Laguerre random polynomials: definition, differential and statistical properties*

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Abstract

In this paper we introduce the Laguerre polynomials as mean square solutions of random differential equations. The study is based on the construction of an infinite random power series solution which becomes a random polynomial under certain conditions to be satisfied by the single involved random coefficient, denoted by A. This approach allows us to introduce the concept of Laguerre polynomials associated to the random variable A retaining their deterministic definition when the probability mass of A is concentrated in a non-negative integer. As a result, we provide a natural way to extend the deterministic Laguerre polynomials to the random framework. In addition, the main statistical functions of the approximate solution stochastic process obtained by truncation of the exact power series solution, which generates random Laguerre polynomials, are also given. Several illustrative examples are provided.

Keywords: Laguerre random polynomials, random differential equation, mean square and mean fourth calculus.

*This work has been partially supported by the Spanish M.C.Y.T. grants MTM2009-08587, DPI2010–20891–C02–01, Universitat Politècnica de València grant PAID06-11-2070 and Mexican CONACYT.

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Utilitas Mathematica **98**(2015), pp. 283-293

1 Motivation and preliminaries

In paper [1] authors studied the Legendre random differential equation (r.d.e.) by applying mean square (m.s.) and mean fourth (m.f.) calculus [2]. A description of its solution stochastic process (s.p.) was provided including an approximation of its main statistical functions such as the average and standard deviation. However, in that contribution any attention was paid to obtain (random) polynomial solutions as it is usually done in its deterministic counterpart [3]. This aims us to obtain the random polynomial solutions of the Laguerre r.d.e.

$$t\ddot{X}(t) + (1-t)\dot{X}(t) + AX(t) = 0, \quad t > 0, \tag{1}$$

where A is assumed to be a random variable (r.v.) which absolute moments with respect to the origin increase at the most exponentially, i.e., we suppose that there exist a nonnegative integer n_0 and positive constants H and M such that

$$\mathrm{E}\left[|A|^n\right] \leq H\, M^n < +\infty, \quad \forall n \geq n_0. \tag{2}$$

Notice that it is equivalent to assume that $\mathrm{E}\left[|A|^n\right] = \mathcal{O}(M^n)$. In the deterministic framework, the coefficient A that multiplies the unknown $X(t)$ in (1) is just a deterministic parameter, say a. The polynomial solutions of Laguerre deterministic differential equations, usually referred to as Laguerre polynomials, play a relevant role in the solution of physical problems. For instance, they appear in Quantum Mechanics in dealing with the study of the hydrogen atom. Specifically, they allow to represent the so-called radial function when solving the Schrödinger's equation. The radial function involves the Bohr radius which in turns depends on the mass of the nucleus. This crucial information is related to coefficient a. Although in practice, this mass has to be measured by means of high-precision methods, it could involve measure errors. As a consequence, it would be more realistic to consider a as a r.v. rather than a deterministic parameter. This leads us to consider the r.d.e. (1). Further scientific examples from which the randomness can also be considered in an analogous way as we have done to motivate the study of the Laguerre random polynomials from the deterministic context, can be found in [4] and in the references therein.

As in the paper [1], we will also require m.s. and m.f. stochastic calculus to develop our study. Given $(\Omega, \mathcal{F}, P)$ a probability space, this means that we will work in the Banach spaces L^p endowed with the norms

$$\|X\|_p = \left(\mathrm{E}\left[X^p\right]\right)^{1/p}, \quad p = 2, 4,$$

whose elements X are second and fourth order real random variables (2-r.v.'s and 4-r.v.'s), respectively. That is, $X : \Omega \to \mathbb{R}$ such that $\mathrm{E}[X^p] <$

