APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 16, no. 2(2015), 109-118
doi:10.4995/agt.2015.3045

Primal spaces and quasihomeomorphisms

Afef Haouati and Sami Lazaar

Department of Mathematics, Faculty of Sciences of Tunis. University Tunis-El Manar. "Campus Universitaire" 2092 Tunis, Tunisia. (haouati.afef@yahoo.fr, salazaar72@yahoo.fr)

Abstract

In [3], the author has introduced the notion of primal spaces. The present paper is devoted to shedding some light on relations between quasihomeomorphisms and primal spaces.
Given a quasihomeomorphism $q : X \to Y$, where X and Y are principal spaces, we are concerned specifically with a main problem: what additional conditions have to be imposed on q in order to render X (resp. Y) primal when Y (resp. X) is primal.

2010 MSC: *54B30; 54D10; 54F65; 54H20.*

Keywords: *Quasihomeomorphism; principal space; sober space.*

1. Introduction

First, we recall some notions which were introduced by the Grothendieck school (see for example [6] and [7]) such as locally closed sets and quasihomeomorphisms.

Let X be a topological space and S be a set of X. S is called *locally closed* if it is an intersection of an open set and a closed set of X. We denote by $\mathcal{L}(X)$ the set of all locally closed sets of X.

Given topological spaces X and Y, a continuous map $q : X \to Y$ is called a *quasihomeomorphism* if $A \mapsto q^{-1}(A)$ defines a bijection from $\mathcal{O}(Y)$ (resp., $\mathcal{F}(Y)$, resp., $\mathcal{L}(Y)$) to $\mathcal{O}(X)$ (resp., $\mathcal{F}(X)$, resp., $\mathcal{L}(X)$) where $\mathcal{O}(X)$ (resp., $\mathcal{F}(X)$, resp., $\mathcal{L}(X)$) is the family of all open (resp., closed, resp., locally closed) sets of X.

On the other hand, another definition of quasihomeomorphism is given by K.W.Yip in [9] as follows. A continuous map $q : X \to Y$ between topological

Received 3 June 2014 – Accepted 29 May 2015

spaces is said to be a quasihomeomorphism if the following equivalent conditions are satisfied:

- For any closed set C of X, $q^{-1}(\overline{q(C)}) = C$.
- For any closed set F of Y, $q(\overline{q^{-1}(F)}) = F$.

Fortunately, the two notions Grothendieck's quasihomeomorphism and Yip's quasihomeomorphism coincide.

Quasihomeomorphisms are used in algebraic geometry and it has recently been shown that this notion arises naturally in the theory of some foliations associated to closed connected manifolds (one may see [6]).

A *principal space* is a topological space in which any intersection of open sets is open. It is also recognized as Alexandroff space.

Let X be a principal space. Then, X provides a quasi-order $\leq$ (i.e a reflexive, transitive relation) given by $x \leq y$ if and only if $x \in \overline{\{y\}}$ which is called the specialization quasi-order (For more informations, one may see [10]).
Conversely, every quasi-order $\leq$ on a space X determines a principal topology. Indeed, for each $x \in X$ we let $x \uparrow$ be the upperset of x defined by $x \uparrow := \{y \in X : x \leq y\}$. Then, the family $\mathcal{B} := \{x \uparrow : x \in X\}$ is a basis of a principal topology on X. Note that the closure $\overline{\{x\}}$ is exactly the downset $\downarrow x := \{y \in X : y \leq x\}$.

Now, let $\mathcal{C}$ be a category. Then, a *flow* in $\mathcal{C}$ is a couple (X, f) where X is an object of $\mathcal{C}$ and $f : X \to X$ is an arrow called iterator. If (X, f) and (Y, g) are flows in $\mathcal{C}$, then a *morphism of flows* from (X, f) to (Y, g) is an arrow q from X to Y such that the following diagram is commutative.

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ q\downarrow & \circlearrowleft & \downarrow q \\ Y & \xrightarrow{g} & Y \end{array}$$

That is $g \circ q = q \circ f$. For more details, one may see [4] and [5].

