambda - \lambda')d(x, y) \leq d(W(x, y, \lambda), W(x, y, \lambda'))$.

A dual argument yields that if $x, y \in X$ and $\lambda', \lambda \in I$ with $\lambda' \geq \lambda$, then $(\lambda' - \lambda)d(y, x) \leq d(W(x, y, \lambda), W(x, y, \lambda'))$.

(b) We also have

$$d(W(x, y, \lambda), W(x, y, \lambda')) \leq \lambda'(1 - \lambda)d(y, x) + (1 - \lambda')\lambda d(x, y)$$

whenever $x, y \in X$ and $\lambda, \lambda' \in I$.

(c) Similarly as in (b) we obtain the following result: For any $x, y \in X$ and $\lambda \in I$ we have that $d(W(x,y,\lambda), W'(x,y,\lambda)) \leq \lambda(1-\lambda)(d(y,x)+d(x,y))$. In particular $d(W(x,y,\lambda), W'(x,y,\lambda)) \leq \frac{1}{4}(d(y,x)+d(x,y))$ whenever $x, y \in X$ and $\lambda \in I$.

Takahashi does not require any continuity property in his definition of a convexity structure for metric spaces. However it is often natural to make the assumption that W satisfies some additional continuity conditions. For instance in the metric case it is assumed in the literature that W is continuous in the third variable.

We next state a result for general convexity structures of T_0-quasi-metric spaces that is analogous to a known result for metric spaces.

Proposition 6. *Let W be a TCS on a T_0-quasi-metric space (X,d). Then for each $x \in X$ and $\lambda \in I$, W is continuous at (x,x,λ) where X carries the topology $\tau(d)$ (or $\tau(d^{-1})$). (It does not matter which topology I carries.)*

2. Asymmetrically normed real vector spaces

Let X be a real vector space and let $\|\cdot| : X \to [0,\infty)$ be a map such that

(1) $\|0| = 0$.

(2) $\|x+y| \leq \|x| + \|y|$ whenever $x, y \in X$.

(3) $\|\alpha x| = \alpha\|x|$ whenever $x \in X$ and $\alpha \geq 0$.

Furthermore suppose that $\|x| = \|-x| = 0$ implies that $x = 0$.

The function $\|\cdot|$ is called an *asymmetric norm* on X. Obviously each asymmetrically normed vector space X induces a T_0-quasi-metric d on X by setting $d(x,y) = \|x-y|$ whenever $x, y \in X$ (compare for instance [6]).

Example 7. Let $\mathbb{R}$ be equipped with its usual real vector space structure. Then $\|x|_u = x$ if $x \geq 0$ and $\|x|_u = 0$ otherwise, defines an asymmetric norm on $\mathbb{R}$ with the induced T_0-quasi-metric u, as defined above. (We shall often write $\|\cdot|$ instead of $\|\cdot|_u$ in the following.)

Proposition 8. *Let C be a convex subset (in the usual linear sense) of a real vector space X equipped with the asymmetric norm $\|\cdot|$. Then $S(x,y,\lambda) = \lambda x + (1-\lambda)y$*

whenever $x, y \in C$ *and* $\lambda \in I$ *defines a synchronized convexity structure for the* T_0*-quasi-metric space* (C, d) *where* $d(x, y) = \|x - y|$ *whenever* $x, y \in C$ *:*

Proof. For any $x, y, u \in C$ and $\lambda \in I$ we have that

$$d(S(x, y, \lambda), u) = \|S(x, y, \lambda) - u| =$$
$$\|S(x - u, y - u, \lambda)| \leq \lambda\|x - u| + (1 - \lambda)\|y - u| = \lambda d(x, u) + (1 - \lambda)d(y, u).$$

Similarly for each $x, y, u \in C$ and $\lambda \in I$, $d(u, (S(x, y, \lambda)) = \|u - S(x, y, \lambda)| =$

$$\|S(u - x, u - y, \lambda)| \leq \lambda\|u - x| + (1 - \lambda)\|u - y| = \lambda d(u, x) + (1 - \lambda)d(u, y).$$

□

Remark 9. If $W(x, y, \lambda)$ and $W'(x, y, \lambda)$ are convexity structures in an asymmetrically normed real vector space $(X, \| \cdot |)$, then for each $\alpha \in I$, $W_\alpha(x, y, \lambda) = \alpha W(x, y, \lambda) + (1 - \alpha)W'(x, y, \lambda)$ is also a convexity structure on X (with respect to the induced T_0-quasi-metric).

3. T_0-quasi-metric spaces with a unique convexity structure

This section is motivated by the investigations in [16] in the case of metric spaces.

Let (X, d) be a T_0-quasi-metric space. Then (X, d) will be called *strictly convex* provided that for each $x, y \in X$ and $\lambda \in I$ there is a unique $w(x, y, \lambda) \in X$ such that

$$d(x, w(x, y, \lambda)) = (1 - \lambda)d(x, y),$$
$$d(w(x, y, \lambda), y) = \lambda d(x, y),$$
$$d(y, w(x, y, \lambda)) = \lambda d(y, x)$$

and

$$d(w(x, y, \lambda), x) = (1 - \lambda)d(y, x).$$

Remark 10. Note that in the terminology of [1] it follows that for any $x, y \in X$, we have that $(x, w(x, y, \lambda), y)$ is collinear in (X, d) as well as in (X, d^{-1}).

Since any TCS $W(x, y, \lambda)$ on (X, d) satisfies the afore-mentioned system of equations, we see that each strictly convex T_0-quasi-metric space admits at most one convexity structure.

Example 11. There exists an example of a T_0-quasi-metric space (X, d) and a TCS W for (X, d^s), which is not a TCS for (X, d).

Let W be a TCS on a T_0-quasi-metric space (X, d). We shall say that (X, d) has a *unique* TCS if for any $w \in X$ such that there exists $(x, y, \lambda) \in X \times X \times I$ with $d(z, w) \leq \lambda d(z, x) + (1 - \lambda)d(z, y)$ and $d(w, z) \leq \lambda d(x, z) + (1 - \lambda)d(y, z)$ whenever $z \in X$, then $w = W(x, y, \lambda)$.

Obviously the condition means exactly that W is the unique TCS on (X, d). Of course for any T_0-quasi-metric space (X, d) if (X, d^s) has a unique TCS, then (X, d) has a unique TCS. Recall also that if (X, d) is a T_0-quasi-metric space with a TCS W and (X, d) is strictly convex in the above mentioned sense, then W is necessarily the unique TCS on (X, d).

Example 12. Let $n \in \mathbb{N}$ and $X = \mathbb{R}^n$ be equipped with its usual real vector space structure and its supremum asymmetric norm $\|(x_i)| = \max_{i \leq n} \|x_i|$ and d its associated T_0-quasi-metric. Then the standard convexity structure $S(x, y, \lambda) = \lambda x + (1 - \lambda)y$ is the unique TCS on (X, d).

Proposition 13. *Let (X, d) be a T_0-quasi-metric space with a unique TCS W such that $\tau(d^s)$ is compact. Then $W : (X, \tau(d^s)) \times (X, \tau(d^s)) \times (I, \tau(u^s)) \to (X, \tau(d^s))$ is a continuous function.*

Lemma 14. *Let W be the unique TCS on a T_0-quasi-metric space (X, d). Then for every $x, y \in X$ and $\alpha, \beta \in I$, we have*

$$W(W(x, y, \beta), y, \alpha) = W(x, y, \alpha\beta).$$

We observe that a convexity structure W in a metric space is said to have property (J) if it satisfies the property that

$W(W(x, y, \beta), y, \alpha) = W(x, y, \alpha\beta)$ whenever $x, y \in X$ and $\alpha, \beta \in I$. We shall use the same name for this property in a T_0-quasi-metric space.

If the convexity structure W on a T_0-quasi-metric space (X, d) is unique, then it satisfies condition (C), that is, $W(y, x, 1-\lambda) = W(x, y, \lambda)$ whenever $x, y \in X$ and $\lambda \in I$.

Example 15. Let $X = I$. Choose $\alpha, \beta \in [0, \infty)$ such that $\alpha + \beta \neq 0$. For $\lambda_1, \lambda_2 \in I$ set $d_{\alpha\beta}(\lambda_1, \lambda_2) = (\lambda_1 - \lambda_2)\alpha$ if $\lambda_1 \geq \lambda_2$ and $d_{\alpha\beta}(\lambda_1, \lambda_2) = (\lambda_2 - \lambda_1)\beta$ if $\lambda_2 > \lambda_1$. Then $(I, d_{\alpha\beta})$ is a T_0-quasi-metric space (called the *quasi-metric segment* $I_{\alpha\beta}$) induced by the restriction of the asymmetric norm $n_{\alpha\beta}$ on $\mathbb{R}$ defined for $\lambda \in \mathbb{R}$ by $n_{\alpha\beta}(\lambda) = \lambda\alpha$ if $\lambda \geq 0$ and $n_{\alpha\beta}(\lambda) = -\lambda\beta$ if $\lambda < 0$. Obviously on $\mathbb{R}$ the asymmetric norm $n_{\alpha\beta}$ induces the T_0-quasi-metric $u_{\alpha\beta}$ defined for $x, y \in \mathbb{R}$ by $u_{\alpha\beta}(x, y) = (x - y)\alpha$ if $x \geq y$ and $u_{\alpha\beta}(x, y) = (y - x)\beta$ if $y > x$. Note that in particular $u_{10} = u$ with the T_0-quasi-metric u as introduced above.

We remark that a TCS W in a metric space (X, d) is said to satisfy condition (I) provided that

$$d(W(x, y, \lambda_1), W(x, y, \lambda_2)) = |\lambda_1 - \lambda_2| d(x, y)$$

whenever $x, y \in X$ and $\lambda_1, \lambda_2 \in I$.

The given formulation of condition (I) is unsuitable for a T_0-quasi-metric space (X, d) that is not metric: If d is a T_0-quasi-metric with properties (C) and (I), then it satisfies

$$|1-0| d(x, y) = d(W(x, y, 1), W(x, y, 0)) = d(W(y, x, 0), W(y, x, 1)) = |0-1| d(y, x)$$

whenever $x, y \in X$ and thus d would be a metric. Hence for a T_0-quasi-metric space (X, d) we suggest that condition (I) is reformulated as condition (I') in the following way:

A T_0-quasi-metric space (X, d) satisfies condition (I') provided that for any $x, y \in X$ and $\lambda_1, \lambda_2 \in I$, $d(W(x, y, \lambda_1), W(x, y, \lambda_2)) = (\lambda_1 - \lambda_2) d(x, y)$ if $\lambda_1 \geq \lambda_2$ and $d(W(x, y, \lambda_1), W(x, y, \lambda_2)) = (\lambda_2 - \lambda_1) d(y, x)$ if $\lambda_1 < \lambda_2$.

Note that a metric space satisfies condition (I) if and only if it satisfies condition (I').

If a TCS W on a T_0-quasi-metric space (X, d) is unique, then for every $a, b \in X$ with $a \neq b$ the function $h : I \to X$ defined by $h(\lambda) = W(a, b, \lambda)$ whenever $\lambda \in I$ is an isometric embedding of $I_{d(a,b)d(b,a)}$ into X. (Hence W satisfies condition (I').)

4. Collections of convex subsets in T_0-quasi-metric spaces

A subset K of a T_0-quasi-metric space (X, d) with TCS W will be said to be *(W-)convex* provided that $W(x, y, \lambda) \in K$ whenever $x, y \in K$ and $\lambda \in I$. Note that in particular X is a convex set and that each convex subset C of (X, d) carries a natural TCS, namely the restriction of W to $C \times C \times I$.

The intersection of any family of convex sets is obviously convex.

Given a subset A of X we can construct the smallest convex subset containing A by iterations as $\bigcup_{n\in\mathbb{N}} C^n(A)$ where $C(A) = \{W(x, y, \lambda) : x, y \in A, \lambda \in I\}$ and $C^{n+1}(A) = C(C^n(A))$ whenever $n \in \mathbb{N}$.

Proposition 16. *Let (X, d) be a T_0-quasi-metric space with a TCS W. Then for any $x \in X$ and $r > 0$ the open balls $B_d(x, r)$ and $B_{d^{-1}}(x, r)$, and the closed balls $C_d(x, r)$ and $C_{d^{-1}}(x, r)$ in X are convex subsets of X.*

We say that a TCS W in a T_0-quasi-metric space (X, d) has property (S) provided that

$$d(W(x, y, \lambda), W(x', y', \lambda)) \leq \lambda d(x, x') + (1 - \lambda)d(y, y')$$

whenever $x, y, x', y' \in X$ and $\lambda \in I$.

Remark 17. (a) Clearly condition (S) together with the condition that for any $x \in X$ and $\lambda \in I$, $W(x, x, \lambda) = x$ imply conditions (1) and (2).

(b) Observe that if we replace d by d^{-1} in condition (S), then we obtain a condition which is equivalent to condition (S).

(c) We note that the standard convexity structure on a convex set of an asymmetrically normed real vector space X has property (S).

(d) Condition (S) together with condition (I') for a TCS W on a T_0-quasi-metric space (X, d) imply continuity of $W : (X, \tau(d)) \times (X, \tau(d)) \times (I, \tau(u^s)) \to (X, \tau(d))$. (The result also holds for $\tau(d^{-1})$ instead of $\tau(d)$.)

A TCS W on a T_0-quasi-metric space (X,d) satisfies property (S') if

$$d(W(x,y,\lambda), W(x,z,\lambda)) \leq (1-\lambda)d(y,z)$$

whenever $x,y,z \in X$ and $\lambda \in I$.

Remark 18. Let W be a TCS on a T_0-quasi-metric space (X,d) having properties (C) and (S'). Then it satisfies property (S).

Let (X,d) be a T_0-quasi-metric space. Furthermore let $\mathcal{P}_0(X)$ be the set of all nonempty subsets of X. Equip $\mathcal{P}_0(X)$ with the Hausdorff quasi-pseudometric d_H, which is defined as follows:

As usual, for any nonempty subset $A \subseteq X$ and $x \in X$ we set $d(A,x) = \inf\{d(a,x) : a \in A\}$ and $d(x,A) = \inf\{d(x,a) : a \in A\}$. For any $A,B \in \mathcal{P}_0(X)$ set $d_{H^-}(A,B) = \sup_{a\in A} d(a,B)$ and $d_{H^+}(A,B) = \sup_{b\in B} d(A,b)$. Furthermore set $d_H = d_{H^+} \vee d_{H^-}$. It is well known that d_H is an (extended) quasi-pseudometric on the set $\mathcal{P}_0(X)$ of nonempty subsets of X. (Here *extended* means that d_H may attain the value ∞, where the triangle inequality is interpreted in the obvious way.) The reader is referred to [5, 12] for further information on this hyperspace construction.

Now we shall assume that the T_0-quasi-metric space (X,d) is equipped with a TCS W that satisfies condition (S). Furthermore we shall work on the subcollection $\mathcal{CB}_0(X)$ of bounded convex elements of $\mathcal{P}_0(X)$. In this case d_H is indeed a quasi-pseudometric: One proves that $d_H(A,B) < \infty$ if $A,B \subseteq X$ are bounded, that is, the diameters $\delta(A)$ and $\delta(B)$ are less than infinity.

