BOHR’S ABSOLUTE CONVERGENCE PROBLEM FOR $\mathcal{H}_p$-DIRICHLET SERIES
IN BANACH SPACES

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The Bohr–Bohnenblust–Hille theorem states that the width of the strip in the complex plane on which an ordinary Dirichlet series $\sum_n a_n n^{-s}$ converges uniformly but not absolutely is less than or equal to $\frac{1}{2}$, and this estimate is optimal. Equivalently, the supremum of the absolute convergence abscissas of all Dirichlet series in the Hardy space $\mathcal{H}_\infty$ equals $\frac{1}{2}$. By a surprising fact of Bayart the same result holds true if $\mathcal{H}_\infty$ is replaced by any Hardy space $\mathcal{H}_p$, $1 \leq p < \infty$, of Dirichlet series. For Dirichlet series with coefficients in a Banach space $X$ the maximal width of Bohr’s strips depend on the geometry of $X$; Defant, García, Maestre and Pérez-García proved that such maximal width equals $1 - \frac{1}{\text{Cot} X}$, where $\text{Cot} X$ denotes the maximal cotype of $X$. Equivalently, the supremum over the absolute convergence abscissas of all Dirichlet series in the vector-valued Hardy space $\mathcal{H}_\infty(X)$ equals $1 - \frac{1}{\text{Cot} X}$. In this article we show that this result remains true if $\mathcal{H}_\infty(X)$ is replaced by the larger class $\mathcal{H}_p(X)$, $1 \leq p < \infty$.

1. Main result and its motivation

Given a Banach space $X$, an ordinary Dirichlet series in $X$ is a series of the form $D = \sum_n a_n n^{-s}$, where the coefficients $a_n$ are vectors in $X$ and $s$ is a complex variable. Maximal domains where such Dirichlet series converge conditionally, uniformly or absolutely are half planes $\{\text{Re} > \sigma\}$, where $\sigma = \sigma_c, \sigma_u$ or $\sigma_a$ are called the abscissa of conditional, uniform or absolute convergence, respectively. More precisely, $\sigma_\alpha(D)$ is the infimum of all $r \in \mathbb{R}$ such that on $\{\text{Re} > r\}$ we have convergence of $D$ of the requested type $\alpha = c, u$ or $a$. Clearly, we have $\sigma_c(D) \leq \sigma_u(D) \leq \sigma_a(D)$, and it can be easily shown that $\sup \sigma_a(D) - \sigma_c(D) = 1$, where the supremum is taken over all Dirichlet series $D$ with coefficients in $X$. To determine the maximal width of the strip on which a Dirichlet series in $X$ converges uniformly but not absolutely is more complicated. The main result of [Defant et al. 2008] states, with the notation given below, that

$$ S(X) := \sup \sigma_a(D) - \sigma_u(D) = 1 - \frac{1}{\text{Cot} X}. $$

Recall that a Banach space $X$ is of cotype $q$, $2 \leq q < \infty$, whenever there is a constant $C \geq 0$ such that for each choice of finitely many vectors $x_1, \ldots, x_N \in X$ we have

$$ \left( \sum_{k=1}^N \|x_k\|_X^q \right)^{1/q} \leq C \left( \int_{X^n} \left\| \sum_{k=1}^N x_k z_k \right\|_X^2 \, dz \right)^{1/2}, $$

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where $\mathbb{T} := \{ z \in \mathbb{C} \mid |z| = 1 \}$ and $\mathbb{T}^N$ is endowed with the $N$-th product of the normalized Lebesgue measure on $\mathbb{T}$; we denote the best of such constants $C$ by $C_q(X)$. As usual we write

$$\text{Cot} X := \inf\{ 2 \leq q < \infty \mid X \text{ is of cotype } q \},$$

and, although this infimum in general is not attained, we call it the optimal cotype of $X$. If there is no $2 \leq q < \infty$ for which $X$ has cotype $q$, then $X$ is said to have no finite cotype, and we put $\text{Cot} X = \infty$. To see an example,

$$\text{Cot} \ell_q = \begin{cases} q & \text{for } 2 \leq q \leq \infty, \\ 2 & \text{for } 1 \leq q \leq 2. \end{cases}$$

The scalar case $X = \mathbb{C}$ in (1) was first studied over a hundred years ago: Bohr [1913a] proved that $S(\mathbb{C}) \leq \frac{1}{2}$, and Bohnenblust and Hille [1931] that $S(\mathbb{C}) \geq \frac{1}{2}$. Clearly, the equality

$$S(\mathbb{C}) = \frac{1}{2}, \quad (3)$$

nowadays called the \textit{Bohr–Bohnenblust–Hille theorem}, fits with (1). Let us give a second formulation of (1). Define the vector space $\mathcal{H}_\infty(X)$ of all Dirichlet series $D = \sum_n a_n n^{-s}$ in $X$ such that

- $\sigma_c(D) \leq 0$,
- the function $D(s) = \sum_n a_n (1/n^s)$ on $\text{Re } s > 0$ is bounded.

Then $\mathcal{H}_\infty(X)$ together with the norm

$$\|D\|_{\mathcal{H}_\infty(X)} = \sup_{\text{Re } s > 0} \left\| \sum_{n=1}^{\infty} a_n \frac{1}{n^s} \right\|_X$$

forms a Banach space. For any Dirichlet series $D$ in $X$ we have

$$\sigma_u(D) = \inf \left\{ \sigma \in \mathbb{R} \mid \sum_n a_n \frac{1}{n^\sigma n^s} \in \mathcal{H}_\infty(X) \right\}, \quad (4)$$

In the scalar case $X = \mathbb{C}$, this is (what we call) \textit{Bohr’s fundamental theorem} [1913b], and for Dirichlet series in arbitrary Banach spaces the proof follows similarly. Together with (4) a simply translation argument gives the following reformulation of (1):

$$S(X) = \sup_{D \in \mathcal{H}_\infty(X)} \sigma_u(D) = 1 - \frac{1}{\text{Cot } X}. \quad (5)$$