Let (X, f) be a flow in the category of sets noted **Set**. O.Echi has defined the topology $\mathcal{P}(f)$ on X with closed sets exactly those A which are f-invariant (A set A of X is called f-invariant if $f(A) \subseteq A$). Clearly $\mathcal{P}(f)$ provides a principal topology on X.
We can easily see that for any set A of X, the closure $\overline{A}$ is exactly $\cup[f^n(A) : n \in \mathbb{N}]$ and in particular for any point $x \in X$, $\overline{\{x\}} = \{f^n(x), n \in \mathbb{N}\}$ denoted $\mathcal{O}_f(x)$ and called the orbit of x by f.
The family $x \uparrow = \{y \in X : f^n(y) = x \text{ for some } n \in \mathbb{N}\}$ is a basis of open sets of $\mathcal{P}(f)$.

According to O.Echi, a *primal* space is a topological space (X, τ) such that there is some mapping $f : X \to X$ with $\tau = \mathcal{P}(f)$ (for more informations see [3]).

In the first section of this paper, we are interested in some dynamical properties of quasihomeomorphisms between principal spaces.

The main goal of the second section is to show that given an onto quasihomeomorphism from a primal space X to a principal space Y, then Y is primal (see Theorem 3.2).

In the third section, we move our focus to one-to-one quasihomeomorphisms and its effects on primal spaces (see Theorem 4.1).

Finally, some particular cases of quasihomeomorphisms are studied and commented.

2. Preliminary results

Let X be a principal space and $\leq$ its specialization quasi-order.
A point $x \in X$ is called *minimal* if it satisfies the property:

$$\text{for each } y \in X,\ y \leq x \Rightarrow x \leq y.$$

Let $f : X \to Y$ be a continuous map between two principal spaces. It follows immediately, from the fact that a map between two principal spaces is continuous if and only if the induced map between the associated quasi-ordered sets is isotone, that for every $x, y \in X$, we have:

$$x \leq y \Rightarrow f(x) \leq f(y).$$

Now, the following proposition shows that the converse holds if f is a quasihomeomorphism.

Proposition 2.1. *Let $f : X \longrightarrow Y$ be a quasihomeomorphism where X and Y are principal spaces. Then, for every $x, y \in X$, we have:*

$$x \leq y \Longleftrightarrow f(x) \leq f(y).$$

Proof. It is sufficient to show the second implication.
For that, let $x, y \in X$ such that $f(x) \leq f(y)$. Since f is a quasihomeomorphism, then there exists a unique closed set F of Y such that $\downarrow y = f^{-1}(F)$.
Now, the fact that $f(y) \in F$ implies $\downarrow f(y) \subseteq F$ and consequently $f(x) \in F$, that is $x \in \downarrow y$ as desired. □

This result leads to the following corollary.

Corollary 2.2. *Let $q : X \longrightarrow Y$ be a quasihomeomorphism where X and Y are principal spaces. For every $x \in X$, if $q(x)$ is minimal in Y then x is minimal in X.*

Proof. Let x be a point in X such that $q(x)$ is a minimal point in Y. Suppose that there exists $x' \in X$ satisfying $x' \leq x$. Then, we have $q(x') \leq q(x)$ and thus, by minimality of $q(x)$, $q(x) \leq q(x')$. Therefore, Proposition 2.1 does the job. □

Question 2.3. *Let $q : X \longrightarrow Y$ be a quasihomeomorphism and x a point in X. If x is minimal, then what about $q(x)$?*

The following proposition shows that if in addition q is an onto quasihomeomorphism from X to Y, then there is an equivalence between x is minimal in X and $q(x)$ is minimal in Y.

Proposition 2.4. *Let $q : X \to Y$ be an onto quasihomeomorphism where X and Y are principal spaces. For any point $x \in X$, the following properties hold:*

1. $q(\downarrow x) = \downarrow q(x)$ *and* $q(x \uparrow) = q(x) \uparrow$
2. x *is minimal in* X *if and only if* $q(x)$ *is minimal in* Y*.*

Proof. Recall that if $q : X \longrightarrow Y$ is a quasihomeomorphism then q is onto iff q is open iff q is closed (see [2, Lemma 1.1]).