For any $A,B \in \mathcal{CB}_0(X)$ and $\lambda \in I$ set $W(A,B,\lambda) = \{W(a,b,\lambda) : a \in A, b \in B\}$. One observes that the (nonempty) set $W(A,B,\lambda)$ is bounded by condition (S).

Suppose that the TCS W satisfies the condition that for any $a,a',b,b' \in X$ and any $\delta,\epsilon \in I$ there are $\epsilon_1,\epsilon_2 \in I$ such that

$$W(W(a,b,\delta), W(a',b',\delta),\epsilon) = W(W(a,a',\epsilon_1), W(b,b',\epsilon_2),\delta).$$

We shall call this condition of a TCS W "condition $(*)$". If W is a TCS satisfying condition $(*)$ in a T_0-quasi-metric space (X,d), then obviously for any $\delta \in I$ and

$A, B \in \mathcal{CB}_0(X)$ the set $W(A, B, \delta)$ is also convex because of the convexity of A and B.

The condition $(*)$ introduced above is satisfied by the standard linear convexity structure $S(x, y, \lambda) = \lambda x + (1 - \lambda)y$ discussed above of an asymmetrically normed real vector space X.

We say that a TCS W on a T_0-quasi-metric space (X, d) has property (M) if $W(W(x, y, \alpha), z, \beta) = W(W(y, z, \frac{\beta(1-\alpha)}{1-\alpha\beta}), x, 1 - \alpha\beta)$ whenever $x, y, z \in X$ and $\alpha, \beta \in I$ with $\alpha\beta \neq 1$. (Note that the latter equation for points also holds in the case $\alpha\beta = 1$, that is, $\alpha = 1 = \beta$, if it is interpreted in the obvious sense that $x = x$, although the fraction remains undefined in this case.)

Each convex subset C of an asymmetrically normed real vector space X with its standard convexity structure satisfies property (M).

Lemma 19. *(a) Condition (M) in a T_0-quasi-metric space implies condition (C) and condition (J).*

(b) Let (X, d) be a T_0-quasi-metric space equipped with a TCS W that satisfies condition (M). Then W satisfies condition $(*)$ introduced above.

Let (X, d) be a T_0-quasi-metric space. Given a subset A of X we shall call $\mathrm{cl}_{\tau(d)}A \cap \mathrm{cl}_{\tau(d^{-1})}A$ the *double closure* of A. Furthermore we shall say $A \subseteq X$ is *doubly closed* if it coincides with its double closure.

Let (X, d) be a T_0-quasi-metric space with a TCS W satisfying condition (S) and let $\mathcal{DCB}_0(X)$ be the set of all nonempty doubly closed convex bounded subsets of X. If $A \in \mathcal{CB}_0(X)$, then its double closure $\mathrm{cl}_{\tau(d)}A \cap \mathrm{cl}_{\tau(d^{-1})}A$ also belongs to $\mathcal{CB}_0(X)$. Furthermore the T_0-quotient of the quasi-pseudometric space $(\mathcal{CB}_0(X), d_H)$ can be identified with its subspace $(\mathcal{DCB}_0(X), d_H)$.

Proposition 20. *Let W be a TCS on a T_0-quasi-metric space (X, d) satisfying conditions (S) and (M). For any $A, B \in \mathcal{DCB}_0(X)$ and $\lambda \in I$ set*

$$P(A, B, \lambda) = cl_{\tau(d)}W(A, B, \lambda) \cap cl_{\tau(d^{-1})}W(A, B, \lambda).$$

Then P is a TCS on the T_0-quasi-metric space $(\mathcal{DCB}_0(X), d_H)$. (Identifying the points of X with singleton subsets we can interpret P as an extension of W, since for each $a, b \in X$ and $\lambda \in I$ we have $P(\{a\}, \{b\}, \lambda) = \{W(a, b, \lambda)\}$.)

5. Convexity structures in q-hyperconvex spaces

Our presentation in this section will follow the construction of the q-hyperconvex hull of a T_0-quasi-metric space given in [10].

Let (X, d) be a T_0-quasi-metric space. A function pair $f = (f_1, f_2)$ on (X, d) where $f_i : X \to [0, \infty)$ $(i = 1, 2)$ is called *ample* provided that $d(x, y) \leq f_2(x) + f_1(y)$ whenever $x, y \in X$. Let P_X denote the set of all ample function pairs on (X, d). For each $f, g \in P_X$ we set

$$D(f, g) = \sup_{x \in X}(f_1(x) \dot{-} g_1(x)) \vee \sup_{x \in X}(g_2(x) \dot{-} f_2(x)).$$

Then D is an extended quasi-pseudometric on P_X.

We shall call a function pair f *minimal* on (X, d) (among the ample function pairs on (X, d)) if it is ample and whenever g is ample on (X, d) and for each $x \in X$ we have $g_1(x) \leq f_1(x)$ and $g_2(x) \leq f_2(x)$ (if this holds we briefly write $g \leq f$), then $g = f$. Zorn's Lemma implies that for each ample function pair f there exists a (in general not unique) minimal ample pair g on (X, d) such that $g \leq f$. By Q_X we shall denote the set of all minimal ample function pairs on (X, d) equipped with the restriction of D to $Q_X \times Q_X$, which we shall also denote by D. Then D is a (real-valued) T_0-quasi-metric on $Q_X \times Q_X$.

For each $x \in X$ we can define the minimal function pair

$$f_x(y) = (d(x, y), d(y, x))$$

(whenever $y \in X$) on (X, d). The map e defined by $x \mapsto f_x$ whenever $x \in X$ defines an isometric embedding of (X, d) into (Q_X, D). Then (Q_X, D) is called the *q-hyperconvex hull* of (X, d).

We have $f = (f_1, f_2) \in Q_X$ if and only if the following equations are satisfied:

$$f_1(x) = \sup\{d(y, x) \dot{-} f_2(y) : y \in X\}$$

and

$$f_2(x) = \sup\{d(x,y) \dot{-} f_1(y) : y \in X\}$$

whenever $x \in X$. In particular pairs satisfying these equations are ample on (X,d).

Some important properties of functions pairs in Q_X are listed next:

(a) $f = (f_1, f_2) \in Q_X$ implies that $f_1(x) - f_1(y) \leq d^{-1}(x,y)$ and $f_2(x) - f_2(y) \leq d(x,y)$ whenever $x, y \in X$.

(b) $\sup_{x\in X}(f_1(x) \dot{-} g_1(x)) = \sup_{x\in X}(g_2(x) \dot{-} f_2(x))$ whenever $f = (f_1, f_2)$ and $g = (g_1, g_2)$ belong to Q_X.

(c) $D(f, f_x) = f_1(x)$ and $D(f_x, f) = f_2(x)$ whenever $x \in X$ and $f = (f_1, f_2) \in Q_X$.

A T_0-quasi-metric space (X,d) is called *q-hyperconvex* provided that for each $f \in Q_X$ there is $x \in X$ such that $f = f_x$. Equivalently, for each family $(x_i)_{i\in I}$ of points in X and families of nonnegative real numbers $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ the following condition holds: If $d(x_i, x_j) \leq r_i + s_j$ whenever $i, j \in I$ then $\bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)) \neq \emptyset$.

Let $f, g \in X$ and $\lambda \in I$. For any $u \in X$ let $r_2(f, g, \lambda)(u) = \lambda d(u, f) + (1-\lambda)d(u, g)$ and $r_1(f, g, \lambda)(u) = \lambda d(f, u) + (1-\lambda)d(g, u)$. Then each function pair $r(f, g, \lambda) = (r_1(f, g, \lambda), r_2(f, g, \lambda))$ is ample on (X,d).

The following result was established in [2] (for the metric case, see [7]):

Let (X,d) be a T_0-quasi-metric space. There exists a retraction map $p : P_X \to Q_X$ that satisfies the conditions

(a) $D(p(f), p(g)) \leq D(f, g)$ whenever $f, g \in P_X$.

(b) $p(f) \leq f$ whenever $f \in P_X$.

Using the result above we can make the choice of minimal pairs below ample pairs in such a way that condition (a) is satisfied, which allows us to establish various properties of constructed convexity structures analogous to those obtained for hyperconvex metric spaces (compare [3]).

Indeed let (X, d) be a q-hyperconvex T_0-quasi-metric space. (Hence we can identify (X, d) with (Q_X, D) via the isometric bijection $x \mapsto f_x$.) Take any retraction $p : P_X \to Q_X$ as described in the previous result.

We define a TCS W on (X, d) as follows: Fix $f, g \in X$ and $t \in I$. Now we set $W(f, g, t) \in X$ equal to the unique point in X that satisfies $e(W(f, g, t)) = w(f, g, t) := p(te(f) + (1 - t)e(g))$. The so defined W is a TCS on (X, d) that evidently has property (C), that is, $W(y, x, 1-\lambda) = W(x, y, \lambda)$ whenever $x, y \in X$ and $\lambda \in I$. One computes that W as a TCS satisfies $d(W(f, g, t), W(f, g, s)) = (t-s)d(f, g)$ if $t \geq s$ and $d(W(f, g, t), W(f, g, s)) = (s-t)d(g, f)$ if $t < s$. Hence we conclude that W satisfies property (I'). One also checks that W satisfies condition (S). It follows that the map $W : (X, \tau(d)) \times (X, \tau(d)) \times (I, \tau(u^s)) \to (X, \tau(d))$ is continuous.

References

[1] C. A. Agyingi, P. Haihambo and H.-P. A. Künzi, Endpoints in T_0-quasi-metric spaces, Topology Appl. 168 (2014), 82–93.

[2] C. A. Agyingi, P. Haihambo and H.-P. A. Künzi, Tight extensions of T_0-quasi-metric spaces, in: V. Brattka, H. Diener, D. Spreen (Eds.), Logic, Computation, Hierarchies, Festschrift in Honour of V.L. Selivanov's 60th Birthday, Ontos Verlag, De Gruyter Berlin, Boston, 2014, pp. 9–22.

[3] M. A. Alghamdi, W. A. Kirk and N. Shahzad, Locally nonexpansive mappings in geodesic and length spaces, Topology Appl. 173 (2014), 59–73.

[4] K. Aoyama, K. Eshita and W. Takahashi, Iteration processes for nonexpansive mappings in convex metric spaces, Proc. of the International Conference on Nonlinear Analysis and Convex Analysis (Okinawa, 2005), 31–39.

[5] G. Berthiaume, On quasi-uniformities in hyperspaces, Proc. Amer. Math. Soc. 66 (1977), 335–343.

[6] S. Cobzaş, Functional Analysis in Asymmetric Normed Spaces, Birkhäuser, Basel, 2013.

[7] A. W. M. Dress, Trees, tight extensions of metric spaces, and the cohomological dimension of certain groups: a note on combinatorial properties of metric spaces, Adv. Math. 53 (1984), 321–402.

[8] P. Fletcher and W.F. Lindgren, Quasi-uniform Spaces, Dekker, New York, 1982.

[9] M. Izadi, A fixed point theorem on quasi-metric spaces of hyperbolic type, Int. J. Contemp. Math. Sciences 7, no. 44 (2012), 2155–2160.

[10] E. Kemajou, H.-P.A. Künzi and O. O. Otafudu, The Isbell-hull of a di-space, Topology Appl. 159 (2012), 2463–2475.

[11] H.-P. A. Künzi, An introduction to quasi-uniform spaces, in: Beyond Topology, eds. F. Mynard and E. Pearl, Contemp. Math., Amer. Math. Soc. 486 (2009), pp. 239–304.

[12] H.-P. A. Künzi and C. Ryser, The Bourbaki quasi-uniformity, Topology Proc. 20 (1995), 161–183.

[13] H.-P. A. Künzi and F. Yıldız, Convexity structures in T_0-quasi-metric spaces, Topology Appl. 200 (2016), 2–18.

[14] H. V. Machado, A characterization of convex subsets of normed spaces, Kōdai Math. Sem. Rep. 25 (1973), 307–320.

[15] W. Takahashi, A convexity in metric space and nonexpansive mappings, I, Kōdai Math. Sem. Rep. 22 (1970), 142–149.

[16] L. A. Talman, Fixed points for condensing multifunctions in metric spaces with convex structure, Kōdai Math. Sem. Rep. 29 (1977), 62–70.

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Fuzzy metrics for colour image similarity

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Abstract

In this paper we propose an application of fuzzy metrics to the problem of measuring the similarity between two colour images. This problem is of paramount importance in many applications of the computer vision field. The commonly used pixelwise similarity measures such as Mean Absolute Error, Peak Signal to Noise Ratio, Mean Squared Error or Normalized Color Difference do not match well with perceptual similarity. From another point of view, we propose a method using fuzzy metrics between small image patches of the images. Experimental results employing a survey of observations show that the global performance of our proposal is competitive with best state of the art methods and that it shows some advantages in performance for images with low correlation among some image channels.

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1. Introduction

Many applications in the fields of image processing and computer vision use image similarity measures for different purposes [2]. In some cases the objective is the very measurement of the similarity itself globally or partially in the images, but other times the similarity is used to assess the performance of an image processing method. For instance, in image filtering, the common process to measure the performance of a filtering method is the following: an original image is corrupted artificially with noise, then it is filtered with the method under study and it is measured how similar is the filtered image to the original one. This allows to properly adjust filter parameters for optimal performance, to assess different filter configurations as well as to compare the performance of different filtering methods. An analogous approach is used in other image processing procedures such as image compression, image demosaicing or video de-interlacing. Therefore, the similarity measure used highly influences the whole process.

The most common similarity measures used in this context are based on a pixelwise approach, such as the Mean Absolute Error (MAE), the Mean Squared Error (MSE), the Peak Signal to Noise Ratio (PSNR) or the Normalized Color Difference (NCD) (which is the MSE in the Lab color space). However, these measures do not match well with perceptual observations and, as the MSE, some of them have other concerns [3].

Recently, in [4, 5] a similarity measure for gray-scale images that matches well with perceptual similarity has been introduced (UQI-Universal Quality Index and SSIM-Single-scale Structural Similarity Index). This method could be applied in color images in a componentwise fashion, that is, independently in each color channel and then averaged. However, it is well-known that the correlation among the color image channels should be taken into account and this approach cannot provide optimal performance [2], as we show in this paper.

In this paper, we introduce a method for color image similarity that matches perceptual similarity. Our method follows a procedure inspired in [4, 5] as follows: the images are processed with sliding patches so that a number of small image portions are compared and the similarity between two images is obtained by averaging the similarities of all portions. In each pair of patches three different factors are compared separately and then combined: contrast, structure and luminance. The particular expressions used in [4, 5] for these three factors cannot be directly generalized from gray-scale images to color images, so we propose our own expressions to measure them. Experimental results employing perceptual similarity observations show that our approach is able to outperform classical similarity measures, is competitive with best state-of-the-art methods, and shows some advantages in performance for images with low correlation among some image channels.

In the following section we detail the proposed method. Section 3 contains the experimental results and discussion. Finally, Section 4 presents the conclusions.

2. Proposed image similarity measure

Let $\mathbf{X}$ denote a RGB image and W be the sliding patch of finite size $q \times q = n$ used to process the image. The image pixels in W, $\mathbf{X}_W$, are denoted as $\mathbf{x}_i(l), i = 1, ..., n$ where $l = 1, 2, 3$ denotes the R, G, and B channels, respectively. Notice that $\mathbf{x}_i$ can be processed as a three component vector.