Following an ingenious idea of Bohr each Dirichlet series may be identified with a power series in infinitely many variables. More precisely, fix a Banach space $X$ and denote by $\mathfrak{P}(X)$ the vector space of all formal power series $\sum \alpha c_\alpha z^\alpha$ in $X$ and by $\mathfrak{D}(X)$ the vector space of all Dirichlet series $\sum_n a_n n^{-s}$ in $X$. Let as usual $(p_n)_n$ be the sequence of prime numbers. Since each integer $n$ has a unique prime
number decomposition \( n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} = p^\alpha \) with \( \alpha_j \in \mathbb{N}_0 \), \( 1 \leq j \leq k \), the linear mapping

\[
\mathcal{B}_X : \mathfrak{H}(X) \to \mathcal{D}(X), \quad \sum_{\alpha \in \mathbb{N}_0^k} c_\alpha z^\alpha \rightarrow \sum_{n=1}^{\infty} a_n n^{-s} \text{ if } a_p = c_\alpha,
\]

is bijective; we call \( \mathcal{B}_X \) the Bohr transform in \( X \). As discovered by Bayart [2002] this (a priori very) formal identification allows us to develop a theory of Hardy spaces of scalar–valued Dirichlet series.

Similarly, we now define Hardy spaces of \( X \)-valued Dirichlet series. Denote by \( dw \) the normalized Lebesgue measure on the infinite-dimensional polytorus \( \mathbb{T}^\infty = \prod_{k=1}^{\infty} \mathbb{T} \), that is, the countable product measure of the normalized Lebesgue measure on \( \mathbb{T} \). For any multiindex \( \alpha = (\alpha_1, \ldots, \alpha_n, 0, \ldots) \in \mathbb{Z}^{(n)} \) (all finite sequences in \( \mathbb{Z} \)) the \( \alpha \)-th Fourier coefficient \( \hat{f}(\alpha) \) of \( f \in L_1(\mathbb{T}^\infty, X) \) is given by

\[
\hat{f}(\alpha) = \int_{\mathbb{T}^\infty} f(w) w^{-\alpha} \, dw,
\]

where we as usual write \( w^\alpha \) for the monomial \( w_1^{\alpha_1} \cdots w_n^{\alpha_n} \). Then, given \( 1 \leq p < \infty \), the \( X \)-valued Hardy space on \( \mathbb{T}^\infty \) is the subspace of \( L_p(\mathbb{T}^\infty, X) \) defined as

\[
H_p(\mathbb{T}^\infty, X) = \{ f \in L_p(\mathbb{T}^\infty, X) \mid \hat{f}(\alpha) = 0 \text{ for all } \alpha \in \mathbb{Z}^{(n)} \setminus \mathbb{N}_0^{(n)} \}.
\]

Assigning to each \( f \in H_p(\mathbb{T}^\infty, X) \) its unique formal power series \( \sum_\alpha \hat{f}(\alpha) z^\alpha \) we may consider \( H_p(\mathbb{T}^\infty, X) \) as a subspace of \( \mathfrak{H}(X) \). We denote the image of this subspace under the Bohr transform \( \mathcal{B}_X \) by

\[
\mathcal{H}_p(X).
\]

This vector space of all (so-called) \( \mathcal{H}_p(X) \)-Dirichlet series \( D \) together with the norm

\[
\| D \|_{\mathcal{H}_p(X)} = \| \mathcal{B}_X^{-1}(D) \|_{H_p(\mathbb{T}^\infty, X)}
\]

forms a Banach space; in other words, through Bohr’s transform \( \mathcal{B}_X \) from (6) we by definition identify

\[
\mathcal{H}_p(X) = H_p(\mathbb{T}^\infty, X), \quad 1 \leq p < \infty.
\]

For \( p = \infty \) we this way of course could also define a Banach space \( \mathcal{H}_\infty(X) \), and it turns out that at least in the scalar case \( X = \mathbb{C} \) this definition then coincides with the one given above; but we remark that these two \( \mathcal{H}_\infty(X) \)’s are different for arbitrary \( X \). It is important to note that by the Birkhoff–Khinchin ergodic theorem the following internal description of the \( \mathcal{H}_p(X) \)-norm for finite Dirichlet polynomials \( D = \sum_k^{a_k} a_k k^{-s} \) holds:

\[
\| D \|_{\mathcal{H}_p(X)} = \lim_{T \to \infty} \left( \frac{1}{2T} \int_{-T}^{T} \left\| \sum_{k=1}^{n} a_k t^k \right\|_X^p \, dt \right)^{1/p}
\]

(see, for example, Bayart [2002] for the scalar case, and the vector-valued case follows exactly the same way).
Motivated by (4) we define for $D \in \mathcal{D}(X)$ and $1 \leq p < \infty$

$$\sigma_{\mathcal{H}_p(X)}(D) := \inf \left\{ \sigma \in \mathbb{R} \left| \sum_{n} a_n \frac{1}{n^\sigma} \in \mathcal{H}_p(X) \right. \right\}, \quad (8)$$

the so-called $\mathcal{H}_p(X)$-abscissa of $D$. In [Aleman et al. ≥ 2014], Aleman, Olsen, and Saksman prove that the sequence (of Dirichlet series) $1/n^\sigma$, $n \in \mathbb{N}$ is a Schauder basis in $\mathcal{H}_p(\mathbb{C})$ for $1 < p < \infty$. Hence, for $1 < p < \infty$ and any Dirichlet series $D \in \mathcal{D}(\mathbb{C})$ we have

$$\sigma_{\mathcal{H}_p(\mathbb{C})}(D) = \inf \left\{ \sigma \in \mathbb{R} \left| \left( \sum_{n=1}^{N} a_n \frac{1}{n^\sigma} \right)_N \right. \right\}, \quad (9)$$

which (in the scalar case) is the perfect analog of Bohr’s fundamental theorem (i.e., the case $p = \infty$ from (4), where uniform convergence is precisely being Cauchy in $\mathcal{H}_p(\mathbb{C})$). In [Defant 2013] it is shown that (9) also holds true for $p = 1$ (although in this case the $1/n^\sigma$ are definitely no Schauder basis in $\mathcal{H}_1(\mathbb{C})$), and even more: The arguments given in [Defant 2013] (inspired by Bohr’s original ideas [1913b]) prove that (9) even holds for any $1 \leq p \leq \infty$ and any $X$-valued Dirichlet series $D \in \mathcal{H}_p(X)$. In view of (1) and (5), it therefore seems natural to study

$$S_p(X) := \sup_{D \in \mathcal{D}(X)} \sigma_a(D) - \sigma_{\mathcal{H}_p(X)}(D) = \sup_{D \in \mathcal{H}_p(X)} \sigma_a(D)$$

(for the second equality use again a simple translation argument). The scalar case is completely understood since, by a result of Bayart [2002],

$$S_p(\mathbb{C}) = 1 - \frac{1}{\operatorname{Cot} X}, \quad (10)$$

which according to Helson [2005] is surprising since $\mathcal{H}_\infty(\mathbb{C})$ is much smaller than $\mathcal{H}_p(\mathbb{C})$.