1. Let x be a point in X. The inclusion $q(\downarrow x) \subseteq \downarrow q(x)$ (resp., $q(x \uparrow) \subseteq q(x) \uparrow$) follows immediately from Proposition 2.1. Conversely, since q is closed (resp., open) then $q(\downarrow x)$ is a closed (resp., open) set containing $q(x)$. So that $q(\downarrow x)$ (resp., $q(x \uparrow)$) contains $\downarrow q(x)$ (resp., $q(x) \uparrow$).
2. According to Corollary 2.2, it is enough to show that if x is minimal in X then $q(x)$ is minimal in Y. Indeed, let $z \in \downarrow q(x)$. Due to Proposition 2.4.(1), we have $q(\downarrow x) = \downarrow q(x)$ so that $z = q(y)$ with $y \leq x$.

   Since x is minimal, then $x \leq y$ which leads to $q(x) \leq z$ as desired.

□

Proposition 2.5. *Let $q : X \longrightarrow Y$ be a quasihomeomorphism where X and Y are primal spaces.*

1. *If X is a T_0-space, then q is one to one.*
2. *If Y is a T_0-space, then q is onto.*
3. *If X and Y are both T_0-spaces, then q is a homeomorphism.*

   Proof.

   It follows immediately from [2, Lemma 3.7].

(2) Let $y \in Y$. Since q is a quasihomeomorphism then there exists $x \in X$ such that $q(x) \in \downarrow y$. Using [3, Theorem 2.3], the primality of Y gives a finite interval $[q(x), y]$. Let $q(x_0)$ be the biggest element of $[q(x), y]$ that have an antecedent in X.

   We claim that $q^{-1}(\downarrow y) = q^{-1}(\downarrow q(x_0))$.

   In fact, since $\downarrow y$ is totally ordered, then $q(z) \leq y$ gives either $q(z) \leq q(x)$ and so $q(z) \leq q(x_0)$ or $q(z) \in [q(x), y]$ which also implies that $q(z) \leq q(x_0)$ because $q(x_0)$ is the biggest element in that interval that have an antecedent. So that $q^{-1}(\downarrow y) \subseteq q^{-1}(\downarrow q(x_0))$. On the other hand, since $q(x_0) \leq y$ then $q^{-1}(\downarrow q(x_0)) \subseteq q^{-1}(\downarrow y)$.

   Now, we have $q^{-1}(\downarrow y) = q^{-1}(\downarrow q(x_0))$ and q is a quasihomeomorphism. So, $\overline{\{q(x_0)\}} = \overline{\{y\}}$.

   Finally, since Y is T_0 then $y = q(x_0)$.

   We conclude that Y is onto.

(3) Combining (1) and (2), we have q is a homeomorphism.

□

3. Primal spaces and onto quasihomeomorphisms

In order to characterize quasihomeomorphisms that conserve the property of primality between principal spaces, we need to recall some notions which were introduced by O.Echi in [3].

Given a quasi-ordered set $(X, \leq)$.

- We say that $(X, \leq)$ is *causal* if for each $x, y \in X$, the interval $[x, y] := \{z \in X : x \leq z \leq y\}$ is finite.
- $(X, \leq)$ will be called a *quasi-forest* if the downset of any point is totally quasi-ordered.

Given a flow (X, f) and $x \in X$. x is said to be a *periodic* point if $f^n(x) = x$ for some $n \in \mathbb{N}$.

O. Echi used the previous concepts to provide in [3] an interesting characterization of primal spaces in order-theoretical terms.

Before stating one of the main goals of this paper, it is of interest to recall this important result.

Theorem 3.1 ([3, Theorem 2.3])**.** *Let X be a principal topological space. Then, X is a primal space if and only if the associated quasi-ordered set $(X, \leq)$ is a causal quasi-forest in which each non-minimal point x has singleton interval $[x, x]$.*

Now, we are in a position to give our main result.

Theorem 3.2. *Let $(X, P(f))$ be a primal space, Y a principal space and $q : X \longrightarrow Y$ a quasihomeomorphism. If q is onto, then Y is primal.*

Proof. According to Theorem 3.1, we have to show that the associated quasi-ordered set $(Y, \leq)$ is a causal quasi-forest whose non minimal points y have singleton intervals $[y, y]$.

- $(Y, \leq)$ is a quasi-forest.