We measure the similarity between images $\mathbf{X}$ and $\mathbf{Y}$ as the average of the similarities of the image patches $\mathbf{X}_W$ and $\mathbf{Y}_W$ obtained when sliding the patch along every image row. To measure the similarity between two patches in the same image location we measure three different similarities: contrast, structure and luminance. In so doing, we need to measure the similarities between all image color pixels $\mathbf{x}_i$ and $\mathbf{y}_i$ in $\mathbf{X}_W$ and $\mathbf{Y}_W$, respectively, and the mean color vector in each patch, $\overline{\mathbf{x}_W}$ and $\overline{\mathbf{y}_W}$. We denote these similarities by $M_{\mathbf{x}_i}$ and $M_{\mathbf{y}_i}$ and we measure them by employing the fuzzy metric used in [7, 8, 9, 10] for its high sensitivity to edges as follows.

$$M_{\mathbf{x}_i} = M(\mathbf{x}_i, \overline{\mathbf{x}_W}, t) = \prod_{l=1}^{3} \frac{\min(x_i(l), \overline{\mathbf{x}_W}(l)) + t}{\max(x_i(l), \overline{\mathbf{x}_W}(l)) + t}, \quad i = 1, ..., n, \tag{1}$$

where $t > 0$ and

$$\overline{\mathbf{x}_W} = \frac{1}{n}\sum_{j=1}^{n} x_j, \quad l = 1, 2, 3 \tag{2}$$

Through an analogous computation in the image $\mathbf{Y}$ we obtained the similarities $M_{\mathbf{y}_i}, i = 1, \ldots, n$. Notice that $M_{\mathbf{x}_i}$ and $M_{\mathbf{y}_i}$ are fuzzy similarities that take value in $[0, 1]$.

2.1. **Contrast.** Contrast can be seen as the largest difference observed in $\mathbf{X}_W$ and $\mathbf{Y}_W$. We can measure contrast in $\mathbf{X}_W$ using $M_{\mathbf{x}_i}$ as $C_{\mathbf{X}_W} = \max(M_{\mathbf{x}_i}) - \min(M_{\mathbf{x}_i}), \quad i = 1, ..., n$, and analogously for $\mathbf{Y}_W$. Then, the fuzzy similarity between the contrasts is given by

$$SC(\mathbf{X}_W, \mathbf{Y}_W) = 1 - |C_{\mathbf{X}_W} - C_{\mathbf{Y}_W}|. \tag{3}$$

2.2. **Structure.** Structure describes how the differences between the pixels in a patch are distributed spatially. Therefore, for this aspect we average the fuzzy similarities of $M_{\mathbf{x}_i}$ and $M_{\mathbf{y}_i}$ as follows.

$$SS(\mathbf{X}_W, \mathbf{Y}_W) = \frac{\sum_{i=1}^{n} 1 - |M_{\mathbf{x}_i} - M_{\mathbf{y}_i}|}{n}. \tag{4}$$

2.3. **Luminance.** To compare image luminance we propose to use spherical coordinates computed from RGB values [11]. Luminance correspond with the radius parameter given by

$$L\mathbf{x}_i = \sqrt{\mathbf{x}_i(1)^2 + \mathbf{x}_i(2)^2 + \mathbf{x}_i(3)^2} \tag{5}$$

The luminance similarity between $\mathbf{X}_W$ and $\mathbf{Y}_W$ is obtained through the corresponding expression in [4] as

$$(6) \qquad SL(\mathbf{X}_W, \mathbf{Y}_W) = \frac{2\overline{L_{\mathbf{X}_W}L_{\mathbf{Y}_W}}}{\overline{L_{\mathbf{X}_W}}^2 + \overline{L_{\mathbf{Y}_W}}^2}$$

where $\overline{L_{\mathbf{X}_W}}$ and $\overline{L_{\mathbf{Y}_W}}$ are the mean luminance in each patch. In the case that $\overline{L_{\mathbf{X}_W}} = \overline{L_{\mathbf{Y}_W}} = 0$ we assign $SL(\mathbf{X}_W, \mathbf{Y}_W) = 1$.

Finally, the similarity between $\mathbf{X}_W$ and $\mathbf{Y}_W$ results from combining the three previous measures as follows

$$(7) \quad S(\mathbf{X}_W, \mathbf{Y}_W) = SC(\mathbf{X}_W, \mathbf{Y}_W)^\alpha \cdot SS(\mathbf{X}_W, \mathbf{Y}_W)^\beta \cdot SL(\mathbf{X}_W, \mathbf{Y}_W)^\gamma$$

where $\alpha, \beta, \gamma > 0$ are parameters used to adjust relative importance of three components. As commented above, the average of all $S(\mathbf{X}_W, \mathbf{Y}_W)$ provides the similarity between $\mathbf{X}$ and $\mathbf{Y}$, that will be high only if the three similarities are high.

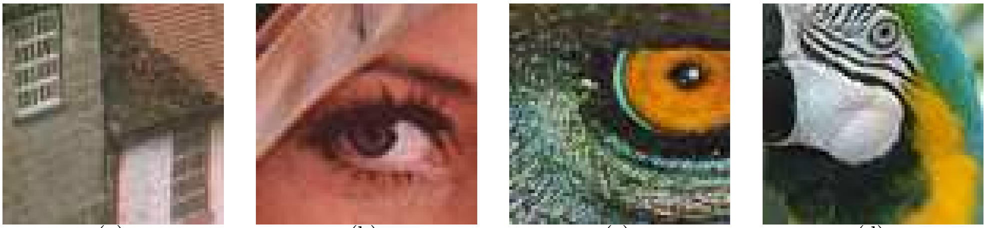
Finally, we would like to point out that in each processing patch the number of operations is proportional to the number of pixels, so for the whole method we have also a linear computational cost.

3. Experimental study

In order to study the performance of our proposal and also to compare with other approaches we make a comparison with respect to a survey of perceptual observations as follows.

We have chosen the four color bmp images in Figure 3: Goldhill, Lenna, Baboon, and Parrots. To better appreciate low resolution differences we have taken a small part of 68x68 pixels of the original images. We have applied a series of 10 different distortions to each of the test images. The distortions applied over the image Parrots along with the software use in each case, which are shown in Figure 4, are the following.

(1) jpg compression of ratio 20% (MS Picture Manager)
(2) Increase brightness by 15% (MS Picture Manager)



(a) (b) (c) (d)

Figure 3. Images for tests: (a) Goldhill, (b) Lenna, (c) Baboon, and (d) Parrots.

(3) Increase contrast by 15% (MS Picture Manager)
(4) Gaussian blur with radius 1.5 (Corel Draw X5)
(5) Addition of 5% of impulsive noise (imnoise function from Matlab)
(6) Addition of white Gaussian noise with standard deviation equals to 10% of the maximum value in the channels (imnoise function from Matlab)
(7) Filtering of original image with [12]
(8) Addition of Gaussian noise as in 6) and filtering with [12]
(9) Filtering of original image with Vector Median Filter (VMF) [13]
(10) Addition of 5% of impulsive noise as in 5) and filtering with Vector Median Filter (VMF) [13]

In the survey, we asked independent observers to rank the 10 distorted images with respect to its similarity to the original image (1*st* the most similar, 10*th* the least). We did this through a questionnaire available on the internet to get as many answers as possible. We received 108 complete answers. We processed them to remove outliers using boxplot and we found 4 outliers that could be due to the observer not paying enough attention or to wrong understanding. Finally, we average the ranks obtained by each of the distorted images and we re-scale the average rankings to the interval [1, 10].

Next, we measure the similarity between all distorted images and the original one with the usual similarity measures MAE, MSE, NCD, as well as with Structural Similarity Index (SSIM) [4, 5] (used by averaging after component-wise application in each channel), FSIMc [14], CMSSIM [6] and the proposed method (Fuzzy Color

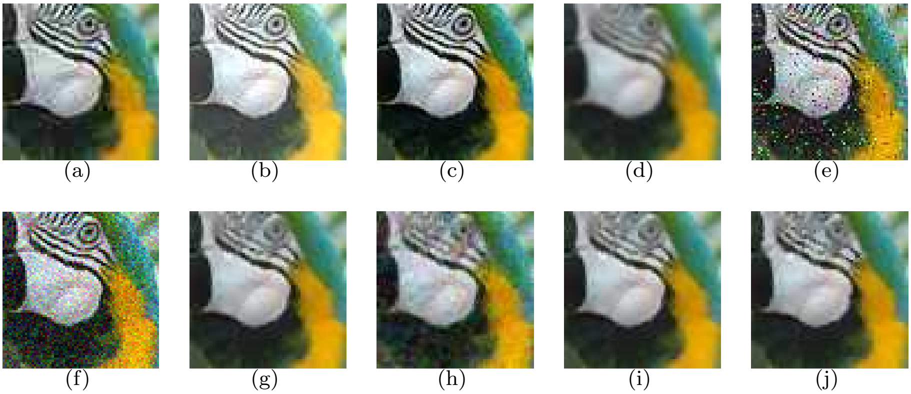


FIGURE 4. Distortions applied to the image Parrots: (a) jpg compression of ratio 20% (MS Picture Manager), (b) Increase brightness by 15% (MS Picture Manager), (c) Increase contrast by 15% (MS Picture Manager), (d) Gaussian blur with radius 1.5 (Corel Draw X5), (e) Addition of 5% of impulsive noise (Matlab according to [2]), (f) Addition of white Gaussian noise with standard deviation equals to 10% of the maximum value in the channels (Matlab according to [2]), (g) Filtering of original image with [12], (h) Addition of Gaussian noise as in (f) and filtering with [12], (i) Filtering of original image with Vector Median Filter (VMF) [13], (j) Addition of 5% of impulsive noise as in (e) and filtering with Vector Median Filter (VMF) [13].

Structural Similarity, FCSS). To assess the match between these measures and the survey perceptual observations, we re-scaled similarity measures results to the interval $[1, 10]$. In this way we can measure the similarity between each measure ranking and the perceptual ranking.

For our proposal we try different parameter settings and one providing a nice overall performance is the following: $t = 256$, patch size $q = 4$ and $\alpha = \beta = \gamma = 1$.

Table 1 show the comparison in terms of the correlation coefficient r between average observer ranking and each measure ranking.

Table 1. Performance comparison in terms of correlation coefficient r between average observer ranking and each measure ranking

Image	MAE	MSE	NCD	SSIM	CMSSIM	FSIM	FCSS
GoldHill	-0.11	0.12	0.57	0.96	0.80	0.96	0.87
Lenna	0.16	0.38	0.64	0.93	0.39	0.93	0.85
Baboon	-0.27	0.22	0.21	0.78	0.35	0.78	0.86
Parrots	-0.29	0.12	0.13	0.74	0.22	0.80	0.80

From these results we can see that performance of SSIM, FSIMc and FCSS is much better than the rest of the methods. CMSSIM only works well for Goldhill image, which suggests that it is too sensitive to the image features. SSIM exhibits a very high performance ($r > 0.9$) in two cases (Goldhill and Lenna) but much lower ($r < 0.8$) in another two cases (Baboon and Parrots). FSIMc performs very well for GoldHill and Lenna ($r > 0.9$), well for Parrots ($r \sim 0.8$), but worse for Baboon image, where its performance drops with respect to FCSS ($r < 0.8$). On the other hand, FCSS exhibits a consistent high performance in all cases ($r \in [0.80, 0.90]$) and it is better than SSIM for Baboon and Parrots images and better than FSIMc for Baboon image.

In order to understand these pretty high differences in the performance of SSIM and FSIMc for different images we analyzed several features of the images and we realized that there is significant differences with respect to their correlations among the image channels. These correlations are shown in Table 2. We see that correlations in Goldhill and Lenna images are high in all cases, whereas in Parrots and Baboon appear some medium and low correlations respectively. This implies that SSIM is only able to provide high performance when the correlation among the color channels is high in all cases. However, when for a couple of channels the correlation is not high, SSIM performs worse. This is most probably due to the component-wise application of SSIM. FSIMc performs better from this point of view and still performs well in the presence of some medium correlations (Parrots), but its performance drops for the Baboon image were the correlation between the R and B channels is very low and the rest are not high. We see that

FSIMc is sensitive to low correlations between channels which probably means that its capability to take into account correlation can be improved. On the other hand, FCSS performance is independent from the correlation among the image channels which in turns indicates proper correlation management. This is interesting for practical applications and also for possible adaptations to other types of multichannel images and future research.

TABLE 2. Correlation in image channels

Channels	Goldhill	Lenna	Baboon	Parrots
RG	0.92	0.89	0.69	0.9
RB	0.89	0.78	0.1	0.5
GB	0.97	0.96	0.7	0.75

4. Conclusions

In this paper we have proposed a method to measure the similarity between two color images that uses fuzzy metrics. The similarity between the images takes into account three factors: structural similarity, contrast similarity, and luminance similarity. The method takes into account the correlation among the image channels by processing the images as vector fields. Experimental results employing a survey of observations show that the global performance of our proposal is competitive with best state of the art methods and that it shows some advantages in performance for images with low correlation among some image channels, which is interesting for future research.

References

[1] R. Engelking, General Topology, Sigma Series in Pure Mathematics **6**, Heldermann Verlag Berlin, 1989.

[2] K. N. Plataniotis, A. N. Venetsanopoulos, Color Image Processing and Applications, Springer-Verlag (2000), pp. 1–45, 51–100, 109–157.

[3] Z. Wang, A. C. Bovik, Mean Squared Error: Love It or Leave It?, IEEE Signal Processing Magazine 26, no. 1 (2009), 98–117.

[4] Z. Wang, A. C. Bovik, A Universal Image Quality Index, IEEE Signal Processing Letters 9 (2002), 81–84.

[5] Z. Wang, A. C. Bovik, H. R. Sheikh, E. P. Simoncelli, Image Quality Assessment: From Error Visibility to Structural Similarity, IEEE Transactions on Image Processing 13 (2004), 600–612.

[6] M. Hassan, C. Bhagvati, Structural Similarity Measure for Color Images, International Journal of Computer Applications 43 (2012), 7–12.

[7] S. Morillas, V. Gregori, G. Peris-Fajarnés, P. Latorre, A fast impulsive noise color image filter using fuzzy metrics, Real-Time Imaging 11 (2005), 417–428.

[8] S. Morillas, V. Gregori, G. Peris-Fajarnés, P. Latorre, A new vector median filter based on fuzzy metrics, Lecture Notes in Computer Science 3656 (2005), 81–90.

[9] S. Morillas, V. Gregori, A. Sapena, Fuzzy bilateral filtering for color images, Lecture Notes in Computer Science 4141 (2006), 138–145.

[10] J. G. Camarena, V. Gregori, S. Morillas, A. Sapena, Two-step fuzzy logic-based method for impulse noise detection in color images, Pattern Recognition Letters 31 (2010), 1842–1849.

[11] E.S. Hore, B. Qiu, H.R. Wu, Noise estimation in spherical coordinates for color image restoration, Optical Engineering 44(4)(2005).