$+\infty$, $p = 2, 4$, respectively, where $\mathrm{E}[\cdot]$ denotes the expectation operator. The convergence in each one of these spaces is the one inferred by their respective norms and they are referred to as m.s. and m.f. convergence, respectively. By applying Cauchy-Schwartz inequality, it is straightforward to see that m.f. convergence implies m.s. convergence. The reason to consider both types of convergence (and therefore both norms) is motivated by the necessity of legitimating algebraic operations involving non-dependent r.v.'s. As a basic but still illustrative example, later we will require to use the following basic property $AX_n \xrightarrow[n\to\infty]{\text{m.s.}} AX$, which holds true when $\{X_n : n \geq 0\}$ in L^4 such that $X_n \xrightarrow[n\to\infty]{\text{m.f.}} X$ and $A \in \mathrm{L}^4$ (see Lemma 6 in [1]). Another usual operational situation to be handled is the norm of the product of two r.v.'s, say X, Y. When both r.v.'s are statistically independent one gets $\|XY\|_p = \|X\|_p \|Y\|_p$ and, in the general case where independence cannot be assumed the following inequality is very useful (see Proposition 9 of [1] for $p = 4$)

$$\begin{gathered}\left\|\prod_{i=1}^{n} Y_i\right\|_p \leq \prod_{i=1}^{n} \left(\|(Y_i)^n\|_p\right)^{1/n}, \ n \geq 1, \\ \{Y_i\}_{i=1}^{n} : \mathrm{E}\left[(Y_i)^{np}\right] < \infty, \ 1 \leq i \leq n. \end{gathered} \tag{3}$$

The paper is organized as follows. Section 2 is addressed to construct a m.s. convergent power series solution to (1) under condition (2). The case where A is a discrete r.v. taking a finite number of integer values leads to the concept of Laguerre random polynomials whose definition and computing is shown in Section 2. We close this section by computing the main statistical functions (average and standard deviation) of the truncated random power series solution to (1) previously constructed, and particularly of the Laguerre random polynomials. Finally, some illustrative examples and conclusions are presented in Section 3.

2 Obtaining the random Laguerre polynomials and their main statistical functions

Let us seek a formal power series solution s.p. to problem (1)

$$X(t) = \sum_{n\geq 0} X_n t^n, \tag{4}$$

where coefficients X_n are 2-r.v.'s to be determined. Assuming that $X(t)$ is termwise m.s. differentiable, by Example 3 of [1], and imposing that (4) is a solution to (1), one gets a *formal* power series solution to the Laguerre r.d.e. (1)

$$X(t) = \sum_{n\geq 0} X_0 P_n^A t^n, \quad P_n^A = \prod_{k=1}^{n} \left(\frac{k-1-A}{k^2} \right). \tag{5}$$

Notice that we have implicitly applied the commutation between the r.v. A and the random infinite sum given by (4), whose coefficients X_n, $n \geq 1$, depend on A. Thus, according to Lemma 6 of [1], we must justify the m.f. convergence of random power series defined by (5). By assuming independence between r.v.'s X_0 and A, one follows

$$\sum_{n\geq 0} \left\| X_0 P_n^A \right\|_4 |t|^n = \sum_{n\geq 0} \| X_0 \|_4 \left\| P_n^A \right\|_4 |t|^n . \tag{6}$$

Bearing in mind the definition of P_n^A, under hypothesis (2), we firstly apply inequality (3) and secondly c_s-inequality (see [5, p.157]) for $X = -A$, $Y = k-1$ and $s = 4n$, this yields:

$$\begin{aligned}
\left\| P_n^A \right\|_4 &\leq \prod_{k=1}^{n} \frac{\left(\|(k-1-A)^n\|_4\right)^{1/n}}{k^2} \\
&= \prod_{k=1}^{n} \frac{1}{k^2} \left(\mathrm{E}\left[|k-1-A|^{4n} \right] \right)^{1/4n} \\
&\leq \prod_{k=1}^{n} \frac{1}{k^2} \left(2^{4n-1} \left\{ \mathrm{E}\left[|A|^{4n} \right] + (k-1)^{4n} \right\} \right)^{1/4n} \\
&\leq \frac{\left(2^{4n-1} \left\{ \mathrm{E}\left[|A|^{4n} \right] + (n-1)^{4n} \right\} \right)^{1/4n}}{(n!)^2}.
\end{aligned}$$