  Let $z \in Y$ and y_1, y_2 be two elements in $\downarrow z$. By the surjectivity of q, there exists $x \in X$ such that $z = q(x)$. According to Proposition 2.4, we have $q(\downarrow x) = \downarrow q(x)$. So there exists $x_1, x_2 \in \downarrow x$ such that $y_1 = q(x_1)$ and $y_2 = q(x_2)$. The primality of X gives $x_1 \leq x_2$ or $x_2 \leq x_1$ and consequently either $q(x_1) \leq q(x_2)$ or $q(x_2) \leq q(x_1)$. Therefore, $\downarrow z$ is totally quasi-ordered.
- $(Y, \leq)$ is causal.

  Let $y_1, y_2 \in Y$. If $y_1 \uparrow \cap \downarrow y_2$ is empty then it is finite. Otherwise, there exists z an element of this intersection which allows one to claim that $y_1 \leq y_2$. We denote by x_1 (resp. x_2) an antecedent of y_1 (resp. y_2). By Proposition 2.1, it follows that $x_1 \leq x_2$.

  Now, we show that $q(x_1 \uparrow \cap \downarrow x_2) = y_1 \uparrow \cap \downarrow y_2$.
  In fact, the continuity of q gives the first inclusion.
  Conversely, since $x_1 \uparrow \cap \downarrow x_2$ is a locally closed subset of X and q is a quasihomeomorphism then there exists a locally closed subset L of Y satisfying $x_1 \uparrow \cap \downarrow x_2 = q^{-1}(L)$. Now, by the surjectivity of q, $q(x_1 \uparrow \cap \downarrow x_2)$ is locally closed in Y. Let U (resp., F) an open (resp., closed) subset of X such that $L = U \cap F$. It is clear that $q(x_1) \uparrow \subset U$ (resp. $\downarrow q(x_2) \subset F$) so that $q(x_1) \uparrow \cap \downarrow q(x_2) \subset q(x_1 \uparrow \cap \downarrow x_2)$.

  Finally, remark that $x_1 \uparrow \cap \downarrow x_2$ is finite which implies that $y_1 \uparrow \cap \downarrow y_2$ is finite.

- Every non minimal point y of Y has a singleton interval $[y, y]$.
  Let $y \in Y \setminus Min(Y)$, using Proposition 2.4.(2), there exists $x \in X \setminus Min(X)$ such that $y = q(x)$.
  By the primality of X, we get $q(\downarrow x \cap x \uparrow) = q(\{x\})$.
  Therefore, $\downarrow y \cap y \uparrow = \{y\}$.

□

Now, in this context, some observations are presented by the following examples.

Examples 3.3.

(1) The following example shows that the surjectivity of q in Theorem 3.2 is necessary.
Indeed, let α, β be two distinct points and set $X = \{\alpha, \beta\}$ equipped with the indiscrete topology.
On the other hand, let Y be an infinite set equipped with the indiscrete topology and define $q : X \to Y$ by $q(\alpha) = q(\beta) = y$ with y an arbitrary element of Y.
We can easily see that X is a primal space and q is a non onto quasihomeomorphism.
However, Y is a principal space which is not primal since it is not causal ($[x, x] = Y$ infinite for every $x \in Y$).

(2) The converse of Theorem 3.2 does not hold. To see this, consider $X = \{\alpha, \beta\}$ and $Y = \{0, 1, 2\}$ both equipped with the indiscrete topology.
Now, let $q : X \longrightarrow Y$ defined by $q(\alpha) = q(\beta) = a$. Therefore, q is a quasihomeomorphism which is not onto despite of the primality of X and Y.

(3) Let X be the set $\{0, 1, 2\}$ and Y the set $\{a, b\}$ with $a \neq b$, both equipped with the indiscrete topology τ. Set f from X to itself by $f(0) = 1, f(1) = 0$ and $f(2) = 0$.
Clearly the map q from X to Y, defined by $q(0) = a$ and $q(1) = q(2) = b$, is an onto quasihomeomorphism. Now, there is a unique map g from Y to itself that satisfies $(Y, \tau) = (Y, \mathcal{P}(f))$; it is defined by $g(a) = b$ and $g(b) = a$.
We can see easily that $g \circ q \neq q \circ f$ and consequently q is not a morphism of flows from (X, f) to (Y, g).