[12] S. Morillas, V. Gregori, A. Hervás, Fuzzy peer groups for reducing mixed gaussian-impulse noise from color images, IEEE Trans. Image Processing 18 (2009), 1452–1466.

[13] J. Astola, P. Haavisto, Y. Neuvo, Vector Median Filters, Proceedings of the IEEE 78 (1990), 678–689.

[14] L. Zhang, L. Zhang, X. Mou, D. Zhang, FSIM: A Feature Similarity Index for Image Quality Assessment, IEEE Trans. Image Processing 20 (2011), 2378–2386.

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Fuzzy metrics for switching filters

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Abstract

In this talk, we revise some aspects of the using of metrics and fuzzy metrics for colour image filtering.

1. Colour image filtering

Colour image processing has received so much attention in the recent years. Early approaches process colour images by straightforwardly applying gray-scale methods to each colour channel. However, many deficiencies arise from this way of processing mainly due to the fact that inter-channel interactions are not taken into account. As a result, a widely studied solution is to process the colour (or multichannel) images in a vector fashion [8]. One of the most well-known filters of this family which are based on vector ordering is the *vector median filter* (VMF) [1] where the colour vectors are ranked using the reduced ordering principle by means

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of a suitable distance or similarity measure. The lowest ranked vectors are those which are close to all the other vectors in the window according to the distance or similarity measure used. On the other hand, atypical vectors, susceptible to be considered as noisy or outliers, occupy the highest ranks. The output of these filters is defined as the lowest ranked vector as follows.

Let $\mathbf{F}$ represent a multichannel image and let W be a window of size $n+1$ (filter length). The image vectors in the filtering window W are denoted as $\mathbf{F}_j,\ j = 0, 1, \ldots, n$. The distance between two vectors $\mathbf{F}_k$ and $\mathbf{F}_j$ is denoted as $\rho(\mathbf{F}_k, \mathbf{F}_j)$, where ρ is a chosen metric. For each vector in the filtering window, a global or accumulated distance to all the other vectors in the window has to be calculated. The scalar quantity $R_k = \sum_{j=0, j\neq k}^{n} \rho(\mathbf{F}_k, \mathbf{F}_j)$, is the accumulated distance associated to the vector $\mathbf{F}_k$. The ordering of the R_k's: $R_{(0)}, R_{(1)}, \ldots, R_{(n)}$, implies the same ordering of the vectors $\mathbf{F}_k$'s: $\mathbf{F}_{(0)}, \mathbf{F}_{(1)}, \ldots, \mathbf{F}(n)$. Given this order, the output of the filter is $\mathbf{F}_{(0)}$.

On the other hand, vector median type techniques that output some of the input vectors are not useful for Gaussian noise smoothing since all the input vectors usually contain some noise. Indeed, thanks to the commonly assumed zero-mean property of Gaussian noise this noise can be smoothed out by locally averaging pixel values. Classical linear filters, such as the *arithmetic mean filter* (AMF) smooth noise but blur edges significantly.

Since then, there have been proposed a series of nonlinear filters to approach this problem with the main idea to use local measures to detect edges and smooth them less than the rest of the image to better preserve their sharpness. This mechanism works well to reduce gaussian noise without blurring edges however these methods would detect impulsive noises as edges to be preserved so, these vector filters are very efficient in reducing impulsive noise but they mostly lack of good signal-preserving ability because the filtering operation is applied to each image pixel regardless whether it is noisy or not.

In order to address this drawback, several approaches have been introduced by using different criteria to preserve the original signal structures, such as edges and fine details. Most of these filters work in two steps: a first step to classify whether a pixel is corrupted or not (detection step) and a second step to replace only the pixels detected as corrupted. These methods have been called *switching filters*.

In particular, the notion of *peer group* is used in the switching filters to detect the noise-likely pixels [2, 3]. This method consists on the construction of the set of the pixels in a filtering window which are close to the central one. The central pixel will be classified as corrupted if its peer group is *small* and it will be classified as noise free if its peer group is *great enough* by means of a parameter that determines whether the peer group is small or not.

Another notion that has been used in the recent literature is the Rank-Ordered Differences (ROD) statistic [4]. In this case, the detection process consists on the calculation of the sum of the lower m distances between the central pixel in the filtering window and the rest of pixels. The ROD statistic provides a measure of how close a pixel is to its m most similar neighbours in W attending to their RGB colour vectors. The logic underlaying of the statistic is that unwanted impulses will vary greatly in intensity in one or more colours from most of their neighbouring pixels, whereas most pixels composing the actual image should have at least some of their neighbouring pixels of similar intensity in each colour, even pixels on an edge in the image. Thus noise-free pixels have a significantly lower ROD value than corrupted pixels.

As a consequence of the ROD statistic notion, a new idea has been implemented in a recent work. In front of the calculation of the m closest distances, one can consider only the m-th closest distance to decide whether a pixel is noise-free or not. In this case, it is not important to consider how close the pixels are to the central one except the m-th.

2. The using of fuzzy metrics

As commented before, to measure similarity (or difference) between pixels one can use the so called metrics. Since early filtering techniques, classical metrics have been implemented and the more used ones are the well-known L_1, L_2 and L_∞, although any Minkowski type metric could be used to measure it. In particular, the L_1 distance measure takes into account the noise appearance in all the vector components but L_∞ takes into account the noise appearance in only one component. With respect to the distance L_2 it works as the L_1 but its efficiency is computationally higher.

Nevertheless, recently fuzzy metrics have been introduced to extend the classical notions to the fuzzy setting.

Definition 1 ([5])**.** A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (nonempty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times \mathbb{R}^+$ satisfying the following conditions, for all $x, y, z \in X,\ s, t > 0$:

(GV1) $M(x, y, t) > 0$
(GV2) $M(x, y, t) = 1$ if and only if $x = y$
(GV3) $M(x, y, t) = M(y, x, t)$
(GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$
(GV5) $M(x, y, \_) : \mathbb{R}^+ \to]0, 1]$ is continuous

If $(X, M, *)$ is a fuzzy metric space we say that $(M, *)$, or simply M, is a fuzzy metric on X. Also, we say that (X, M) or, simply, X is a fuzzy metric space. The value $M(x, y, t)$ represents the degree of nearness between x and y (with respect to the parameter t) and according to (GV2) $M(x, y, t)$ is close to 0 when x is far from y.

The three most commonly used continuous t-norms in fuzzy logic are the minimum, denoted by $\wedge$, the usual product, denoted by $\cdot$ and the Lukasievicz t-norm, denoted by $LUC(G)$, $(xLUC(G)y = \max\{0, x + y - 1\})$. They satisfy the inequality $xLUC(G)y \leq x \cdot y \leq x \wedge y$ and for each (continuous) t-norm $*$ it is satisfied $x * y \leq x \wedge y$.

Definition 2 ([6])**.** A fuzzy metric M on X is said to be *stationary* if M does not depend on t, i.e. if for each $x, y \in X$, the function $M_{x,y}(t) = M(x, y, t)$ is constant. In this case we write $M(x, y)$ instead of $M(x, y, t)$,

The interest of fuzzy metrics is mainly due to the following two main advantages with respect to classical metrics: First, values given by fuzzy metrics are in the interval]0,1] regardless the nature of the distance concept being measured. This implies that it is easy to combine different distance criteria that may originally be in quite different ranges but fuzzy metrics take to a common range. In this way, the combination of several distance criteria may be done in a straightforward way. Second, fuzzy metrics match perfectly with the employment of other fuzzy techniques since the value given by a fuzzy metric can be directly employed or interpreted as a fuzzy certainty degree. This allows to straightforwardly include fuzzy metrics as part of other complex fuzzy systems.

Notice that a metric space (X, d) can be normalized to take values in the range [0,1]. It would be enough to take the metric $d^* = \dfrac{d}{K+d}$ for a $K > 0$ but, in this case, to measure similarity one should consider the function $(1 - d^*)$ and it is easy to observe that it coincides with a stationary version of the well-known standard fuzzy metric for the usual product. Now, one can consider other fuzzy metric examples to measure similarity and not only refer to the standard fuzzy metric.

Actually, this kind of metrics have been implemented in several types of filtering processes with a better performance in the quality. Moreover, fuzzy metrics take values in the interval $]0, 1]$ and this is an advantage in front of classical metrics since one can mix several fuzzy metrics to generate a new one to measure several criteria simultaneously. In fact, it has been introduced in the literature a fuzzy metric to simultaneously measure colorimetric difference and spatial distance.

In particular, to determine colorimetric difference between two pixels it has been used the so called quotient fuzzy metric (for the usual product), defined by

$$M_q(\mathbf{x}, \mathbf{y}) = \prod_{i=1}^{3} \frac{\min\{x_i, y_i\} + K}{\max\{x_i, y_i\} + K}$$

where K is a parameter which can be adjusted attending to each image characteristics (in general, it is used the value $K = 1024$ to guarantee certain symmetry in the RGB domain).

To evaluate spatial distance between two pixel in the image matrix, the most used fuzzy metric is the so called standard fuzzy metric [5] defined by

$$M_d(\mathbf{x},\mathbf{y}) = \frac{K}{K + d(\mathbf{x},\mathbf{y})}$$

where d is a classical metric (usually some of the above commented).

So, attending to the fuzzy metrics properties, one can define, in this case, a new fuzzy metric by the expression

$$M(\mathbf{x},\mathbf{y}) = M_q(\mathbf{x},\mathbf{y}) \cdot M_d(\mathbf{x},\mathbf{y})$$

to evaluate, simultaneously, colorimetric difference and spatial distance.

The previous fact is a clear advantage in front of classical metrics. Nevertheless, the M_q fuzzy metric has the important handicap that it is so *drastic* when measuring similarity between two pixels since an extreme value in one only component will decrease significantly the corresponding final product and, in consequence, both pixels will considered very different.

3. A new similarity measure

In some cases colorimetric difference between two pixels is distorted because the triple product. For example, in a gray-scale system, the value 90 is close to 100 and the fuzzy distance is 0.9 (without the correction parameter). Nevertheless, in the three channel colour space RGB, the fuzzy distance between the pixels $(90, 90, 90)$ and $(100, 100, 100)$ decreases to 0.73.

In order to overcome the commented inconvenience, we introduce a new similarity measure that study in some sense similarity between two pixels regarding the M_d idea but not to be so drastic when considering the three components RGB.

So we consider the following function:

$$M_1(\mathbf{x},\mathbf{y}) = \frac{\sum_{i=1}^{3} \min\{x_i, y_i\}}{\sum_{i=1}^{3} \max\{x_i, y_i\}}$$

Notice that, in this case, the colorimetric difference between the previously considered pixels is $M_1((90,90,90),(100,100,100)) = 0.9$ that can be considered to better agree with the gray-scale case.

Proposition 3. *The similarity measure M_1 satisfies the following conditions:*

(i) $M_1(\mathbf{x},\mathbf{y}) \geq 0$
(ii) $M_1(\mathbf{x},\mathbf{y}) = M(\mathbf{y},\mathbf{x})$
(iii) $M_1(\mathbf{x},\mathbf{y}) = 1$ *if and only if* $\mathbf{x} = \mathbf{y}$

This function will compare similarity between pixels with a better sensibility and not to be so drastic. Nevertheless, it would be very interesting to prove that it satisfies the corresponding axioms to be a stationary fuzzy metric.

If we consider $*$ as the usual product we have proved that this measure does not satisfy axiom $(GV4)$. Indeed, consider the following colour pixels (in a two-space range): $\mathbf{x} = (1,6)$, $\mathbf{y} = (5,8)$ and $\mathbf{z} = (7,2)$. Then we have

$$M_1(\mathbf{x},\mathbf{z}) = \frac{1+2}{7+6} = \frac{3}{13}$$

$$M_1(\mathbf{x},\mathbf{y}) \cdot M_1(\mathbf{y},\mathbf{z}) = \frac{1+6}{5+8} \cdot \frac{5+2}{7+8} = \frac{7}{13} \cdot \frac{7}{15} = \frac{49}{195}$$

It remains an open question to verify if this measure satisfies axiom $(GV4)$ for $*$ as the Lukasievicz t-norm.

References

[1] J. Astola, P. Haavisto, Y. Neuvo, Vector Median Filters, In: Proc. IEEE 78, no. 4 (1990), 678–689.

[2] B. Smolka, A. Chydzinski, Fast detection and impulsive noise removal in color images, Real-Time Imaging. 11, no. 5-6 (2005), 389–402.

[3] J. G. Camarena, V. Gregori, S. Morillas, A. Sapena, Fast detection and removal of impulsive noise using Peer Groups and Fuzzy Metrics, J. Vis. Commun. Image R. 19, no. 1 (2008), 20–29.

[4] R. Garnett, T. Huegerich, C. Chui, W. He, A universal noise removal algorithm with an impulse detector, IEEE Trans. Image Process. 14, no. 11 (2005), 1747–1754.

[5] A. George, P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets and Systems 64 (1994), 395–399.

[6] V. Gregori, S. Romaguera, Characterizing completable fuzzy metric spaces, Fuzzy Sets and Systems. 144 (2004), 411–420.

[7] V. Gregori, S. Romaguera, Some properties of fuzzy metric spaces, Fuzzy Sets and Systems 115 (2000), 485–489.

[8] K. N. Plataniotis, A. N. Venetsanopoulos, Color Image Processing and Applications, Springer-Verlag, Berlin (2000).

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Probabilistic uniform structures

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Abstract

Uniform spaces can be defined in various equivalent ways by means of entourages, uniform covers and pseudometrics. The latter allows to identify a uniformity the family of all pseudometrics generating a coarser uniformity. Hence, it is natural to wonder if this is also true in the realm of fuzzy uniform spaces. Recently, it has been proved that a uniformity is categorically equivalent to a family of fuzzy pseudometrics generating a coarser uniformity. Here we will show that a probabilistic uniformity can be also defined by means of a certain family of fuzzy pseudometrics. This new approach to probabilistic uniformities sheds light on their relationship with classical uniformities.

1. Introduction

It is well-known that, although not every uniformity is metrizable, every entourage of a uniformity belongs to a coarser uniformity generated by a certain pseudometric. This allows to establish a categorical identification between a uniformity and

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a family of pseudometrics called a *gauge* or a *uniform structure*. In 2010, Gutiérrez García, S. Romaguera and M. Sanchis [6] showed that this is also true when you consider fuzzy pseudometrics, i. e. every uniformity is, categorically speaking, equivalent to a *fuzzy uniform structure*, i. e. a family of fuzzy pseudometrics in the sense of Kramosil and Michalek, satisfying certain properties.

On the other hand, probabilistic uniform spaces were introduced in [7] as a fuzzy counterpart of the concept of uniform space (see, for example, [13] for a discussion about several notions of fuzzy uniformity).

Consequently, it is natural to wonder whether, as classical uniformities, every probabilistic uniformity is equivalent to a certain family of fuzzy pseudometrics. Here, we show explicitly that this is true by establishing a category isomorphism theorem. Furthermore, as we will see, this isomorphism provides a more understandable way of establishing the relationship between classical uniformities and probabilistic uniformities.

2. Uniformities and fuzzy uniform structures

We start recalling some well-known facts about uniformities that will be useful later on. Our basic reference is [1].

We will denote by Unif the topological category whose objects are the uniform spaces (defined in terms of entourages) and whose morphisms are the uniformly continuous functions.