The following theorem unifies and generalizes (1), (3) as well as (10), and it is our main result.

**Theorem 1.1.** For every $1 \leq p \leq \infty$ and every Banach space $X$ we have

$$S_p(X) = 1 - \frac{1}{\operatorname{Cot} X}.$$ 

The proof will be given in Section 3. But before we start let us give an interesting reformulation in terms of the monomial convergence of $X$-valued $H_p$-functions on $\mathbb{T}^\infty$. Fix a Banach space $X$ and $1 \leq p \leq \infty$, and define the set of monomial convergence of $H_p(\mathbb{T}^\infty, X)$:

$$\text{mon } H_p(\mathbb{T}^\infty, X) = \left\{ z \in B_{c_0} \mid \sum_{\alpha} \| \hat{f}(\alpha) z^\alpha \|_X < \infty \text{ for all } f \in H_p(\mathbb{T}^\infty, X) \right\}.$$ 

Philosophically, this is the largest set $M$ on which for each $f \in H_p(\mathbb{T}^\infty, X)$ the definition $g(z) = \sum_{\alpha} \hat{f}(\alpha) z^\alpha$, $z \in M$ leads to an extension of $f$ from the distinguished boundary $\mathbb{T}^\infty$ to its “interior” $B_{c_0}$ (the open unit ball of the Banach space $c_0$ of all null sequences). For a detailed study of sets of monomial convergence in the scalar case $X = \mathbb{C}$ see [Defant et al. 2009], and in the vector-valued case [Defant and Sevilla-Peris 2011].
We later need the following two basic properties of monomial domains (in the scalar case see [Defant et al. 2008, p. 550; 2014, Lemma 4.3], and in the vector-valued case the proofs follow similar lines).

**Remark 1.2.** (1) Let \( z \in \text{mon } H_p(\mathbb{T}^\infty, X) \). Then \( u = (z_{\sigma(n)})_n \in \text{mon } H_p(\mathbb{T}^\infty, X) \) for every permutation \( \sigma \) of \( \mathbb{N} \).

(2) Let \( z \in \text{mon } H_p(\mathbb{T}^\infty, X) \) and \( x = (x_n)_n \in \mathbb{D}^\infty \) be such that \( |x_n| \leq |z_n| \) for all but finitely many \( n \)'s. Then \( x \in \text{mon } H_p(\mathbb{T}^\infty, X) \).

Given \( 1 \leq p \leq \infty \) and a Banach space \( X \), the following number measures the size of \( \text{mon } H_p(\mathbb{T}^\infty, X) \) within the scale of \( \ell_r \)-spaces:

\[
M_p(X) = \sup \{ 1 \leq r \leq \infty | \ell_r \cap B_{c_0} \subset \text{mon } H_p(\mathbb{T}^\infty, X) \}. 
\]

The following result is a reformulation of Theorem 1.1 in terms of vector-valued \( H_p \)-functions on \( \mathbb{T}^\infty \) through Bohr’s transform \( \mathfrak{B}_X \). The proof is modeled along ideas from Bohr’s seminal article [1913a, Satz IX].

**Corollary 1.3.** For each Banach space \( X \) and \( 1 \leq p \leq \infty \) we have

\[
M_p(X) = \frac{\text{Cot } X}{\text{Cot } X - 1}.
\]

**Proof.** We are going to prove that \( S_p(X) = 1/M_p(X) \), and as a consequence the conclusion follows from Theorem 1.1. We begin by showing that \( S_p(X) \leq 1/M_p(X) \). We fix \( q < M_p(X) \) and \( r > 1/q \); then we have that \((1/p_n^q)_n \in \ell_q \cap B_{c_0} \) and, by the very definition of \( M_p(X) \), \( \sum \alpha \| \hat{f}(\alpha)(1/p^r)^\alpha \|_X < \infty \) converges absolutely for every \( f \in H_p(\mathbb{T}^\infty, X) \). We choose now an arbitrary Dirichlet series

\[
D = \mathfrak{B}_X f = \sum_n a_n n^{-s} \in \mathfrak{H}_p(X) \quad \text{with } f \in H_p(\mathbb{T}^\infty, X).
\]

Then

\[
\sum_n \| a_n \|_X \frac{1}{n^r} = \sum_{\alpha} \| a_{p^\alpha} \|_X \left( \frac{1}{p^r} \right)^\alpha = \sum_{\alpha} \| \hat{f}(\alpha) \|_X \left( \frac{1}{p^r} \right)^\alpha < \infty.
\]

Clearly, this implies that \( S_p(X) \leq r \). Since this holds for each \( r > 1/q \), we get that \( S_p(X) \leq 1/q \), and since this now holds for each \( q < M_p(X) \), we have \( S_p(X) \leq 1/M_p(X) \). Conversely, let us take some \( q > M_p(X) \); then there is \( z \in \ell_q \cap B_{c_0} \) and \( f \in H_p(\mathbb{T}^\infty, X) \) such that \( \sum_{\alpha} \| \hat{f}(\alpha)z^\alpha \|_X < \infty \) converges absolutely. By Remark 1.2 we may assume that \( z \) is decreasing, and hence \( (z_n n^{1/q})_n \) is bounded. We choose now \( r > q \) and define \( w_n = 1/p_n^{1/r} \). By the prime number theorem we know that there is a universal constant \( C > 0 \) such that

\[
0 < \frac{z_n}{w_n} = z_n p_n^{1/r} = z_n n^{1/q} p_n^{1/q} = z_n n^{1/q} \left( \frac{p_n}{n} \right)^{1/r} \leq \frac{1}{n^{1/q-1/r}} \leq C z_n n^{1/q} \left( \frac{\log n}{n} \right)^{1/r}.
\]