(4) Let (X, f) as in (3) and $q : X \longrightarrow X$ the identity map. Clearly, there is exactly two maps g_1 and g_2 from X to itself such that $(X, \tau) = (X, \mathcal{P}(g_i)), i \in \{1, 2\}$.
If we choose $g = g_1$, we have q is a morphism of flows from (X, f) to (X, g) but not if we choose $g = g_2$.

Before giving an interesting consequence of the previous Theorem, we recall the T_0-reflection of a topological space.

Given a topological space X, we define the equivalence relation on X by

$$x \sim y \text{ if and only if } \overline{\{x\}} = \overline{\{y\}}.$$

The resulting quotient space $X/_\sim$ is a T_0-space called the T_0-reflection of X and the following properties hold:

- The canonical onto map $\mu_X : X \longrightarrow T_0(X)$ is a quasihomeomorphism ([8]).
- X is a principal space if and only if $T_0(X)$ is also.

Corollary 3.4. *The T_0-reflection of a primal space is primal.*

One may see by example 3.3.(4) that we have to impose additional conditions in order to render the quasihomeomorphism cited in Theorem 3.2 a morphism of flows from (X, f) to (Y, g).

Thus, we provide the following proposition.

Proposition 3.5. *Let (X, f) (resp., (Y, g)) be a flow in* **Set**. *We equip (X, f) (resp., (Y, g)) with the topology $\mathcal{P}(f)$ (resp., $\mathcal{P}(g)$) and let $q : X \longrightarrow Y$ be a quasihomeomorphism.*

If Y is a T_0-space, then q is a morphism of flows from (X, f) to (Y, g).

Proof. According to Proposition 2.5, we have q is onto.
Let $x \in X$ and we show that $goq(x) = qof(x)$. We denote $y = q(x)$. The surjectivity of q allows us to denote x' an antecedent of $g(y)$ with $x' \leq x$ since $q(x') \leq q(x)$.

<u>First</u> <u>case :</u> $x \in Min(X)$.

We have: $x' \leq x \Longrightarrow x \leq x'$. Which leads to $f(x) \leq x'$ so that $q(f(x)) \leq q(x')$ (1). On the other hand, since x is minimal we have also $x \leq f(x)$. So $x' \leq f(x)$ which gives $q(x') \leq q(f(x))$ (2). Now, we have from (1) and (2) $\overline{\{q(x')\}} = \overline{\{q(f(x))\}}$. Since Y is T_0 then $q(x') = g(q(x)) = q(f(x))$ as desired.

<u>Second</u> <u>case :</u> $x \notin Min(X)$.

In this case, we have $(\downarrow x) \setminus \{x\} = \downarrow f(x)$. So, either $x' = x$ or $x' \in \downarrow f(x)$. We start with studying the problem when $x' = x$. In that case, $q(x) = q(x') = g(q(x))$ which means that $\overline{\{q(x)\}} = \{q(x)\}$. Now, $f(x) \leq x \Rightarrow q(f(x)) \leq q(x) \Rightarrow q(f(x)) \in \overline{\{q(x)\}}$ which implies that $q(f(x)) = q(x) = g(q(x))$ as desired.

Therefore, we give attention now to the case when $x' \in \downarrow f(x)$. Indeed, in that case, $q(x') \leq q(f(x))$ $(*)$. If $q(f(x)) = y$, then we have $q(f(x)) \leq q(x)$ and $q(x) \leq q(f(x))$. By Proposition 2.1, $f(x) \leq x$ and $x \leq f(x)$. Yet, since $x \notin Min(X)$ then $f(x) = x$. Which means that $\overline{\{x\}} = \{x\}$ and thus $x' = x$. This leads to $g(q(x)) = q(x') = q(x) = q(f(x))$, as desired.

Otherwise, if $q(f(x)) \neq y$, then $q(f(x)) \in (\downarrow y) \setminus \{y\}$. Using Proposition 2.4.(2), we have $y \notin Min(Y)$, so that $(\downarrow y) \setminus \{y\} = \downarrow g(y)$. Thus, $q(f(x)) \leq g(y)$ $(**)$

Now, it follows from $(*)$ and $(**)$ that $\overline{\{q(f(x))\}} = \overline{\{g(y)\}}$. Since Y is T_0 then $g(q(x)) = g(y) = q(f(x))$. Which completes the proof.