Uniformities admit several equivalent definitions. One of the most usual is the definition by means of pseudometrics.

Definition 1. Let X be a nonempty set. A gauge or a **uniform structure** on X is a nonempty family $\mathcal{D}$ of pseudometrics on X such that:

(G1) if $d, q \in \mathcal{D}$ then $d \vee q \in \mathcal{D}$;
(G2) if e is a pseudometric on X and for each $\varepsilon > 0$ there exist $d \in \mathcal{D}$ and $\delta > 0$ such that $d(x, y) < \delta$ implies $e(x, y) < \varepsilon$ for all $x, y \in X$, then $e \in \mathcal{D}$.

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If we say that a function $f : (X, \mathcal{D}) \to (Y, \mathcal{Q})$ between two spaces endowed with a uniform structure is uniformly continuous whenever $f : (X, \bigvee_{d\in\mathcal{D}} \mathcal{U}_d) \to (Y, \bigvee_{q\in\mathcal{Q}} \mathcal{U}_q)$ is uniformly continuous, then we can consider the category SUnif whose objects are the spaces endowed with a uniform structure and whose morphisms are the uniformly continuous functions. It is well-known that Unif and SUnif are isomorphic categories as the next theorem shows.

Theorem 2. *Let $\mathcal{U}$ and $\mathcal{D}$ be a uniformity and a uniform structure on a nonempty set X respectively. Define:*

- $\mathcal{D}_{\mathcal{U}}$ *as the family of all pseudometrics d on X such that $\mathcal{U}_d \subseteq \mathcal{U}$;*
- $\mathcal{U}_{\mathcal{D}}$ *as the uniformity $\bigvee_{d\in\mathcal{D}} \mathcal{U}_d$.*

Then the mappings:

- $\Delta : \mathsf{Unif} \to \mathsf{SUnif}$ *given by* $\Delta((X,\mathcal{U})) = (X, \mathcal{D}_{\mathcal{U}})$;
- $\Lambda : \mathsf{SUnif} \to \mathsf{Unif}$ *given by* $\Lambda((X,\mathcal{D})) = (X, \mathcal{U}_{\mathcal{D}})$;

which leave morphisms unchanged are covariant functors such that $\Delta \circ \Lambda = 1_{\mathsf{SUnif}}$ and $\Lambda \circ \Delta = 1_{\mathsf{Unif}}$.

In [6] the authors studied a fuzzy notion of the concept of uniform structure giving a new category isomorphic to Unif. We recall the necessary notions to establish this isomorphism. In the sequel we wilI use the following notation: $I = [0,1]$, $I_0 = (0,1]$ and $I_1 = [0,1)$.

Definition 3. A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a **continuous t-norm** if $([0,1], *)$ is an Abelian topological monoid with unit 1, such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0,1]$.

Definition 4 ([6])**.** A **fuzzy pseudometric** (in the sense of Kramosil and Michalek) on a nonempty set X is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X \times X \times [0, +\infty)$ such that for every $x, y, z \in X$ and $t, s > 0$:

(FM1) $M(x, y, 0) = 0$;
(FM2) $M(x, x, t) = 1$;
(FM3) $M(x, y, t) = M(y, x, t)$;

(FM4) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$;
(FM5) $M(x,y,\cdot) : [0,\infty) \to [0,1]$ is left continuous.

If the fuzzy pseudometric $(M,*)$ also satisfies:

(FM2') $M(x,y,t) = 1$ for all $t > 0$ if and only if $x = y$

then $(M,*)$ is said to be a **fuzzy metric** on X [10].

A **fuzzy (pseudo)metric** space is a triple $(X,M,*)$ such that X is a nonempty set and $(M,*)$ is a fuzzy (pseudo)metric on X.

Remark 5. Every fuzzy pseudometric $(M,*)$ on a nonempty set X generates a topology τ_M on X which has as a base the family $\{B_M(x,\varepsilon,t) : x \in X, \varepsilon \in (0,1), t > 0\}$ where $B_M(x,\varepsilon,t) = \{y \in X : M(x,y,t) > 1-\varepsilon\}$. Furthermore (cf. [4]) every fuzzy (pseudo)metric space $(X,M,*)$ is (pseudo)metrizable and it has a compatible uniformity $\mathcal{U}_M$ with a countable base given by

$$U_n^M = \{(x,y) \in X \times X : M(x,y,1/n) > 1 - 1/n\}$$

(we will omit the superscript M if no confusion arises).

Definition 6 ([3])**.** A function $f : (X,M,*) \to (Y,N,\star)$ between two fuzzy metric spaces is said to be **uniformly continuous** if for every $\varepsilon \in (0,1)$ and $t > 0$ there exist $\delta \in (0,1)$ and $s > 0$ such that

$$\text{if } M(x,y,s) > 1-\delta \text{ then } N(f(x),f(y),t) > 1-\varepsilon$$

where $x,y \in X$.

This is equivalent to assert that $f : (X,\mathcal{U}_M) \to (Y,\mathcal{U}_N)$ is uniformly continuous.

Example 7 ([6], cf. [2])**.** Let (X,d) be a pseudometric space. Let M_d be the fuzzy set on $X \times X \times [0,\infty)$ given by

$$M_d(x,y,t) = \begin{cases} \frac{t}{t+d(x,y)} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}.$$

For every continuous t-norm $*$, $(M_d,*)$ is a fuzzy pseudometric on X which is called the *standard fuzzy pseudometric* induced by d.

Furthermore, we notice that $\mathcal{U}_d = \mathcal{U}_{M_d}$ (cf. [5, Lemma 5]) where $\mathcal{U}_d$ is the uniformity generated by d.

Definition 8 ([6])**.** Let X be a nonempty set and let $*$ be a continuous t-norm. A **fuzzy uniform structure** for $*$ is a pair $(\mathcal{M}, *)$ where $\mathcal{M}$ is a nonempty family of fuzzy pseudometrics with respect $*$ on X such that:

(FU1) if $(M,*), (N,*) \in \mathcal{M}$ then $(M \wedge N, *) \in \mathcal{M}$;
(FU2) if $(M,*)$ is a fuzzy pseudometric on X, and if for each $\varepsilon \in (0,1)$ and each $t > 0$ there exist $(N,*) \in \mathcal{M}$, $\delta \in (0,1)$ and $s > 0$ such that $N(x,y,s) \geq 1-\delta$ implies $M(x,y,t) \geq 1-\varepsilon$ for all $x,y \in X$, then $(M,*) \in \mathcal{M}$.

A **fuzzy uniform space** is a triple $(X, \mathcal{M}, *)$ such that X is a nonempty set and $(\mathcal{M}, *)$ is a fuzzy uniform structure on X.

Definition 9 ([6])**.** Let $(X, \mathcal{M}, *)$ and $(Y, \mathcal{N}, \star)$ be two fuzzy uniform spaces. A mapping $f : X \to Y$ is said to be **uniformly continuous** if for each $N \in \mathcal{N}$, $\varepsilon \in (0,1)$ and $t > 0$ there exist $M \in \mathcal{M}$, $\delta \in (0,1)$ and $s > 0$ such that $N(f(x), f(y), t) > 1-\varepsilon$ whenever $M(x,y,s) > 1-\delta$.

Then we can consider the category FUnif whose objects are the fuzzy uniform spaces and whose morphisms are the uniformly continuous functions. Besides, if $*$ is a continuous t-norm, we denote by $\mathsf{FUnif}(*)$ the full subcategory of FUnif whose objects are the fuzzy uniform spaces of the form $(X, \mathcal{M}, *)$. In [6] it is proved that the category $\mathsf{FUnif}(*)$ is isomorphic to Unif as follows:

Theorem 10 ([6])**.** *Let $(X, \mathcal{U})$ be a uniform space and $(X, \mathcal{M}, *)$ be a fuzzy uniform space. Let us consider:*

- $(\varphi_*(\mathcal{D}_\mathcal{U}), *)$ *the fuzzy uniform structure* $\varphi_*(\mathcal{D}_\mathcal{U}) = \{(M,*) : \mathcal{U}_M \subseteq \mathcal{U}\}$;
- $\psi(\mathcal{M})$ *the uniformity* $\mathcal{U}_\mathcal{M} = \bigvee_{(M,*)\in\mathcal{M}} \mathcal{U}_M$.

Then:

(i) $\Phi_* : \mathsf{Unif} \to \mathsf{FUnif}(*)$ *is a covariant functor sending each* $(X, \mathcal{U})$ *to* $(X, \varphi_*(\mathcal{D}_\mathcal{U}), *)$;
(ii) $\Psi : \mathsf{FUnif}(*) \to \mathsf{Unif}$ *is a covariant functor sending each* $(X, \mathcal{M}, *)$ *to* $(X, \psi(\mathcal{M})) = (X, \mathcal{U}_\mathcal{M})$;

(iii) $\Phi_* \circ \Psi = 1_{\mathsf{FUnif}(*)}$ *and* $\Psi \circ \Phi_* = 1_{\mathsf{Unif}}$.

3. Probabilistic uniform structures

Probabilistic uniformities were first considered by Höhle and Katsaras [7, 9] as a fuzzy counterpart of uniformities. On his behalf, Lowen introduced in [12] for the t-norm $\wedge$ a different type of fuzzy uniformities, now called Lowen uniformities or Lowen-Höhle uniformities [13], which were also studied by Höhle [8] for an arbitrary t-norm. We recall their definitions.

Definition 11 ([11, 12])**.** Let X be a nonempty set.

- A **prefilter** $\mathcal{F}$ on X is a filter on the lattice I^X.
- A **prefilter base** $\mathcal{B}$ on X is a filter base on the lattice I^X. We denote

$$\langle B \rangle = \{F \in I^X : B \leq F \text{ for some } B \in \mathcal{B}\}.$$

- A prefilter $\mathcal{F}$ on X is said to be **saturated** if for every $\{F_\varepsilon : \varepsilon \in I_0\} \subseteq \mathcal{F}$ we have that $\sup_{\varepsilon \in I_0}(F_\varepsilon - \varepsilon) \in \mathcal{F}$.

Definition 12 ([7, Definition 2.1],[9],[12])**.** A **probabilistic $*$-uniformity** on a nonempty set X is a pair $(\mathcal{U}, *)$, where $*$ is a continuous t-norm and $\mathcal{U}$ is a prefilter on $X \times X$ such that:

(PU1) $U(x, x) = 1$ for all $U \in \mathcal{U}$;
(PU2) if $U \in \mathcal{U}$ then $U^{-1} \in \mathcal{U}$ where $U^{-1}(x, y) = U(y, x)$;
(PU3) for each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that

$$V^2 \leq U$$

where $V^2(x, y) = \sup_{z \in X} V(x, z) * V(z, y)$.

In this case we say that $(X, \mathcal{U}, *)$ is a **probabilistic $*$-uniform space**.

Furthermore, if $(\mathcal{U}, *)$ is a saturated probabilistic uniformity on X, then it is called a **Lowen $*$-uniformity**.

In general, we will not make reference to the t-norm $*$ if no confusion arises.

Definition 13. A function $f : (X, \mathfrak{U}, *) \to (Y, \mathcal{V}, \star)$ between two probabilistic uniform spaces is said to be **fuzzy uniformly continuous** if $(f \times f)^{-1}(V) \in \mathfrak{U}$ for all $V \in \mathcal{V}$, i. e. for every $V \in \mathcal{V}$ we can find $U \in \mathfrak{U}$ such that

$$U(x, y) \leq V(f(x), f(y)) \text{ for all } x, y \in X.$$

Then we can consider the category PUnif whose objects are the probabilistic uniform spaces and whose morphisms are the fuzzy uniformly continuous functions. For a fixed continuous t-norm $*$, PUnif$(*)$ is the full subcategory of PUnif whose objects are the probabilistic uniform spaces with respect to the continuous t-norm $*$.

Lowen [12] provided a pair of adjoint functors ω_* and ι between the category Unif and the category LUnif$(*)$ of probabilistic uniform spaces with respect to a fixed continuous t-norm $*$, which is a coreflective subcateogry of PUnif$(*)$ [13].

Theorem 14 ([12]). *Let X be a nonempty set, $\mathcal{U}$ be a uniformity on X and $(\mathfrak{U}, *)$ be a Lowen uniformity on X. Define*

$$\omega(\mathcal{U}) = \{F \in I^{X \times X} : F^{-1}((\varepsilon, 1]) \in \mathcal{U} \text{ for all } \varepsilon \in I_1\}$$

and

$$\iota(\mathfrak{U}) = \{U^{-1}((\varepsilon, 1]) : U \in \mathfrak{U}, \varepsilon \in I_1\}.$$

Then:

(1) $(\omega(\mathcal{U}), *)$ *is a Lowen uniformity on* X;
(2) $\iota(\mathfrak{U})$ *is a uniformity on* X;
(3) $\iota(\omega(\mathcal{U})) = \mathcal{U}$;
(4) $(\omega(\iota(\mathfrak{U})), *)$ *is the coarsest Lowen uniformity generated by a uniformity and which is finer than* $\mathfrak{U}$.

Furthermore, if we consider the mappings $\omega_* :$ Unif $\to$ LUnif$(*)$ *and* $\iota :$ LUnif $\to$ Unif, *which leave morphisms unchanged, and* $\omega_*((X, \mathcal{U})) = (X, \omega(\mathcal{U}), *)$ *and* $\iota((X, \mathfrak{U}, *)) = (X, \iota(\mathfrak{U}))$ *then they are fully faithful and faithful functors respectively. Therefore* Unif *is isomorphic to a full subcategory of* LUnif$(*)$.

We next introduce a new fuzzy uniform structure which, as we will see in the next section, is very related with probabilistic uniformities.

Definition 15. Let X be a nonempty set and let $*$ be a continuous t-norm. A **probabilistic $*$-uniform structure** on X is a pair $(\mathcal{M}, *)$ where $\mathcal{M}$ is a family of fuzzy pseudometrics on X with respect to $*$ such that:

- if $(M,*),(N,*) \in \mathcal{M}$ then $(M \wedge N, *) \in \mathcal{M}$;
- if $(M,*)$ is a fuzzy pseudometric on X such that for all $t > 0$, there exist $(N,*) \in \mathcal{M}$ and $s > 0$ verifying

$$N(x,y,s) \leq M(x,y,t)$$

  for all $x, y \in X$, then $(M,*) \in \mathcal{M}$.

A space with a probabilistic $*$-uniform structure is a triple $(X, \mathcal{M}, *)$ such that X is a nonempty set and $(\mathcal{M}, *)$ is a probabilistic $*$-uniform structure on X.

Definition 16. Let $(X, \mathcal{M}, *)$ and $(Y, \mathcal{N}, \star)$ be two spaces endowed with two probabilistic uniform structures. A mapping $f : X \to Y$ is said to be **fuzzy uniformly continuous** if for every $(N, \star) \in \mathcal{N}$ and $t > 0$ there exist $(M,*) \in \mathcal{M}$ and $s > 0$ such that $M(x,y,s) \leq N(f(x), f(y), t)$ for all $x, y \in X$.