Taking into account hypothesis (2), one gets

$$\left\| P_n^A \right\|_4 \leq \frac{\left(2^{4n-1} \left\{ M H^{4n} + (n-1)^{4n} \right\} \right)^{1/(4n)}}{(n!)^2}, \ \forall n \geq n_0.$$

Notice that we can always choose an integer $n_1 \geq n_0 \geq 1$, large enough such that: $(n-1)^{4n} \geq HM^{4n}$ for each $n \geq n_1$, therefore

$$\left\| P_n^A \right\|_4 \leq \frac{2(n-1)}{(n!)^2}, \ \forall n \geq n_1.$$

So, for each t, the series given by (6) can be majorized by

$$\sum_{n\geq n_1} \alpha_n, \quad \alpha_n = \frac{2(n-1)}{(n!)^2} \| X_0 \|_4 |t|^n ,$$

which is convergent on the whole real line as can be directly checked by D'Alembert test. Therefore, random series $X(t)$ given by (5) is m.f. convergent for each t, then by Lemma 5 of [1] it is m.s. convergent. Notice that the previous reasoning also shows that series solution $X(t)$ is m.s. uniformly convergent, therefore taking into account Example 3 of [1] and Theorem 11 of [6], the formal differentiation required to obtain (5) is justified. Summarizing the following result has been established:

Theorem 2.1 *Let us assume that r.v. A satisfies condition (2) and it is independent of r.v. $X_0 = X(0)$. Then the random differential equation (1) admits the power series solution (5) which is mean square convergent for every t.*

Remark 2.2 *Due to the lack of explicitness of the absolute moments of some r.v.'s, condition (2) could be difficult to check in practice. As a consequence, applicability of Theorem 2.1 could seem limited. Fortunately, if A is a r.v. with finite codomain, say $a_1 \leq A(\omega) \leq a_2$, for each $\omega \in \Omega$, condition (2) holds true since*

$$E[|A|^n] = \int_{a_1}^{a_2} |a|^n f_A(a)\, da \leq H^n, \quad \textit{where } H = \max(|a_1|, |a_2|),$$

where $f_A(a)$ denotes the probability density function of the continuous r.v. A. Substituting the integral for a sum, previous conclusion remains true if A is a discrete r.v. Notice that important r.v.'s such as binomial, hypergeometric, uniform or beta have finite codomain. Otherwise, we can take advantage of the so-called truncation method (see [5]) to deal with unbounded r.v.'s such as exponential or Gaussian by censuring adequately their codomain. This approximation can be further improved by enlarging enough the truncated codomain (see second part of Example 3.1).

Next, we address to introduce the Laguerre polynomials in the random framework. First, we recall the following

Definition 2.3 *Given a collection of r.v.'s $\{X_k : k \geq 0\}$ and $t \in \mathcal{T}$, $\sum_{k \geq 0} X_k t^k$ is called a random power series. If $P[\{\omega \in \Omega : X_k(\omega) = 0, \forall k > m\}] = 1$, then $\sum_{k=0}^{m} X_k t^k$ is said to be a random polynomial in t of degree m.*

Taking into account previous development, this will be made in a natural way by following an analogous approach to its deterministic counterpart. Therefore, we are going to present the Laguerre random polynomials as finite series solutions to r.d.e. (1). First at all, notice that from (5) one deduces that if there exists a non-negative integer n such that $\mathrm{P}[A = n] = 1$, that is, A is a (degenerate) discrete r.v. whose total probability mass is concentrated in the non-negative integer value n, then the r.d.e. (1) has a (random) polynomial solution. For every n, each (random degenerate) solution can be interpreted as the correspondent Laguerre polynomial of degree n that one presents in the deterministic scenario. However, in the random framework one appears further situations that deserve to be considered. In fact, let us assume that A is a continuous r.v., taking into account that $\mathrm{P}[A = n] = 0$ for every integer $n \geq 0$, then with probability 1, one can conclude that there are not random polynomial solutions to r.d.e. (1). Whereas if A is a non-negative discrete r.v. that takes different integer values, then there will exist with probability 1, random polynomial solutions (see comments in this issue in Remark 2.6). This situation generalizes the concept of Laguerre-type polynomial solution from the deterministic framework (note that this situation contains the previous one where A was a non-negative degenerated r.v. concentrated in an integer value). Previous considerations motivate the following result:

Corollary 2.4 *Let us consider the r.d.e. (1) where the r.v. A takes only a finite number of non-negative integer values, $0 \leq n \leq N < \infty$, that is, $\mathrm{P}[A = n] = p_n \geq 0$, with $\sum_{n=0}^{N} p_n = 1$. In this context, this r.d.e. (1) has a random polynomial solution $L_N^A(t)$ of degree N given by*

$$L_N^A(t) = \sum_{n=0}^{N} \prod_{k=1}^{n} \left(\frac{k-1-A}{k^2} \right) t^n.$$

Definition 2.5 *In the context of Corollary 2.4, $L_N^A(t)$ is referred to as the Laguerre random polynomial of degree N associated to r.v. A.*

Remark 2.6 *With respect to Definition 2.5 and keeping the notation, it is important to point out that under conditions of Corollary 2.4, a Laguerre random polynomial solution can be interpreted as a collection of deterministic polynomials, which, for each $n : 0 \leq n \leq N$, has a probability p_n of be sampled, being $p_N(t)$ the Laguerre deterministic polynomial of degree N. For a fixed r.v. A, the degree N of the Laguerre random polynomial $L_N^A(t)$ is the greatest of all degrees corresponding to each (deterministic) polynomial. For instance, if A has the probability mass: $p_0 = 1/8$, $p_1 = 1/4$, $p_2 = 1/3$, $p_3 = 7/24$, then each of these values generates the sample polynomials 1, $1 - 3t$, $1 - 3t + (3t^2)/2$ and $1 - 3t + (3t^2)/2 - t^3/6$, respectively. Notice that the last one is just the Laguerre (deterministic) polynomial of degree 3 and it has a probability of $7/24$ of being sampled.*

Remark 2.7 *Let us assume that A is the discrete r.v. taking every non-negative integer value such that $P[A=n]=2^{-(n+1)}$, $n=0,1,2,\ldots$, then for each n one obtains a sample polynomial solution, but the degree of the random polynomial solution cannot be defined according to Definition 2.5.*

Next we address to compute approximations of the average and the standard deviation of the m.s. solution defined by (5) including the particular case in which this series becomes the Laguerre random polynomials. In this latter case, the obtained formulaes will not be approximation but exact. The aforementioned approximation will be expressed in terms of the data $\mathrm{E}[X_0]$, $\mathrm{E}\left[(X_0)^2\right]$ and certain statistical moments related to algebraic transformations of the random coefficient A that will be specified later. Notice that the solution $X(t)$ is an infinite series, then in practice its truncation is demanded to keep computationally feasible. So we will consider the truncation of order M

$$X_M(t)=\sum_{n=0}^{M} X_0 P_n^A t^n, \quad P_n^A=\prod_{k=1}^{n}\left(\frac{k-1-A}{k^2}\right). \tag{7}$$

Assuming that r.v.'s A and $X(0)=X_0$ are independent, then taking the expectation operator in (7) and considering whether r.v. A is discrete, with probability mass function $p_A(a)$, or continuous, with probability density function $f_A(a)$, one gets

$$\mathrm{E}[X_M(t)]=\mathrm{E}[X_0]\sum_{n=0}^{M}\mathrm{E}\left[P_n^A\right]t^n,$$

where

$$\mathrm{E}\left[P_n^A\right]=\begin{cases}\displaystyle\sum_{a:p_A(a)>0}\prod_{k=1}^{n}\left(\frac{k-1-A}{k^2}\right)p_A(a),\\ \displaystyle\int_{-\infty}^{\infty}\prod_{k=1}^{n}\left(\frac{k-1-A}{k^2}\right)f_A(a)\,da.\end{cases}$$