□

We state a useful remark.

Remark 3.6. The condition "Y is a T_0-space" in Proposition 3.5 is a sufficient condition but not a necessary condition. To see this, consider the example 3.3.(4).

4. Primal spaces and one-to-one quasihomeomorphisms

In this section, our interest is directed towards the characterization of quasihomeomorphisms $q : X \to Y$ that render X primal when Y is.

Thus, we present the following result.

Theorem 4.1. *Let (X, τ) be a principal space, $(Y, P(g))$ a primal space and $q : X \longrightarrow Y$ a quasihomeomorphism.*
If q is one-to-one, then X is primal.

Proof. Let $\leq$ be the specialization quasi-order on X.

- $(X, \leq)$ is a quasi-forest.
  Let $x \in X$ and $x_1, x_2 \in \downarrow x$. Then, $q(x_1), q(x_2) \in \downarrow q(x)$. Since Y is primal, we have either $q(x_1) \leq q(x_2)$ or $q(x_2) \leq q(x_1)$. Using Proposition 2.1, this leads to $x_1 \leq x_2$ or $x_2 \leq x_1$.
- $(X, \leq)$ is causal.
  Let $x_1, x_2 \in X$. Since q is one-to-one then the cardinal of $[x_1, x_2]$ is equal to the cardinal of its image by q. Using Proposition 2.1, we have $q([x_1, x_2]) \subseteq [q(x_1), q(x_2)]$. Now, since Y is primal then $[q(x_1), q(x_2)]$ is finite and so $[x_1, x_2]$ is also.
- $\forall x \in X \setminus Min(X), [x, x] = \{x\}$.
  Let $x \in X \setminus Min(X)$. By Corollary 2.2, $q(x) \in Y \setminus Min(Y)$. Using the primality of Y, we have $[q(x), q(x)] = \{q(x)\}$. Let $z \in [x, x]$, then $q(z) \in [q(x), q(x)]$ which leads to $q(z) = q(x)$. Yet, q is one-to-one. So, $z = x$. Thus, $[x, x] = \{x\}$.

Using Theorem 3.1, we conclude that X is a primal space. □

Now, we give some straightforward remarks.

Remarks 4.2.

(1) The injectivity of the quasihomeomorphism q cited in Theorem 4.1 is necessary to conclude that X is primal. To see this, consider $X = \mathbb{N}$, $Y = \{\alpha, \beta\}$ both equipped with the indiscrete topology and $q : X \to Y$ such that $q(0) = \alpha$ and $q(\mathbb{N}^*) = \{\beta\}$.
(2) The converse of Theorem 4.1 fails. Indeed, consider the example cited in Remark 3.3.(2).

Next, we will localize our interest to the consequences of the previous Theorem.

Recall that a set F of a topological space X is said to be *irreducible* if for each open sets U and V of X such that $F \cap U \neq \varnothing$ and $F \cap V \neq \varnothing$, we have

$F \cap U \cap V \neq \varnothing$ (equivalently, if C_1 and C_2 are two closed sets of X such that $F \subseteq C_1 \cup C_2$, then $F \subseteq C_1$ or $F \subseteq C_2$).

A topological space X is called *sober* if each nonempty irreducible closed set F of X has a unique generic point(i.e there exists a unique $x \in X$ such that $F = \overline{\{x\}}$).