In the sequel, we will denote by PSUnif the category whose objects are the spaces with a probabilistic uniform structure and whose morphisms are the fuzzy uniformly continuous functions. $\mathsf{PSUnif}(*)$ will denote the full subcategory of PSUnif whose objects are the spaces with a probabilistic $*$-uniform structure where $*$ is a fixed continuous t-norm $*$.

4. PSUnif and PUnif are isomoprhic

In [6] it was proved that the categories $\mathsf{FUnif}(*)$ and Unif are isomorphic. Here we will show that the categories PSUnif and PUnif are also isomorphic.

Proposition 17. *Let us consider the map* $\mathfrak{S} : \mathsf{PUnif} \to \mathsf{PSUnif}$ *given by*

$$\mathfrak{S}((X, \mathcal{U}, *)) = (X, \mathfrak{s}(\mathcal{U}), *) = (X, \mathcal{M}_{\mathcal{U}}, *)$$

where $(\mathfrak{s}(\mathcal{U}),*) = (\mathcal{M}_{\mathcal{U}},*)$ *is the probabilistic uniform structure of all fuzzy pseudometrics* $(M,*)$ *on* X *such that* $M(\cdot,\cdot,t) \in \mathcal{U}$ *for all* $t > 0$ *and*

$$\mathfrak{S}(f) = f$$

for every morphism f *in* PUnif. *Then* $\mathfrak{S}$ *is a covariant fully faithful functor.*

Proposition 18. *Let us consider the map* $\Upsilon : \mathsf{PSUnif} \to \mathsf{PUnif}$ *given by*

$$\Upsilon((X,\mathcal{M},*)) = (X,\upsilon(\mathcal{M}),*) = (X,\mathcal{U}_{\mathcal{M}},*)$$

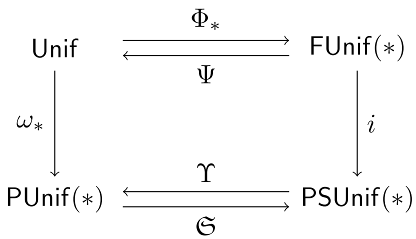
where $(\mathcal{U}_{\mathcal{M}},*)$ *is the probabilistic uniformity which has as base the family* $\{M(\cdot,\cdot,t) : t > 0, (M,*) \in \mathcal{M}\}$ *and*

$$\Upsilon(f) = f$$

for every morphism f *in* PSUnif. *Then* Υ *is a fully faithful covariant functor.*

Theorem 19. $\mathfrak{S} \circ \Upsilon = 1_{\mathsf{PSUnif}}$ *and* $\Upsilon \circ \mathfrak{S} = 1_{\mathsf{PUnif}}$ *so the categories* PSUnif *and* PUnif *are isomorphic.*

Theorem 20. *The following diagram commutes:*



where i *denotes the inclusion functor.*

Corollary 21. *Lowen's functor* ω_* *can be factorized as follows:*

$$\omega_* = \Upsilon \circ i \circ \Phi_*.$$

References

[1] R. Engelking, General topology, vol. 6 of Sigma Series in Pure Mathematics, Heldermann Verlag Berlin, 1989.

[2] A. George and P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets Syst. 64 (1994), 395–399.

[3] A. George and P. Veeramani, Some theorems in fuzzy metric spaces, J. Fuzzy Math. 3 (1995), 933–940.

[4] V. Gregori and S. Romaguera, Some properties of fuzzy metric spaces, Fuzzy Sets Syst. 115 (2000), 485–489.

[5] V. Gregori and S. Romaguera, On completion of fuzzy metric spaces, Fuzzy Sets Syst. 130 (2002), 399–404.

[6] J. Gutiérrez-García, S. Romaguera and M. Sanchis, Fuzzy uniform structures and continuous t-norms, Fuzzy Sets Syst. 161 (2010), 1011–1021.

[7] U. Höhle, Probabilistic uniformization of fuzzy topologies, Fuzzy Sets Syst. 1 (1978), 311–332.

[8] U. Höhle, Probabilistic metrization of fuzzy uniformities, Fuzzy Sets Syst. 8 (1982), 63–69.

[9] A. Katsaras, Fuzzy proximity spaces, J. Math. Anal. Appl. 68 (1979), 100–110.

[10] I. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetica 11 (1975), 326–334.

[11] R. Lowen, Convergence in fuzzy topological spaces, General Topology Appl. 10 (1979), 147–160.

[12] R. Lowen, Fuzzy uniform spaces, J. Math. Anal. Appl. 82 (1981), 370–385.

[13] D. Zhang, A comparison of various uniformities in fuzzy topology, Fuzzy Sets Syst. 140 (2003), 399–422.

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Generating a probability measure from a fractal structure. The distribution function

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Abstract

In this work we show how to define a probability measure with the help of a fractal structure. One of the keys of this approach is the use of the completion of the fractal structure. In this completion we define an order and describe a theory of the distribution function in this context.

1. Introduction

This work collects and advances some results on a research line on the construction of a probability measure with the help of a fractal structure, which is in current development ([3], [4], [5]).

The idea is to define a pre-measure on the elements of the fractal structure or some induced structure and be able to extend it to a probability measure on the Borel σ-algebra of the space or its completion.

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2. Fractal structures and non archimedean quasi metrics

Fractal structures were introduced in [1] to study non archimedean quasi metrization, but they have been used in other fields. We refer the reader to [6] for a survey on fractal structures.

Let X be a set and Γ_1 and Γ_2 be coverings of X. Γ_2 is said to be a strong refinement of Γ_1 if it is a refinement (that is, each element of Γ_2 is contained in some element of Γ_1) and for each $A \in \Gamma_1$ it is satisfied that $A = \cup\{B \in \Gamma_2 : B \subseteq A\}$.

Definition 1. A fractal structure $\mathbf{\Gamma}$ on a set X is a countable family of coverings $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ such that each cover Γ_{n+1} is a strong refinement of Γ_n for each $n \in \mathbb{N}$. Cover Γ_n is called level n of the fractal structure.

A quasi pseudo metric on a set X is a function $d : X \times X \to [0, \infty[$ such that:

(1) $d(x, x) = 0$, for each $x \in X$.
(2) $d(x, z) \leq d(x, y) + d(y, z)$ for each $x, y, z \in X$.

d is called a pseudo metric if it also satisfies that $d(x, y) = d(y, x)$ for each $x, y \in X$. A quasi pseudo metric (resp. a pseudo metric) is said to be a T_0 quasi metric (resp. a metric) if $d(x, y) = d(y, x) = 0$ implies that $x = y$, for each $x, y \in X$.

If d is a quasi (pseudo) metric, the function defined by $d^{-1}(x, y) = d(y, x)$ is also a quasi (pseudo) metric, called conjugate quasi (pseudo) metric of d. Furthermore, the function $d^*(x, y) = \max\{d(x, y), d^{-1}(x, y)\}$ is a (pseudo) metric.

A quasi pseudo metric is said to be non archimedean if $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ for each $x, y, z \in X$.

If d is a non archimedean quasi (pseudo) metric, then d^{-1} is also a non archimedean quasi (pseudo) metric and d^* is a non archimedean (pseudo) metric.

A fractal structure $\mathbf{\Gamma}$ induces a non archimedean quasi pseudo metric $d_{\mathbf{\Gamma}}$ given by:

$$d_{\mathbf{\Gamma}}(x, y) = \begin{cases} \frac{1}{2^n} & if \quad y \in U_{xn} \setminus U_{x,n+1} \\ \\ 1 & if \qquad y \notin U_{x1} \end{cases}$$

where $U_{xn} = X \setminus \bigcup\{A \in \Gamma_n : x \notin A\}$ for each $x \in X$ and $n \in \mathbb{N}$.

In this work, we will assume that the induced topology is T_0, and hence $d_{\mathbf{\Gamma}}$ is a non archimedean T_0-quasi metric and $d^*_{\mathbf{\Gamma}}$ is a non archimedean metric (also called ultrametric).

Given $x \in X$ and $n \in \mathbb{N}$, we will denote by $U^*_{xn} = \{y \in X; d^*(x,y) \leq \frac{1}{2^n}\}$ the closed ball, with respect to the ultrametric d^*, centered at x with radius $\frac{1}{2^n}$. The collection of these balls will be denoted by $\mathcal{G} = \{U^*_{xn} : x \in X; n \in \mathbb{N}\}$.

Conversely, a non archimedean quasi pseudo metric d induces a fractal structure $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$, where $\Gamma_n = \{B_{d^{-1}}(x, \frac{1}{2^n}) : x \in X\}$ and $B_{d^{-1}}(x, \frac{1}{2^n})$ is the ball with respect to the conjugate quasi pseudo metric d^{-1}, given as usual by $B_{d^{-1}}(x, \frac{1}{2^n}) = \{y \in X : d^{-1}(x,y) \leq \frac{1}{2^n}\}$.

2.1. **Completion of a fractal structure.** Following [1], we can define an extension of X as follows.

Let $G_n = \{U^*_{xn} : x \in X\}$. Note that G_n is a partition of X. Then we can define the projection $\rho_n : X \to G_n$ by $\rho_n(x) = U^*_{xn}$, and the bonding maps $\phi_n : G_{n+1} \to G_n$ given by $\phi_n(\rho_{n+1}(x)) = \rho_n(x)$. We will denote by $\widetilde{X} = \varprojlim G_n = \{(g_1, g_2, ...) \in \prod_{n=1}^{\infty} G_n : \phi(g_{n+1}) = g_n, \forall n \in \mathbb{N}\}$. Now, we can embed X into $\widetilde{X}$ by using the map $\rho : X \to \widetilde{X}$ defined as $\rho(x) = (\rho_n(x))_{n\in\mathbb{N}}$.

The construction of the bicompletion of a fractal structure $\mathbf{\Gamma}$ is given in [3] by defining level n of the extended fractal structure $\widetilde{\Gamma}$ as $\widetilde{\Gamma}_n = \{\widetilde{A} : A \in \Gamma_n\}$, where $\widetilde{A} = \{(\rho_k(x_k))_{k\in\mathbb{N}} \in \widetilde{X} : x_n \in A\}$ for each $A \in \Gamma_n$ and $n \in \mathbb{N}$.

We will denote by $\widetilde{U}^*_{xn} = \{y \in \widetilde{X} : \widetilde{d}^*(x,y) < \frac{1}{2^n}\}$, where $\widetilde{d}^*$ is the ultrametric induced by $\widetilde{\Gamma}$ on $\widetilde{X}$. Following a similar notation, we will denote the collection of these balls by $\widetilde{\mathcal{G}} = \{\widetilde{U}^*_{xn} : x \in X; n \in \mathbb{N}\} = \{\widetilde{U}^*_{xn} : x \in \widetilde{X}; n \in \mathbb{N}\}$.

3. Defining a probability measure on $\widetilde{X}$ and X

In this section we show how to define a probability measure on $\widetilde{X}$ by defining it on $\mathcal{G}$ or $\widetilde{\mathcal{G}}$ (this section is further developed in [4]).

Let ω be a pre-measure $\omega : \mathcal{G} \to [0,1]$. We will say that ω satisfies the mass distribution conditions if:

(1) $\sum\{\omega(U^*_{x1}) : U^*_{x1} \in G_1\} = 1$.
(2) $\omega(U^*_n) = \sum\{\omega(U^*_{y,n+1}) : U^*_{y,n+1} \in G_{n+1}; y \in U^*_{xn}\}$ for each $U^*_{xn} \in G_n$ and each $n \in \mathbb{N}$.

ω can be extended to $\widetilde{\mathcal{G}}$ by letting $\widetilde{\omega}(\widetilde{U}^*_{xn}) = \omega(U^*_{xn})$, for each $x \in X$ and $n \in \mathbb{N}$. Note that $\widetilde{\omega}$ also satisfies the mass distribution conditions.

From $\widetilde{\omega}$ and by using Method I and II of construction of outer measures (see [2]) it can be proved the next

Proposition 2. *$\widetilde{\omega}$ can be extended to a probability measure μ on the Borel sigma-algebras of $(\widetilde{X}, \widetilde{d}^*)$ and $(\widetilde{X}, \widetilde{d})$.*

The second way to define a probability measure on $\widetilde{X}$ is by using the elements of Γ_n, instead of $\mathcal{G}$. We will assume that $\mathbf{\Gamma}$ is a tiling fractal structure (that is, each level is a tiling covering which means the elements are regularly closed and their interior are pairwise disjoint).

So now ω is a pre-measure $\omega : \bigcup_{n\in\mathbb{N}} \Gamma_n \to [0,1]$ satisfying the mass distribution conditions, which are now:

(1) $\sum_{A\in\Gamma_1} \omega(A) = 1$.
(2) $\omega(A) = \sum_{B\in\Gamma_{n+1}, B\subseteq A} \omega(B)$, for each $A \in \Gamma_n$ and each $n \in \mathbb{N}$.

From ω, we can define a pre-measure on $\mathcal{G}$ as follows:

$$\omega(U^*_{xn}) = \begin{cases} \omega(A) & \text{if } x \in i_n(A) \\ \\ 0 & \textit{otherwise} \end{cases}$$

where $i_n(A)$ is the set of points which belongs to A, but does not belong to any other B with $B \in \Gamma_n$. Note that $i_n(A)$ is not empty and it contains the interior of A, since $\mathbf{\Gamma}$ is a tiling.

Proposition 3. $\omega : \mathcal{G} \to [0,1]$ *satisfies the mass distribution conditions. If* μ *is the extension of* ω *to the Borel sigma-algebras of* $(\widetilde{X}, \widetilde{d}^*)$ *and* $(\widetilde{X}, \widetilde{d})$*, then* $\omega : \bigcup_{n\in\mathbb{N}} \Gamma_n \to [0,1]$ *satisfies that* $\mu(\widetilde{A}) = \omega(A)$ *for all* $A \in \Gamma_n$ *and* $n \in \mathbb{N}$*.*

The next goal is to find conditions such that ω can be extended to a probability measure on the Borel σ-algebras of (X, d^*) and (X, d).

A fractal structure $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ is said to be Cantor complete if for each sequence $(A_n)_{n\in\mathbb{N}}$ with $A_n \in \Gamma_n$ and $A_{n+1} \subseteq A_n$ for each $n \in \mathbb{N}$, it holds that $\bigcap_{n\in\mathbb{N}} A_n \neq \emptyset$.

A sufficient conditions to obtain a probability measure on X is the following one, which uses the set of critical points $\mathcal{C}_n$ at level n, defined as $\mathcal{C}_n = \bigcup\{A \cap B : A, B \in \Gamma_n; A \neq B\}$ for each $n \in \mathbb{N}$.

Theorem 4. *Let* $\mathbf{\Gamma}$ *be a Cantor complete fractal structure on* X *(we assume that* $\mathbf{\Gamma}$ *is tiling if we define* ω *on* $\bigcup_{n\in\mathbb{N}} \Gamma_n$*) and suppose that for each* $n \in \mathbb{N}$ $\omega(St(\mathcal{C}_n, \Gamma_m)) \to 0$*. Then* ω *can be extended to a probability measure on the Borel* σ*-algebras of* (X, d^*) *and* (X, d)*.*

4. Distribution function

In this section we elaborate a theory of a distribution function in the context of this work (this section is further developed in [5]).

First, we define an order in $\widetilde{X}$ from the collection of balls G_n.