The last term tends to 0 as \( n \to \infty \); hence \( z_n \leq w_n \) but for a finite number of \( n \)'s. By Remark 1.2 this implies that \( \sum_{\alpha} \hat{f}(\alpha)w^\alpha \) does not converge absolutely. But then \( D = \mathfrak{B}_X f = \sum_n a_n n^{-s} \in \mathfrak{H}_p(X) \)
satisfies
\[ \sum_n \|a_n\|_X \frac{1}{n^{1/r}} = \sum_\alpha \|a_{p^\alpha}\|_X \left( \frac{1}{p^{1/r}} \right)^\alpha = \sum_\alpha \|\hat{f}(\alpha)\|_X w^\alpha = \infty. \]

This gives that \( \sigma_a(D) \geq 1/r \) for every \( r > q \), hence \( \sigma_a(D) \geq 1/q \). Since this holds for every \( q > M_p(X) \), we finally have \( S_p(X) \geq 1/M_p(X) \).

We shall use standard notation and notions from Banach space theory, as presented, for example, in [Lindenstrauss and Tzafriri 1977; 1979]. For everything needed on polynomials in Banach spaces see, for example, [Dineen 1999; Floret 1997].

## 2. Relevant inequalities

The main aim here is to prove a sort of polynomial extension of the notion of cotype. Recall the definition of \( C_q(X) \) from (2). Moreover, from Kahane’s inequality we know that there is a (best) constant \( K \geq 1 \) such that, for each Banach space \( X \) and each choice of finitely many vectors \( x_1, \ldots, x_N \in X \),

\[ \left( \int_{T^N} \left\| \sum_{k=1}^N x_k z_k \right\|_X^2 \, dz \right)^{1/2} \leq K \int_{T^N} \left\| \sum_{k=1}^N x_k z_k \right\|_X \, dz. \]

As usual we write \( |\alpha| = \alpha_1 + \cdots + \alpha_N \) and \( \alpha! = \alpha_1! \cdots \alpha_N! \) for every multiindex \( \alpha \in \mathbb{N}_0^N \).

**Proposition 2.1.** Let \( X \) be a Banach space of cotype \( q \), \( 2 \leq q < \infty \), and

\[ P : \mathbb{C}^N \rightarrow X, \quad P(z) = \sum_{\alpha \in \mathbb{N}_0^N, |\alpha|=m} c_\alpha z^\alpha \]

be an \( m \)-homogeneous polynomial. Let

\[ T : \mathbb{C}^N \times \cdots \times \mathbb{C}^N \rightarrow X, \quad T(z^{(1)}, \ldots, z^{(m)}) = \sum_{i_1, \ldots, i_m=1}^N a_{i_1,\ldots,i_m} z_1^{(i_1)} \cdots z_m^{(i_m)} \]

be the unique \( m \)-linear symmetrization of \( P \). Then

\[ \left( \sum_{i_1,\ldots,i_m} \|a_{i_1,\ldots,i_m}\|_X^q \right)^{1/q} \leq (C_q(X)K)^m \frac{m^m}{m!} \int_{T^N} \|P(z)\|_X \, dz. \]

Before we give the proof let us note that [Bombal et al. 2004, Theorem 3.2] is an \( m \)-linear result that, combined with polarization, gives (with the previous notation)

\[ \left( \sum_{i_1,\ldots,i_m} \|a_{i_1,\ldots,i_m}\|_X^q \right)^{1/q} \leq C_q(X)^m \frac{m^m}{m!} \sup_{z \in \mathbb{D}^N} \|P(z)\|. \]

Our result allows us to replace (up to the constant \( K \)) the \( \|\cdot\|_{\infty} \) norm with the smaller norm \( \|\cdot\|_1 \). We prepare the proof of Proposition 2.1 with three lemmas. The first one is a complex version of [Defant et al. 2010, Lemma 2.2] with essentially the same proof; we include it for the sake of completeness.
Lemma 2.2. Let $X$ be a Banach space of cotype $q$, $2 \leq q < \infty$. Then, for every $m$-linear form

$$T : \mathbb{C}^N \times \cdots \times \mathbb{C}^N \to X, \quad T(z^{(1)}, \ldots, z^{(m)}) = \sum_{i_1, \ldots, i_m=1}^N a_{i_1,\ldots,i_m} z_{i_1}^{(1)} \cdots z_{i_m}^{(m)},$$

we have

$$\left( \sum_{i_1,\ldots,i_m=1}^N \|a_{i_1,\ldots,i_m}\|_X^q \right)^{1/q} \leq (C_q(X)K)^m \int_{\mathbb{T}^N} \cdots \int_{\mathbb{T}^N} \|T(z^{(1)}, \ldots, z^{(m)})\|_X dz^{(1)} \cdots dz^{(m)}.$$

Proof. We prove this result by induction on the degree $m$. For $m = 1$ the result is an immediate consequence of the definition of cotype $q$ and Kahane’s inequality. Assume that the result holds for $m - 1$. By the continuous Minkowski inequality we then conclude that for every choice of finitely many vectors $a_{i_1,\ldots,i_m} \in X$ with $1 \leq i_j \leq N$, $1 \leq j \leq m$ we have

$$\sum_{i_1,\ldots,i_m} \|a_{i_1,\ldots,i_m}\|_X^q = \sum_{i_1,\ldots,i_{m-1}} \sum_{i_m} \|a_{i_1,\ldots,i_m}\|_X^q \leq C_q(X)^qK^q \left( \sum_{i_1,\ldots,i_{m-1}} \left( \int_{\mathbb{T}^N} \|a_{i_1,\ldots,i_m}z_{i_m}^{(m)}\|_X^{q \cdot 1/q} \right)^q \right) \leq C_q(X)^qK^q \left( \int_{\mathbb{T}^N} \left( \sum_{i_1,\ldots,i_{m-1}} \|a_{i_1,\ldots,i_m}z_{i_m}^{(m)}\|_X^q \right)^{1/q} \right)^q \leq C_q(X)^qK^q \left( \int_{\mathbb{T}^N} \left( \sum_{i_1,\ldots,i_{m-1}} \|a_{i_1,\ldots,i_m}z_{i_m}^{(m)}\|_X \right) \right)^q \leq C_q(X)^qK^q \left( \int_{\mathbb{T}^N} \left( \sum_{i_1,\ldots,i_{m-1}} \|a_{i_1,\ldots,i_m}z_{i_m}^{(m)}\|_X \right) \right)^q,$$

which is the conclusion. 