Taking into account that $\mathrm{Var}[X_M(t)]=\mathrm{E}\left[(X_M(t))^2\right]-(\mathrm{E}[X_M(t)])^2$, for computing the variance (or equivalently, the standard deviation) we only need to calculate $\mathrm{E}\left[(X_M(t))^2\right]$. Using again independence between A and X_0, from (7) one gets

$$\mathrm{E}\left[(X_M(t))^2\right]=\mathrm{E}\left[(X_0)^2\right]\left\{\sum_{n=0}^{M}\mathrm{E}\left[\left(P_n^A\right)^2\right]t^{2n}+2\sum_{n=1}^{M}\sum_{m=0}^{n-1}\mathrm{E}\left[P_n^A P_m^A\right]t^{n+m}\right\},$$

and $[0, 10]$, *which contain the* 99.3262% *and* 99.995% *of the total probability mass, respectively. We observe that approximations for averages* ($\mu_{X_M}^{[0,5]}(t)$ *and* $\mu_{X_M}^{[0,10]}(t)$) *and standard deviations* ($\sigma_{X_M}^{[0,5]}(t)$ *and* $\sigma_{X_M}^{[0,10]}(t)$), *for a common truncation order* M *are quite similar, although they improve as the length of the censured interval increases. These values are also compared with the ones provided by Monte Carlo method.*

t	$\mu_{X_{10}}^{[0,5]}(t)$	$\mu_{X_{20}}^{[0,5]}(t)$	$\mu_{X_{10}}^{[0,10]}(t)$	$\mu_{X_{20}}^{[0,10]}(t)$	$\widetilde{\mu}_X^m(t)$, $m = 10^5$
0.00	1	1	1	1	1.00089
0.25	0.770457	0.770457	0.764836	0.764836	0.765381
0.50	0.563023	0.563023	0.556	0.556	0.556532
0.75	0.37497	0.37497	0.369176	0.369176	0.369831
1.00	0.203616	0.203616	0.20047	0.20047	0.201248
2.00	-0.3681728	-0.368172	-0.360765	-0.360765	-0.360953
3.00	-0.894135	-0.894157	-0.887424	-0.887447	-0.892175
4.00	-1.60168	-1.60226	-1.60038	-1.60095	-1.61362
5.00	-2.86326	-2.87074	-2.85681	-2.86424	-2.88683

Table 3: Comparison of the average in Example 3.1 using random power series (taking $[0, 5]$ and $[0, 10]$ as censured intervals) and Monte Carlo methods when $A \sim \text{Exp}(1)$.

t	$\sigma_{X_{10}}^{[0,5]}(t)$	$\sigma_{X_{20}}^{[0,5]}(t)$	$\sigma_{X_{10}}^{[0,10]}(t)$	$\sigma_{X_{20}}^{[0,10]}(t)$	$\widetilde{\sigma}_X^m(t)$, $m = 5 \times 10^5$
0.00	1	1	1	1	0.999512
0.25	0.819237	0.819237	0.819441	0.819441	0.818774
0.50	0.74039	0.74039	0.743921	0.743921	0.743789
0.75	0.726797	0.726797	0.728998	0.728998	0.729442
1.00	0.746931	0.746931	0.746042	0.746042	0.746783
2.00	0.996511	0.996510	0.999257	0.999257	0.998757
3.00	1.67333	1.67334	1.67114	1.67115	1.6699
4.00	3.06809	3.06872	3.06051	3.06114	3.05902
5.00	5.78033	5.79088	5.7667	5.77721	5.77292

Table 4: Comparison of the standard deviation in Example 3.1 using random power series (taking $[0, 5]$ and $[0, 10]$ as censured intervals) and Monte Carlo methods when $A \sim \text{Exp}(1)$.

In this paper we have shown that mean square and fourth calculus constitute powerful tools to introduce random Laguerre polynomials as solutions of random differential equation (1). Our approach permits to extend the Laguerre polynomials to the random scenario retaining its deterministic definition as a particular case. In addition, we have computed their main statistical functions such as the average and the standard deviation.

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