Definition 5. Let us suppose that G_n is countable for each $n \in \mathbb{N}$. Then we can enumerate $G_1 = \{g_1, g_2, \ldots\}$. Now we enumerate G_2 such that $g_i = g_{i1} \cup g_{i2} \cup \cdots$ for each $g_i \in G_1$, and define the lexicographical order in G_2. Recursively, we define an order in G_n for each $n \in \mathbb{N}$.

This order induces an order in $\widetilde{X}$ given by $x \leq y$ if and only if $\widetilde{U}^*_{xn} \leq \widetilde{U}^*_{yn}$ in G_n for each $n \in \mathbb{N}$.

Proposition 6. *This order is a complete total order with a bottom. If* $\mathbf{\Gamma}$ *is finite, then it also has a top.*

Definition 7. The cumulative distribution function (in short, cdf) of a probability measure μ is a function $F : \widetilde{X} \to [0,1]$ defined by $F(x) = \mu(\leq x)$, where $\leq x = \{y \in \widetilde{X} : y \leq x\}$.

Proposition 8. *Let F be a cdf. Then:*

(1) *F is non-decreasing.*
(2) *F is right τ_{d^*}-continuous.*
(3) *$\lim_{x\to\infty} F(x) = 1$ (this means that for each $\varepsilon > 0$ and $x \in \widetilde{X}$ there exists $y \in \widetilde{X}$ with $x \leq y$ and such that $1 - F(y) < \varepsilon$).*

Finally, we prove that any function satisfying the conditions of the previous proposition is the distribution function of a probability measure on $\widetilde{X}$ defined with the help of a fractal structure.

Theorem 9. *Let $F : \widetilde{X} \to [0,1]$ be a non-decreasing, right τ_{d^*}-continuous function such that $\lim_{x\to\infty} F(x) = 1$. Then there exists a pre-measure $\omega : \mathcal{G} \to [0,1]$, satisfying the mass distribution conditions, such that F is the cdf of μ, where μ is the extension of $\widetilde{\omega}$ to the Borel σ-algebras of $(\widetilde{X}, \widetilde{d^*})$ and $(\widetilde{X}, \widetilde{d})$.*

References

[1] F. G. Arenas and M. A. Sánchez-Granero, A Characterization of Non-archimedeanly Quasimetrizable Spaces, Rend. Istit. Mat. Univ. Trieste, Suppl. XXX (1999), 21–30.

[2] G. A. Edgar, Measure, Topology and Fractal Geometry, Springer New York, 1990.

[3] J. F. Gálvez-Rodríguez and M. A. Sánchez-Granero, The completion of a fractal structure, preprint.

[4] J.F. Gálvez-Rodríguez and M. A. Sánchez-Granero, Generating a probability measure from a fractal structure, preprint.

[5] J. F. Gálvez-Rodríguez and M. A. Sánchez-Granero, The distribution function of a probability measure on a space with a fractal structure, preprint.

[6] M. A. Sánchez-Granero, Fractal structures, in: Asymmetric Topology and its Applications, in: Quaderni di Matematica, vol. 26, Aracne, 2012, 211–245.

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A brief survey on transitivity and Devaney's chaos: autonomous and nonautonomous discrete dynamical systems

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Abstract

This is intended as a brief survey on topological transitivity and Devaney's chaos for autonomous (respectively, noautonomous) discrete dynamical systems on metric spaces. Our main is to give an overview of basic results on these topics. We emphasize the differences between autonomous and nonautonomous systems. The outcomes are purely "topological" and they do not reflect differentiable dynamics or ergodic theory aspects of these topics.

1. Introduction

All spaces under consideration are metric spaces. Given a metric space $X := (X, d)$ and a continuous function $f : X \to X$, the pair (X, f) is called *a(n) (autonomous) discrete dynamical system* (discrete dynamical system or DS, for short). Notice

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that it is equivalent to consider an action of the usual topological semigroup of the natural on the metric space X and, if f is a homeomorphism, then it is equivalent to consider an action of the usual topological group of the integers on X. One of the most interesting and useful notions in the theory of discrete dynamical systems is the concept of orbit of a point x, that is, the sequence

$$\operatorname{orb}_f(x) = \{x, f(x), f^2(x), \ldots, f^n(x), \ldots\}$$

where f^n stands for the n-th iterate of the function f.

Discrete dynamical systems can be generalized in the following way. Let X be a metric space, $f_n : X \to X$ a continuous map for each positive integer n, and f_∞ the sequence $(f_1, f_2, \ldots, f_n, \ldots)$. The pair (X, f_∞) is called a *nonautonomous discrete dynamical system* (NDS, for short) in which the *orbit of a point* $x \in X$ under f_∞ is defined as the set

$$\operatorname{orb}_{f_\infty}(x) = \{x, f_1(x), f_1^2(x), \ldots, f_1^n(x), \ldots\},$$

where

$$f_1^n = f_n \circ f_{n-1} \circ \ldots \circ f_2 \circ f_1,$$

for each positive integer n, and f_1^0 is the identity on X. In particular, when f_∞ is the constant sequence $(f, f, \ldots, f, \ldots)$, the pair (X, f_∞) is the usual (autonomous) discrete dynamical system given by the map f on X.

In this note we deal with basic results on topological transitivity and Devaney's chaos for DS and NDS. Our presentation is organized in such a way that the differences between DS, on the one hand, and NDS, on the other, are strongly emphasized. A detailed exposition, more suited to the purposes of the present note, is given in [10].

Let (X, f) be DS. We will consider the following two conditions:

(TT) for every pair of nonempty open sets U and V in X, there is a positive integer n such that $f^n(U) \cap V \neq \emptyset$, and

(DO) there is a point x_0 such that the orbit of x_0 is dense in X.

As is customary, we adopt the condition (TT) as the definition of *topological transitivity*, but note that some authors take (DO) instead. Any point with dense orbit is called *a transitive point*. As a motivation for the notion of topological transitivity of (X, f) one may think of a real physical system, where a state is never given or measured exactly, but always up to a certain error. So instead of points one should study (small) open subsets of the phase space and describe how they move in that space.

The concept of topological transitivity goes back Birkhoff. According to [7], he used it in 1920, cf. [4] (see also [3]).

Remember that, given a DS (X, f), a point $x \in X$ is *periodic* if there exists a non-negative integer n such that $f^n(x) = x$. A DS (X, f) is called Devaney's chaotic [5] if it satisfies the following three conditions:

(i) (X, f) is transitive;
(ii) the periodic points are dense in X;
(iii) (X, f) has sensitive dependence on initial conditions, that is, there exists $\varepsilon > 0$ such that for all $\delta > 0$ and all $x \in X$ there is $y \in X$ and $n > 0$ with $d(x, y) < \delta$ and $d(f^n(x), f^n(y)) \geq \varepsilon$.

As far as we know the first to formulate (iii) was Guckenheimer [8] in his study on maps of the interval (he required the condition to hold for a set of positive Lebesgue measure). The phrase *sensitive dependence on initial conditions* was used by Ruelle [12] to indicate some exponential rate of divergence of orbits of nearby points.

The corresponding conditions and definitions for NDS are self-explanatory and we omit the details. NDS were introduced in [9], and are related to nonautonomous difference equations: Indeed, a general form of a nonautonomous difference equation is the following: Given a compact metric space (X, d) and a sequence of continuous functions $(f_n : X \to X)_{n \in \mathbb{N}}$, for each $x \in X$ we set

$$\begin{cases} x_0 = x, \\ x_{n+1} = f_n(x_n). \end{cases}$$

These kind of nonautonomous difference equations have been considered by several mathematicians (see for instance, among others, [15], [17]). The most classical examples are when $X = [0,1]$ is the unit interval, and d is the usual euclidean metric. Observe that the orbit of a point forms a solution of a nonautonomous difference equation.

Our notation and terminology is standard and follows [6].

2. Discrete dynamical systems

We begin by describing the relationship between conditions (TT) and (DO). The two conditions are independent in general. We will present the usual examples of this fact. First consider $X = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ equipped with the usual metric and let f be the continuous function on X onto itself defined by $f(0) = 0$ and $f(1/n) = 1/(n+1)$, $n \geq 1$. The point $x = 1$ is (the only) transitive point for (X, f) but the system is not topologically transitive (take, say, $U = \{1/3\}$ and $V = \{1\}$). Thus, (DO) does not imply (TT). We show that neither (TT) implies (DO). To this end take the unit interval $\mathbb{I}$ and the tent map $g(x) = 1 - |2x - 1|$ from $\mathbb{I}$ into itself. Let X be the set of all periodic points of g and consider the dynamical system (X, f) where $f = g|_X$. Then the system (X, f) does not satisfy the condition (DO), since X is infinite (dense in $\mathbb{I}$) while the orbit of any periodic point is finite. But the condition (TT) is fulfilled. This follows from the fact that for any nondegenerate subinterval J of $\mathbb{I}$ there is a positive integer n with $g^n(J) = \mathbb{I}$ (see [11]). Hence, whenever M and N are nonempty open subintervals of $\mathbb{I}$, there is a periodic orbit of g which intersects both M and N. This gives (TT) for (X, f).

Nevertheless, under some additional assumptions on the phase space X the two conditions (TT) and (DO) are equivalent. In fact, we have

Theorem 1 ([16]). *Let (X, f) be a discrete dynamical system. If X has no isolated points, then (DO) implies (TT). If X is separable and second category, then (TT) implies (DO).*

We now turn our attention to Devaney's chaos. Taking as a point of starting the definition of Devaney's chaos, an obvious question to ask is whether Devaney's chaos is independent of the choice of the metric d on X. The following result answers this question in a drastic way.

Theorem 2 ([2]). *Let (X, f) be a discrete dynamical system with X an infinite set. If (X, f) is transitive and the set of periodic points is dense in X, then (X, f) has sensitive dependence on initial conditions.*

The previous result implies that condition (iii) in Devaney's definition of chaos is in fact superfluous. It is worth nothing (and easy to see) that neither (i) nor (ii) are redundant. However, in [14] Sarkovskiĭ proved that transitivity implies that the periodic points are dense in the case of the dynamical systems on the interval. Sarkovskiĭ's result can be improved by considering a more general class of spaces. A connected space X has a *disconnecting interval* if there is an open subset I of X, homeomorphic to an open interval, such that $X \setminus I$ is not connected.

Theorem 3 ([1]). *If in the dynamical system (X, f) the space X is connected and has a disconnecting interval and f is transitive, then the periodic points are dense in X.*

3. Nonautonomous dynamical systems

In this section we describe how the previous results fit into the theory of NDS. Concerning the relationship between conditions (TT) and (DO) we have the two following results related to Theorem 1.

Theorem 4 ([13]). *Suppose that X is a separable space with the Baire property. If (X, f_∞) is a transitive NDS, then there exists a point $x \in X$ whose orbit is dense.*

Example 5 ([13]). There is a non-transitive NDS $(\mathbb{I}, g_\infty)$ which has a dense orbit.

The relevant results on Devaney's chaos presented by Theorem 2 and Theorem 3 fail to be true in the realm of NDS.

Example 6 ([13]). There is a NDS $(\mathbb{I}, f_\infty)$ which is transitive and has a dense set of periodic points, but it does not have sensitive dependence on initial conditions.

Example 7 ([13]). There is a transitive NDS $(\mathbb{I}, g_\infty)$ with sensitive dependence on initial conditions such that the set of periodic points is not dense in $\mathbb{I}$.

References

[1] Ll. Alsedà, S. Kolyada, J. Llibre and L'. Snoha, Entropy and periodic points for transitive maps, Trans. Amer. Math. Soc. 351, no. 4 (1999), 1551–1573.

[2] J. Banks, J. Brooks, G. Cairns, G. Davis, P. Stacey, On Devaney's definition of chaos, The American Mathematical Monthly 99, no. 4 (1992) 332–334.

[3] G. Birkhoff, Dynamical Systems, Amer. Math. Soc., Providence, RI, 1927.

[4] G. D. Birkhoff, Collected Mathematical Papers, vol. 2, New York, 1950.

[5] R. L. Devaney, An introduction to chaotic dynamical systems, Second edition. Addison-Wesley Studies in Nonlinearity. Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1989

[6] R. Engelking, General Topology, Sigma Series in Pure Mathematics 6, Heldermann Verlag Berlin, 1989.

[7] L. W. Goodwyn, The product theorem for topological entropy, Trans. Amer. Math. Soc. **158** (1971), 445–452.

[8] J. Guckenheimer, Sensitive dependence to initial conditions for one-dimensional maps, Comm. Math. Phys. 70, no. 2 (1979), 133–160

[9] S. Kolyada, L. Snoha, Topological entropy of nonautononous dynamical systems, Random Comput. Dyn. 4 (1996), 205–233.

[10] S. Kolyada, L. Snoha, Some aspects of topological transitivityÔÇöa survey. Iteration theory (ECIT 94) (Opava), 3–35, Grazer Math. Ber., 334, Karl-Franzens-Univ. Graz, Graz, 1997.

[11] R. A. Holmgren, A First Course in Discrete Dynamical Systems, Springer-Verlag, New York, 1994.

[12] D. Ruelle, Dynamical systems with turbulent behavior, Mathematical problems in theoretical physics (Proc. Internat. Conf., Univ. Rome, Rome, 1977), pp. 341–360, Lecture Notes in Phys., 80, Springer, Berlin-New York, 1978.

[13] I. Sánchez, M. Sanchis, H. Villanueva, Chaos in hyperspaces of nonautonomous discrete systems, submitted.

[14] A. N. Sharkovskii, Nonwandering points and the centre of a continuous mapping of the line into itself, Dopovidi Ukrain. Acad. Sci. 7 (1964), 865–868 (in Ukrainian).

[15] Y. Shi, Chaos in nonautonomous discrete dynamcial systems approached by their induced continuous functionpings, Int. J. Bifurcation and Chaos 22, no. 11 (2012), 1–12.

[16] S. Silverman, On maps with dense orbits and the denition of chaos, Rocky Mountain Jour. Math. 22 (1992), 353–375.

[17] H. Zhu, L. Liu and J. Wang, A note on stronger forms of sensitivity for inverse limit dynamical systems, Advances in Difference equations 101 (2015), 1–9.

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Some fixed point theorems in fuzzy metric spaces from Banach's principle

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Abstract

Here we present the concept of p-metric that use to obtain some well-known fixed point theorems in fuzzy metric spaces from the classical Banach's principle

1. Introduction

p-sums or p-Yager-sums are defined as $\oplus_p : \mathbb{R}^+ \times \mathbb{R}^+ :\to [0,1]$; $\oplus_p(a,b) = \min\{1, (a^p + b^p)^{1/p}\}$, $p > 0$. In particular $\oplus_1(a,b) = \min\{1, a+b\}$, that is referred in the literature as the bounded sum or the Lukasiewicz sum. Moreover $\oplus_p$ is known as a Yager t-norm if it is defined on $[0,1]$. It is well known that the theory of fuzzy logic and fuzzy sets focus its attention on the set $[0,1]$. On the other hand, if we have a metric d on a non-empty set X, the topology induced by d on X coincides with the topology induced by d_1, where d_1 is the metric d bounded by 1. So we could generalize the concept of metric by using the concept of p-sum to give the

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definition of p-metric on a set X as a binary fuzzy relation $D : X \times X \to [0,1]$ satisfying the following properties for all $x, y, z \in X$ and for some $p > 0$: i) $D(x,y) = 0$ if and only if $x = y$, ii) $D(x,y) = D(y,x)$ and iii) $D(x,z) \leq D(x,y) \oplus_p D(y,z)$. We will know this triple $(X, D, \oplus_p)$ as a p-metric space. For each $x \in X$ and $r > 0$ we can define the open ball $B_D(x,r) = \{y \in X : D(x,y) < r\}$ and it is obvious that $B_D(x, r_1) \subseteq B_D(x, r_2)$ provided that $r_1 \leq r_2$. Consequently, we may define a topology τ_D on X as $\tau_D = \{A \subseteq X$: for each $x \in A$ there exists $r > 0$ such that $B_D(x,r) \subseteq A\}$.