The following two lemmas are needed to produce a polynomial analog of the preceding result.

Lemma 2.3. Let $X$ be a Banach space, and $f : \mathbb{C} \to X$ a holomorphic function. Then for $R_1$, $R_2$, $R \geq 0$ with $R_1 + R_2 \leq R$ we have

$$\int_{\mathbb{T}} \int_{\mathbb{T}} \|f(R_1z_1 + R_2z_2)\|_X dz_1 dz_2 \leq \int_{\mathbb{T}} \|f(Rz)\|_X dz.$$

Proof. By the rotation invariance of the normalized Lebesgue measure on $\mathbb{T}$ we get

$$\int_{\mathbb{T}} \int_{\mathbb{T}} \|f(R_1z_1 + R_2z_2)\|_X dz_1 dz_2 = \int_{\mathbb{T}} \int_{\mathbb{T}} \|f(R_1z_1z_2 + R_2z_2)\|_X dz_1 dz_2 = \int_{\mathbb{T}} \int_{\mathbb{T}} \|f(z_2(R_1z_1 + R_2))\|_X dz_1 dz_2 = \int_{\mathbb{T}} \int_{\mathbb{T}} \|f(z_2z_1(R_1+R_2))\|_X dz_2 dz_1 = \int_{\mathbb{T}} \int_{\mathbb{T}} \|f(z_2r(z_1)R)\|_X dz_2 dz_1 = \int_0^{2\pi} \int_0^{2\pi} \|f(r(e^{it})Re^{it})\|_X d\theta dt ds,$$
where \( r(z) = (1/R)|R_1z + R_2|, \ z \in \mathbb{T} \). We know that for each holomorphic function \( h : \mathbb{C} \to X \) we have
\[
\int_{\mathbb{T}} \| h(z) \|_X \, dz = \sup_{0 \leq r \leq 1} \int_0^{2\pi} \| h(re^{it}) \|_X \frac{dt}{2\pi}
\]
(see, for example, Blasco and Xu [1991, p. 338]). Define now \( h(z) \), and note that \( 0 \leq r(z) \leq 1 \) for all \( z \in \mathbb{T} \). Then
\[
\int_{\mathbb{T}} \int_{\mathbb{T}} \| f(R_1z_1 + R_2z_2) \|_X \, dz_1 \, dz_2 = \int_0^{2\pi} \int_0^{2\pi} \| h(re^{is})e^{it} \|_X \frac{dt}{2\pi} \frac{ds}{2\pi}
\]
\[
\leq \int_0^{2\pi} \int_{\mathbb{T}} \| h(z) \|_X \frac{ds}{2\pi} = \int_{\mathbb{T}} \| f(Rz) \|_X \, dz.
\]
This completes the proof.

A sort of iteration of the preceding result leads to the next:

**Lemma 2.4.** Let \( X \) be a Banach space, and \( f : \mathbb{C}^N \to X \) a holomorphic function. Then, for every \( m \),
\[
\int_{\mathbb{T}^N} \int_{\mathbb{T}^N} \| f(z^{(1)} + \cdots + z^{(m)}) \|_X \, dz^{(1)} \cdots \, dz^{(m)} \leq \int_{\mathbb{T}^N} \| f(mz) \|_X \, dz.
\]

**Proof.** We fix some \( m \), and do induction with respect to \( N \). For \( N = 1 \) we obtain from Lemma 2.3 that
\[
\int_{\mathbb{T}} \int_{\mathbb{T}} \int_{\mathbb{T}} \int_{\mathbb{T}} \| f(z^{(1)} + \cdots + z^{(m-2)} + z^{(m-1)} + z^{(m)}) \|_X \, dz^{(m-1)} \, dz^{(m)} \, dz^{(1)} \cdots \, dz^{(m-2)}
\]
\[
= \int_{\mathbb{T}} \int_{\mathbb{T}} \int_{\mathbb{T}} \int_{\mathbb{T}} \| g_{z^{(1)}, \ldots, z^{(m-2)}}(2w) \|_X \, dw \, dz^{(1)} \cdots \, dz^{(m-2)}
\]
\[
= \int_{\mathbb{T}} \int_{\mathbb{T}} \int_{\mathbb{T}} \int_{\mathbb{T}} \int_{\mathbb{T}} \int_{\mathbb{T}} \int_{\mathbb{T}} \| f(z^{(1)} + \cdots + z^{(m-2)} + 2w) \|_X \, dw \, dz^{(m-2)} \, dz^{(1)} \cdots \, dz^{(m-3)}
\]
\[
\leq \int_{\mathbb{T}} \int_{\mathbb{T}} \int_{\mathbb{T}} \int_{\mathbb{T}} \int_{\mathbb{T}} \| f(z^{(1)} + \cdots + z^{(m-3)} + 3w) \|_X \, dw \, dz^{(m-3)} \, dz^{(1)} \cdots \, dz^{(m-3)}
\]
\[
\leq \cdots \leq \int_{\mathbb{T}} \| f(mz) \|_X \, dz.
\]
We now assume that the conclusion holds for \( N - 1 \) and write each \( z \in \mathbb{T}^N \) as \( z = (u, w) \), with \( u \in \mathbb{T}^{N-1} \) and \( w \in \mathbb{T} \). Then, using the case \( N = 1 \) in the first inequality and the inductive hypothesis in the second,
we have
\[
\int_{T^N} \cdots \int_{T^N} \| f(z^{(1)} + \cdots + z^{(m)}) \|_X \, d\z^{(1)} \cdots d\z^{(m)}
\]
\[
= \int_{T^{N-1}} \cdots \int_{T^{N-1}} \left( \int_{T} \cdots \int_{T} \| f((u^{(1)}, w_1) + \cdots + (u^{(m)}, w_m)) \|_X \, dw_1 \cdots dw_N \right) \, du^{(1)} \cdots du^{(m)}
\]
\[
\leq \int_{T^{N-1}} \cdots \int_{T^{N-1}} \left( \int_{T} \cdots \int_{T} \| f((u^{(1)}, mw) + \cdots + (u^{(m)}, mw)) \|_X \, dw \right) \, du^{(1)} \cdots du^{(m)}
\]
\[
= \int_{T} \left( \int_{T^{N-1}} \cdots \int_{T^{N-1}} \| f((u^{(1)}, mw) + \cdots + (u^{(m)}, mw)) \|_X \, du \right) \, dw
\]
\[
\leq \int_{T} \left( \int_{T^{N-1}} \| f((mu, mw) + \cdots + (mu, mw)) \|_X \, du \right) \, dw
\]
\[
= \int_{T^N} \| f(mz) \|_X \, dz,
\]
as desired.