Let $S(X)$ be the set of all nonempty irreducible closed sets of X. Let U be an open set of X; set $\widetilde{U} = \{C \in S(X) : U \cap C \neq \varnothing\}$. Then, the collection $\{\widetilde{U} : U$ is an open set of $X\}$ provides a topology on $S(X)$ and the following properties hold:

- The map $\theta_X : X \to S(X)$ which carries $x \in X$ to $\theta_X(x) = \overline{\{x\}}$ is a quasihomeomorphism.
- $S(X)$ is a sober space.
- Let $f : X \longrightarrow Y$ be a continuous map. Let $\mathbf{S}(f) : \mathbf{S}(X) \longrightarrow \mathbf{S}(Y)$ be the map defined by $\mathbf{S}(f)(C) = \overline{C}$, for each irreducible closed subset C of X. Then $\mathbf{S}(f)$ is continuous.
- The topological space $S(X)$ is called the *sobrification* of X, and the assignment $S(X)$ defines a functor from the category of topological spaces to itself.
- Let $f : X \longrightarrow Y$ be a continuous map. Then, the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \theta_X \downarrow & \circlearrowright & \downarrow \theta_Y \\ \mathbf{S}(X) & \xrightarrow{\mathbf{S}(f)} & \mathbf{S}(Y) \end{array}$$

  is commutative.

In [2], the author has proved that $S(f)$ is a homeomorphism if and only if f is a quasihomeomorphism. Now, since $\mu_X : X \longrightarrow T_0(X)$ is a quasihomeomorphism then $S(\mu_X)$ is a homeomorphism and consequently $S(T_0(X))$ is homeomorphic to $S(X)$. This result allows one to present the following corollary.

Corollary 4.3. *Let X be a topological space. If $S(X)$ is a primal space then $T_0(X)$ is primal.*

Proof. This follows immediately from Theorem 4.1 using the one-to-one quasihomeomorphism $\theta_{T_0(X)} : T_0(X) \to S(T_0(X)), x \mapsto \overline{\{x\}}$ and considering that $S(T_0(X))$ is homeomorphic to $S(X)$. □

Now, Proposition 3.5 motivates the following question:
Suppose that $q : (X, \mathcal{P}(f)) \longrightarrow (Y, \mathcal{P}(g))$ is a quasihomeomorphism between two primal spaces. Does the condition "X is a T_0-space" allows one to claim that q is morphism of flows from (X, f) to (Y, g) ?

The following example gives the answer.

Example 4.4. Let $X = \{\alpha, \beta\}$ with $\alpha \neq \beta$ and $Y = \{0, 1, 2, 3\}$. Set f (resp., g) from X (resp., Y) to itself by $f(\alpha) = \beta$ and $f(\beta) = \beta$ (resp., $g(0) = 1, g(1) = 2, g(2) = 3$ and $g(3) = 1$). The quasihomeomorphism defined by $q(\alpha) = 0$ and

$q(\beta) = 2$ is not a morphism of flows from (X, f) to (Y, g). Although X is a T_0- space.

ACKNOWLEDGEMENTS. *The authors gratefully acknowledge the many helpful corrections, comments and suggestions of the anonymous referee.*

REFERENCES

[1] K. Belaid, O. Echi and S. Lazaar, $T_{(\alpha,\beta)}$*-spaces and the Wallman Compactification*, Int. J. Math. Math. Sc. **68** (2004), 3717–3735.

[2] O. Echi, *Quasi-homeomorphisms, Goldspectral spaces and Jacspectral spaces*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (**8**)**6** (2003), 489–507.

[3] O. Echi, *The category of flows of Set and Top*, Topology Appl. **159** (2012), 2357–2366.

[4] J. F. Kennisson, *The cyclic spectrum of a Boolean flow*, Theory Appl. Categ. **10** (2002), 392–409.

[5] J. F. Kennisson, *Spectra of finitly generated Boolean flow*, Theory Appl. Categ. **16** (2006), 434–459.

[6] A. Grothendieck and J. Dieudonné, *Eléments de Géométrie Algébrique*, Die Grundlehren der mathematischen Wissenschaften, vol. **166**, Springer-Verlag, New York, 1971.

[7] A. Grothendieck and J. Dieudonné, *Eléments de Géométrie Algébrique. I. Le langage des schḿas*, Inst. Hautes Études Sci. Publ. Math. No. **4**, 1960.

[8] M. H. Stone, *Applications of Boolean algebra to topology*, Mat. Sb. **1** (1936), 765–772.

[9] K. W. Yip, *Quasi-homeomorphisms and lattice-equivalences of topological spaces*, J. Austral. Math. Soc. **14** (1972), 41–44.

[10] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove and D. S. Scott, *Continuous Lattices and Domains*, Cambridge Univ, Press, 2003.