It is easy to see that $d(x,y) = D^p(x,y)$ for all $x, y \in X$ is a metric on X and that $\tau_d = \tau_D$. Reciprocally, if d is a 1−bounded metric on X then $D(x,y) = d^{1/p}(x,y)$ for all $x, y \in X$ is a p-generalized metric on X for $\oplus_p$, $p > 0$.

In fuzzy theory, the concepts of intersection and union are generalized in terms of t-norms and t-conorms, respectively. A t-norm is a binary operation $*$: $[0,1] \times [0,1] \to [0,1]$ satisfying the following conditions:(i) $*$ is associative and commutative; (ii) $a * 1 = a$ for every $a \in [0,1]$, (iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ with $a, b, c, d \in [0,1]$. A t-conorm is a binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ satisfying the following conditions:(i) $\diamond$ is associative and commutative; (ii) $a \diamond 0 = a$ for every $a \in [0,1]$, (iii) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ with $a, b, c, d \in [0,1]$. If $*$ ($\diamond$) is continuous we will call to $*$ ($\diamond$) as a continuous t-nom (t-conorm). If $*$ is a (continuous) t-norm we can define a (continuous) t-conom $\diamond_*$as follows: $a \diamond_* b = 1 - [(1-a) * (1-b)]$ for all $a, b \in [0,1]$. It is well known that the p-Yager sums are continuous t-conorms.

In [1] L. Zadeh introduced the concept of similarity relation as follows

Definition 1. A similarity relation on a set X is a pair $(E, *)$ such that $*$ is a t-norm and E is a fuzzy set in $X \times X$ such that for all $x, y, z \in X$:

(E1) $E(x,y) = 1$ if and only if $x = y$, (E2) $E(x, y) = E(y, x)$

(E3) $E(x, z) \geq E(x, y) * E(y, z)$.

If we define $D(x, y) = 1 - E(x, y)$ for all $x, y \in X$, then $D(x, z) \leq D(x, y) \diamond_* D(y, z)$. So if $\diamond_* \leq \oplus_p$ for some $p > 0$ then $(X, D, \oplus_p)$ is a p-generalized metric space.

In [2] Kramosil and Michalek introduced its celebrated notion of fuzzy metric as follows:

Definition 2. A fuzzy metric on a set X is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X \times X \times [0, \infty)$ such that for all $x, y, z \in X$:

(FM1) $M(x, y, 0) = 0$;

(FM2) $x = y$ if and only if $M(x, y, t) = 1$ for all $t > 0$;

(FM3) $M(x, y, t) = M(y, x, t)$;

(FM4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$ for all $t, s \geq 0$;

(FM5) $M(x, y, \_) : \mathbb{R}^+ \to [0, 1]$ is left continuous.

By a fuzzy metric space we mean a triple $(X, M, *)$ such that X is a set and $(M, *)$ is a fuzzy metric on X.

A fuzzy metric space $(X, M, *)$ is said to be stationary [5] if M does not depend on t, so a similarity relation $(E, *)$ on a set X is a stationary fuzzy metric space $(X, E, *)$ by defining $E(x, y, 0) = 0$ and $*$ is continuous. On the other hand, if $(X, M, *)$ is a stationary fuzzy metric space such that $\diamond_* \leq \oplus_p$ for some $p > 0$ then $(X, D, \oplus_p)$ is a p-generalized metric space where $D(x, y) = 1 - M(x, y)$ for all $x, y \in X$. Reciprocally, if $(X, D, \oplus_p)$ is a p-generalized metric space then $(X, M, *)$ is a stationary fuzzy metric space, where $M(x, y) = 1 - D(x, y)$ for all $x, y \in X$ and $\diamond_* \leq \oplus_p$.

Given a fuzzy metric $(M, *)$ on a set X we can define a open ball for each $x \in X$, $t > 0$ and $\varepsilon \in (0,1)$ as $B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$. Consequently, we may define a topology τ_M on X as $\tau_M = \{A \subseteq X :$ for each $x \in A$ there exists $\varepsilon \in (0,1)$ and $t > 0$ such that $B_M(x, \varepsilon, t) \subseteq A\}$.

A Cauchy sequence in a fuzzy metric space $(X, M, *)$ is a sequence $\{x_n\}_{n \in \mathbb{N}}$ in X such that for each $\varepsilon \in (0,1)$ and $t > 0$ there exists an $n_0 \in \mathbb{N}$ satisfying $M(x_n, x_m, t) > 1 - \varepsilon$ whenever $n, m \geq n_0$.

A fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence $\{x_n\}_{n \in \mathbb{N}}$ converges with respect to the topology τ_M, i.e, if there exists $y \in X$ such that for each $t > 0$, $\lim_n M(y, x_n, t) = 1$.

It is obvious that every stationary fuzzy metric space $(X, M, *)$ such that $\diamond_* \leq \oplus_p$ for some $p > 0$ is metrizable. In fact $d(x, y) = D^p(x, y) = (1 - M(x, y))^p$ for all $x, y \in X$ is a metric on X and that $\tau_d = \tau_D = \tau_M$.In particular if $\diamond_* \leq \oplus_1$ then $d(x, y) = 1 - M(x, y)$ is a metric on X. Taking into account the general case, it is also well known [4] that every fuzzy metric space is metrizable, however it is not easy to construct from a fuzzy metric $(M, *)$ on a set X a metric d inducing the same topology on X.

2. Metrics from fuzzy metrics

In [2] the authors remarked that FM5 condition is chosen to enable to understand $M(x, y, t)$ as the degree of our belief that the distance between x and y is smaller that t. As it is said above if $(X, M, *)$ is a stationary fuzzy metric space such that $\diamond_* \leq \oplus_1$ then $d(x, y) = 1 - M(x, y)$ is a metric on X. In the general case of a fuzzy metric space $(X, M, *)$ such that $\diamond_* \leq \oplus_1$ and taking into account the previous remark it seems to be quite natural to define "the distance " $d(x, y)$ as $\sup\{t \geq 0 : 1 - M(x, y, t) \geq t\}$. In this direction Radu obtained in [7] the following theorem.

Theorem 1. Let $(X, M, *)$ be a fuzzy metric space such that $\diamond_* \leq \oplus_1$. For each $x, y \in X$ put

$$d_R(x, y) = \sup\{t \geq 0 : M(x, y, t) \leq 1 - t\}.$$

Then d_R is a metric on X such that

$$d_R(x, y) < \varepsilon \Leftrightarrow M(x, y, \varepsilon) > 1 - \varepsilon,$$

for all $\varepsilon \in (0, 1)$. Therefore, the topologies induced by $(M, *)$ and d_R coincide on X. In particular, $(X, M, *)$ is complete if and only if (X, d_R) is complete.

If $(X, M, *)$ is a stationary fuzzy metric space such that $\diamond_* \leq \oplus_p$ for some $p > 0$, then $d(x, y) = D^p(x, y) = (1 - M(x, y))^p$ for all $x, y \in X$ is a metric on X,as it is said above. In the general case of a fuzzy metric space $(X, M, *)$ such that $\diamond_* \leq \oplus_p$ for some $p > 0$ and following the previous construction, it seems natural to define $d(x, y)$ as $\sup\{t \geq 0 : (1 - M(x, y, t))^p \geq t\}$. This agree with the following example in [8].

Example 1. Let $(X, M, *)$ be a fuzzy metric space such that $\diamond_* \leq \oplus_p$ for some $p \in (0, 1)$. The function $d : X \times X \to \mathbb{R}^+$, defined as

$$d(x, y) = \sup\{t \geq 0 : M(x, y, t) \leq 1 - t^{1/p}\},$$

is a metric on X such that

$$d(x, y) < \varepsilon \Leftrightarrow M(x, y, \varepsilon) > 1 - \varepsilon^{1/p},$$

for all $\varepsilon \in (0, 1)$. Therefore, the topologies induced by $(M, *)$ and d coincide on X. In particular, $(X, M, *)$ is complete if and only if (X, d) is complete.

This example is result of the following theorem in [8].

Theorem 2. Let $(X, M, *)$ be a fuzzy metric space. Suppose that there exists a function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

(c1) α is strictly increasing on $[0, 1]$;

(c2) $0 < \alpha(t) \leq t$ for all $t \in (0, 1)$ and $\alpha(t) > 1$ for all $t > 1$;

(c3) $\alpha(t + s) \geq \alpha(t) \diamond_* \alpha(t)$;

Then the function $d_\alpha : X \times X \to \mathbb{R}^+$ defined as

$$d_\alpha(x, y) = \sup\{t \geq 0 : M(x, y, t) \leq 1 - \alpha(t)\},$$

is a metric on X such that $d_\alpha(x, y) \leq 1$ for all $x, y \in X$. If, in addition, the function α is left continuous on $(0, 1]$, then

$$d_\alpha(x, y) < \varepsilon \Leftrightarrow M(x, y, \varepsilon) > 1 - \alpha(\varepsilon)$$

for all $\varepsilon \in (0, 1)$. Thus the topologies induced by $(M, *)$ and d_α coincide on X. Moreover, $(X, M, *)$ is complete if and only if (X, d_α) is complete.

3. Some remarks on fixed point theorems

In [9] the following fixed point theorem on fuzzy metric spaces was established

Theorem 3. Let $(X, M, *)$ be a complete fuzzy metric space such that $\diamond_* \leq \oplus_p$ for some $p > 0$. If T is a self-map on X such that there is $k \in (0, 1)$ satisfying

$M(Tx, Ty, t) \geq 1 - k + kM(x, y, t)$

for all $x, y \in X$ and $t > 0$, then T has a unique fixed point.

Next we show that this theorem can be proved by means of the classical Banach contraction principle. Indeed, the previous contraction condition can be rewritten as follows

$1 - M(Tx, Ty, t) \leq k(1 - M(x, y, t))$

$\Leftrightarrow [1 - M(Tx, Ty, t)]^p \leq [k(1 - M(x, y, t))]^p$

$\Leftrightarrow [1 - M(Tx, Ty, t)]^p \leq k^p[(1 - M(x, y, t))]^p$. So we can write

$\sup\{t \geq 0 : (1 - M(Tx, Ty, t))^p \geq t\} \leq k^p \sup\{t \geq 0 : (1 - M(x, y, t))^p \geq t\}$, i.e, following the notation in Example 1

$d(Tx, Ty) \leq k^p d(x, y)$. Since (X, d) is complete, by the Banach contraction principle T has a unique fixed point.

George and Veeramani introduced in [3] the following modification of Kramosil and Michalek's notion of fuzzy metric space.

Definition 3. A GV-fuzzy metric on a set X is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X \times X \times (0, \infty)$ such that for all $x, y, z \in X$ and $t, s > 0$:

(GV1) $M(x, y, t) > 0$;

(GV2) $x = y$ if and only if $M(x, y, t) = 1$;

(GV3) $M(x, y, t) = M(y, x, t)$;

(GV4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$;

(GV5) $M(x, y, \_) : (0, \infty) \to (0, 1]$ is continuous.

It is interesting to remark the fact that every GV-fuzzy metric space $(X, M, *)$ can be considered as a fuzzy metric space in the sense of Kramosil and Michalek, simply putting $M(x, y, 0) = 0$ for all $x, y \in X$, so the previous results remain valid for GV-fuzzy metric spaces.

In [6] Gregori and Sapena introduced the following notion of contraction in a GV-fuzzy metric spaces, that has been widely used in the related literature.

Definition 4. Let $(X, M, *)$ be a GV-fuzzy metric space and $T : X \to X$ a self-map. We will say that T is fuzzy contractive if there exists $k \in (0, 1)$ such that

$$\frac{1}{M(Tx, Ty, t)} - 1 \leq k(\frac{1}{M(x, y, t)} - 1)$$

for all $x, y \in X$ and $t > 0$.

The following theorem can be deduced from the results that appear in [6]. Nevertheless we use here our approach to deduce it, and by using the classical Banach contraction principle.

Theorem 3. Let $(X, M, \wedge)$ be a complete GV-fuzzy metric space.Then every fuzzy contractive self-map T on X has a unique fixed point.

Proof. Let $\alpha(t) = \frac{t}{t+1}$ for $t \in [0,1]$ and 2 for $t > 1$, then α satisfies the conditions of Theorem 2, then the function $d_\alpha : X \times X \to \mathbb{R}^+$ defined as

$d_\alpha(x,y) = \sup\{t \geq 0 : M(x,y,t) \leq 1 - \frac{t}{t+1}\}$, or , equivalently $d_\alpha(x,y) = \sup\{t \geq 0 : \frac{1}{M(x,y,t)} - 1 \geq t\}$

is a metric on X , thus (X, d_α) is a complete metric space. Let T a fuzzy contractive self-map, then

$\frac{1}{M(Tx,Ty,t)} - 1 \leq k(\frac{1}{M(x,y,t)} - 1)$

so, we have

$\sup\{t \geq 0 : \frac{1}{M(Tx,Ty,t)} - 1 \geq t\} \leq k \sup\{t \geq 0 : \frac{1}{M(x,y,t)} - 1 \geq t\}$, i.e

$d_\alpha(Tx, Ty) \leq k d_\alpha(x,y)$. By the Banach contraction principle, T has a unique fixed point. □

References

[1] L. Zadeh, Similarity relations and fuzzy orderings, Information Sciences 3 (1971), 159–176.

[2] I. Kramosil and J. Michalek, Fuzzy metrics and Statistical metric spaces, Kybernetica 2, no. 2 (1975), 336–344.

[3] A. George and P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets and Systems 64 (1994), 395–399.

[4] V. Gregori and S. Romaguera, Some properties of fuzzy metric spaces, Fuzzy Sets and Systems 115 (2000), 485–489.

[5] V. Gregori and S. Romaguera, Characterizing complete fuzzy metric spaces, Fuzzy Sets and Systems 144 (2004), 411–420.

[6] V. Gregori and A. Sapena, On fixed point theorems in fuzzy metric spaces, Fuzzy Sets and Systems 125 (2002), 245–253.

[7] V. Radu, On the triangle inequality in PM-spaces, STPA, West University of Timisoara 39 (1978).

[8] F. Castro-Company, S. Romaguera and P. Tirado, On the construction of metrics from fuzzy metrics and its application to the fixed point theory of multivalued mappings, Fixed Point Theory and Applications (2015) 2015:226.

[9] F. Castro-Company and P. Tirado, On Yager and Hamacher t-norms and fuzzy metric spaces, International Journal of Intelligent systems 29 (2014), 1179–1180.