\[\square\]

**Proof of the inequality from Proposition 2.1.** By the polarization formula we know that for every choice of $z^{(1)}, \ldots, z^{(m)} \in T^N$ we have
\[
T(z^{(1)}, \ldots, z^{(m)}) = \frac{1}{2^m m!} \sum_{\varepsilon_i = \pm 1} \varepsilon \cdot \varepsilon \cdot P \left( \sum_{i=1}^{N} \varepsilon \cdot z^{(i)} \right)
\]
(see, for example, [Dineen 1999] or [Floret 1997]). Hence we deduce from Lemma 2.4
\[
\int_{T^N} \cdots \int_{T^N} \| T(z^{(1)}, \ldots, z^{(m)}) \|_X \, d\z^{(1)} \cdots d\z^{(m)}
\]
\[
= \frac{1}{2^m m!} \sum_{\varepsilon_i = \pm 1} \int_{T^{N-1}} \cdots \int_{T^{N-1}} \| P \left( \sum_{i=1}^{N} \varepsilon \cdot z^{(i)} \right) \|_X \, d\z^{(1)} \cdots d\z^{(m)}
\]
\[
= \frac{1}{m!} \int_{T^{N-1}} \cdots \int_{T^{N-1}} \| P \left( \sum_{i=1}^{N} z^{(i)} \right) \|_X \, d\z^{(1)} \cdots d\z^{(m)}
\]
\[
\leq \frac{1}{m!} \int_{T^N} \| P \left( m \cdot z \right) \|_X \, dz = \frac{m^m}{m!} \int_{T^N} \| P \left( z \right) \|_X \, dz.
\]
Then by Lemma 2.2 we obtain
\[
\left( \sum_{i_1, \ldots, i_m} \| a_{i_1, \ldots, i_m} \|_X^q \right)^{1/q} \leq (C_q (X) K)^m \int_{T^\infty} \cdots \int_{T^\infty} \| T(z^{(1)}, \ldots, z^{(m)}) \|_X \, d\z^{(1)} \cdots d\z^{(m)}
\]
\[
= (C_q (X) K)^m \frac{m^m}{m!} \int_{T^N} \| P(z) \|_X \, dz,
\]
which completes the proof of Proposition 2.1. \[\square\]
A second proposition is needed which allows us to reduce the proof of our main result (Theorem 1.1) to the homogeneous case. It is a vector-valued version of a result of [Cole and Gamelin 1986, Theorem 9.2] with a similar proof (here only given for the sake of completeness).

**Proposition 2.5.** There is a contractive projection

\[ \Phi_m : H_p(\mathbb{T}^N, X) \to H_p(\mathbb{T}^N, X), \quad f \mapsto f_m, \]

such that, for all \( f \in H_p(\mathbb{T}^N, X) \),

\[ \hat{f}(\alpha) = \hat{f}_m(\alpha) \quad \text{for all } \alpha \in \mathbb{N}_0^N \text{ with } |\alpha| = m. \]  

(11)

**Proof.** Let \( \mathcal{P}(\mathbb{C}^N, X) \subset H_p(\mathbb{T}^N, X) \) be the subspace of all finite polynomials \( f = \sum_{\alpha \in \Lambda} c_{\alpha} z^\alpha; \) here \( \Lambda \) is a finite set of multiindices in \( \mathbb{N}_0^N \) and the coefficients \( c_{\alpha} \in X \). Define the linear projection \( \Phi^0_m \) on \( \mathcal{P}(\mathbb{C}^N, X) \) by

\[ \Phi^0_m(f)(z) = f_m(z) = \sum_{\alpha \in \Lambda, |\alpha|=m} \hat{f}(\alpha) z^\alpha; \]

clearly, we have (11). In order to show that \( \Phi^0_m \) is a contraction on \( (\mathcal{P}(\mathbb{C}^N, X), \| \cdot \|_p) \) fix some function \( f \in \mathcal{P}(\mathbb{C}^N, X) \) and \( z \in \mathbb{T}^N \), and define

\[ f(z \cdot) : \mathbb{T} \to X, \quad w \mapsto f(zw). \]

Clearly, we have

\[ f(zw) = \sum_k f_k(z) w^k, \]

and hence

\[ f_m(z) = \int_{\mathbb{T}} f(zw) w^{-m} \, dw. \]

Integration, Hölder’s inequality and the rotation invariance of the normalized Lebesgue measure on \( \mathbb{T}^N \) give

\[ \int_{\mathbb{T}^N} \| f_m(z) \|_X^p \, dz = \int_{\mathbb{T}^N} \left( \int_{\mathbb{T}} f(zw) w^{-m} \, dw \right)^p \, dz \]
\[ \leq \int_{\mathbb{T}^N} \left( \int_{\mathbb{T}} \| f(zw) \|_X \, dw \right)^p \, dz \]
\[ \leq \int_{\mathbb{T}} \int_{\mathbb{T}^N} \| f(zw) \|_X^p \, dz \, dw = \int_{\mathbb{T}^N} \| f(z) \|_X^p \, dz, \]

which proves that \( \Phi^0_m \) is a contraction on \( (\mathcal{P}(\mathbb{C}^N, X), \| \cdot \|_p) \). By Fejér’s theorem (vector-valued) we know that \( \mathcal{P}(\mathbb{C}^N, X) \) is a dense subspace of \( H_p(\mathbb{T}^N, X) \). Hence \( \Phi^0_m \) extends to a contractive projection \( \Phi_m \) on \( H_p(\mathbb{T}^N, X) \). This extension \( \Phi_m \) still satisfies (11) since the mapping \( H_p(\mathbb{T}^N, X) \to X, \ f \mapsto \hat{f}(\alpha) \) is continuous for each multiindex \( \alpha \). \( \square \)
3. Proof of the main result

We are now ready to prove Theorem 1.1. Let $1 \leq p < \infty$, and recall from (1) that

$$1 - \frac{1}{\text{Cot } X} = S_\infty(X) \leq S_p(X);$$

see Remark 3.1 for a direct argument. Hence it suffices to concentrate on the upper estimate in Theorem 1.1: Since we obviously have $S_p(X) \leq S_1(X)$, we are going to prove that

$$S_1(X) \leq 1 - \frac{1}{\text{Cot } X}. \quad (12)$$

Suppose first that $X$ has no finite cotype, i.e., $\text{Cot } X = \infty$. For $D = \sum_n a_n n^{-s} \in \mathcal{H}_1(X)$ we take $f \in H_1(\mathbb{T}^\infty, X)$ with $D = \mathcal{B}_X f$. Note that

$$\|\hat{f}(\alpha)\|_X \leq \int_{\mathbb{T}^\infty} \|f(w)w^{-\alpha}\|_X dw = \|f\|_{L_1(\mathbb{T}^\infty, X)} < \infty;$$

hence, by the definition of $\mathcal{B}_X$, the coefficients of $D$ are also bounded by $\|f\|_{L_1(\mathbb{T}^\infty, X)}$. As a consequence, for every $\sigma > 1$ we have

$$\sum_{n=1}^{\infty} \|a_n\|_X \frac{1}{n^\sigma} \leq \sum_{n=1}^{\infty} \|f\|_{L_1(\mathbb{T}^\infty, X)} \frac{1}{n^\sigma} < \infty.$$ 

This means that $S_1(X) \leq 1$ and as a consequence (12) holds.

Now if $X$ has finite cotype, take $q > \text{Cot } X$ and $\varepsilon > 0$, and put $s = (1 - 1/q)(1 + 2\varepsilon)$. Choose an integer $k_0$ such that $p_{k_0}^{s/q'} > eC_q(X)K(\sum_{j=1}^{\infty} 1/p_j^{1+\varepsilon})^{1/q'}$ and define

$$\tilde{p} = (p_{k_0}, \ldots, p_{k_0}, p_{k_0+1}, p_{k_0+2}, \ldots).$$

We are going to show that there is a constant $C(q, X, \varepsilon) > 0$ such that for every $f \in H_1(\mathbb{T}^\infty, X)$ we have

$$\sum_{\alpha \in \mathbb{N}_0^{(k_0)}} \|\hat{f}(\alpha)\|_X \frac{1}{\tilde{p}^{s\alpha}} \leq C(q, X, \varepsilon)\|f\|_{H_1(\mathbb{T}^\infty, X)}, \quad (13)$$

This finishes the argument: By Remark 1.2 the sequence $1/p^s$ is in mon $H_1(\mathbb{T}^\infty, X)$. But in view of Bohr’s transform from (6), this means that for every Dirichlet series $D = \sum_n a_n n^{-s} = \mathcal{B}_X f \in \mathcal{H}_1(X)$ with $f \in H_1(\mathbb{T}^\infty, X)$ we have

$$\sum_{n=1}^{\infty} \|a_n\|_X \frac{1}{n^s} = \sum_{\alpha \in \mathbb{N}_0^{(k_0)}} \|\hat{f}(\alpha)\|_X \frac{1}{\tilde{p}^{s\alpha}} < \infty.$$ 

Therefore $\sigma_n(D) \leq (1 - 1/q)(1 + 2\varepsilon)$ for each such $D$ which, since $\varepsilon > 0$ was arbitrary, is what we wanted to prove.

It remains to check (13); the idea is to show first that (13) holds for all $X$-valued $H_1$-functions which only depend on $N$ variables: There is a constant $C(q, X, \varepsilon) > 0$ such that for all $N$ and every
\[ f \in H_1(\mathbb{T}^N, X) \] we have
\[
\sum_{\alpha \in \mathbb{N}_0^N} \| \hat{f}(\alpha) \|_X \frac{1}{\hat{p}^{\alpha}} \leq C(q, X, \varepsilon) \| f \|_{H_1(\mathbb{T}^N, X)}. \tag{14}
\]

In order to understand that (14) implies (13) (and hence the conclusion), assume that (14) holds and take some \( f \in H_1(\mathbb{T}^\infty, X) \). Given an arbitrary \( N \), define
\[
 f_N : \mathbb{T}^N \to X, \quad f_N(w) = \int_{T^\infty} f(w, \tilde{w}) \, d\tilde{w}.
\]
Then it can be easily shown that \( f_N \in L_1(\mathbb{T}^N, X) \), \( \| f_N \|_1 \leq \| f \|_1 \), and \( \hat{f}_N(\alpha) = \hat{f}(\alpha) \) for all \( \alpha \in \mathbb{Z}^N \). If we now apply (14) to this \( f_N \), we get
\[
\sum_{\alpha \in \mathbb{N}_0^N} \| \hat{f}(\alpha) \|_X \frac{1}{\hat{p}^{\alpha}} \leq C(q, X, \varepsilon) \| f \|_{H_1(\mathbb{T}^\infty, X)},
\]
which, after taking the supremum over all possible \( N \) on the left side, leads to (13).

We turn to the proof of (14), and here in a first step will show the following: For every \( N \), every \( m \)-homogeneous polynomial \( P : \mathbb{C}^N \to X \) and every \( u \in \ell_{q'} \) we have
\[
\sum_{\alpha \in \mathbb{N}_0^N, |\alpha| = m} \| \hat{P}(\alpha) u^\alpha \|_X \leq (eC_q(X)K)^m \int_{\mathbb{T}^N} \| P(z) \|_X \, dz \left( \sum_{j=1}^\infty |u_j|^{q'} \right)^{m/q'}. \tag{15}
\]
Indeed, take such a polynomial \( P(z) = \sum_{\alpha \in \mathbb{N}_0^N, |\alpha| = m} \hat{P}(\alpha) z^\alpha \), \( z \in \mathbb{T}^N \), and look at its unique \( m \)-linear symmetrization
\[
T : \mathbb{C}^N \times \cdots \times \mathbb{C}^N \to X, \quad T(z^{(1)}, \ldots, z^{(m)}) = \sum_{i_1, \ldots, i_m=1}^N a_{i_1, \ldots, i_m} z_i^{(1)} \cdots z_i^{(m)}.
\]
Then we know from Proposition 2.1 that
\[
\left( \sum_{i_1, \ldots, i_m=1}^N \| a_{i_1, \ldots, i_m} \|_X^q \right)^{1/q} \leq (eC_q(X)K)^m \int_{\mathbb{T}^N} \| P(z) \|_X \, dz.
\]
Hence (15) follows by Hölder’s inequality:
\[
\sum_{\alpha \in \mathbb{N}_0^N, |\alpha| = m} \| \hat{P}(\alpha) u^\alpha \|_X = \sum_{i_1, \ldots, i_m=1}^N \| a_{i_1, \ldots, i_m} \|_X |u_{i_1} \cdots u_{i_N}| \leq (eC_q(X)K)^m \int_{\mathbb{T}^N} \| P(z) \|_X \, dz \left( \sum_{j=1}^\infty |u_j|^{q'} \right)^{m/q'}.
\]
We finally give the proof of (14): Take \( f \in H_1(\mathbb{T}^N, X) \), and recall from Proposition 2.5 that for each integer \( m \) there is an \( m \)-homogeneous polynomial \( P_m : \mathbb{C}^N \to X \) such that \( \| P_m \|_{H_1(\mathbb{T}^N, X)} \leq \| f \|_{H_1(\mathbb{T}^N, X)} \).
and \( \hat{P}_m(\alpha) = \hat{f}(\alpha) \) for all \( \alpha \in \mathbb{N}_0^N \) with \( |\alpha| = m \). From (15), the definition of \( s \), and the fact that \( \max\{p_{k_0}, p_j\} \leq \tilde{p}_j \) for all \( j \) we have

\[
\sum_{\alpha \in \mathbb{N}_0^N} \| \hat{f}(\alpha) \|_X \frac{1}{p^{s\alpha}} = \sum_{m=1}^{\infty} \left( eC_q(X)K \right)^m \| P_m \|_{H_1(T^N, X)} \left( \sum_{j=1}^{\infty} \frac{1}{p_j^{s'q'}} \right)^{m/q'} \\
= \sum_{m=1}^{\infty} \left( eC_q(X)K \right)^m \| f \|_{H_1(T^N, X)} \left( \sum_{j=1}^{\infty} \frac{1}{p_j^{1+2\varepsilon}} \right)^{m/q'} \\
= \sum_{m=1}^{\infty} \left( eC_q(X)K \right)^m \| f \|_{H_1(T^N, X)} \left( \sum_{j=1}^{\infty} \frac{1}{p_j^{1+\varepsilon}} \right)^{m/q'} \\
\leq \| f \|_{H_1(T^N, X)} \sum_{m=1}^{\infty} \left( eC_q(X)K \sum_{j=1}^{\infty} \frac{1}{1/p_j^{(1+\varepsilon)}} \right)^{m/q'} \left( \frac{p_j^{s'q'}}{p_{k_0}^{s'q'}} \right) \frac{1}{<1} .
\]

This completes the proof of Theorem 1.1. \( \square \)

**Remark 3.1.** We end this note with a direct proof of the fact

\[
1 - \frac{1}{\text{Cot } X} \leq S_p(X), \quad 1 \leq p < \infty ,
\]

in which we do not use the inequality

\[
1 - \frac{1}{\text{Cot } X} \leq S_\infty (X)
\]

from [Defant et al. 2008] (here repeated in (1)). The proof of (17) given in that reference shows in a first step that \( 1 - 1/\Pi(X) \leq S_\infty (X) \) where

\[
\Pi(X) = \inf \{ r \geq 2 \mid \text{id}_X \text{ is } (r, 1)\text{-summing} \},
\]

and then, in a second step, applies a fundamental theorem of Maurey and Pisier stating that \( \Pi(X) = \text{Cot } X \).

The following argument for (16) is very similar to the original one from [Defant et al. 2008] but does not use the Maurey–Pisier theorem (since we here consider \( H_p(X), 1 \leq p < \infty \) instead of \( H_\infty (X) \)); By the proof of Corollary 1.3, inequality (16) is equivalent to

\[
M_p(X) \leq \frac{\text{Cot } X}{\text{Cot } X - 1}.
\]

Take \( r < M_p(X) \), so that \( \ell_r \cap B_{c_0} \subset \text{mon } H_p(\mathbb{T}^\infty, X) \). Let \( H^1_{p}(\mathbb{T}^\infty, X) \) be the subspace of \( H_p(\mathbb{T}^\infty, X) \) formed by all 1-homogeneous polynomials (i.e., linear operators). We can define a bilinear operator
$\ell_r \times H^1_p(\mathbb{T}^\infty, X) \to \ell_1(X)$ by $(z, f) \mapsto (z_j f(e^j))_j$ which, by a closed graph argument, is continuous. Therefore, there is a constant $M$ such that for all $z \in \ell_r$ and all $f \in H^1_p(\mathbb{T}^\infty, X)$ we have

$$\sum_j |z_j| \|f(e^j)\|_X \leq M \|z\|_{\ell_r} \|f\|_{H^1_p(\mathbb{T}^\infty, X)}.$$  

Taking the supremum over all $z \in B_{\ell_r}$ we obtain for all $f \in H^1_p(\mathbb{T}^\infty, X)$

$$\left( \sum_j \|f(e^j)\|_X^{r'} \right)^{1/r'} \leq M \|f\|_{H^1_p(\mathbb{T}^\infty, X)}.$$  

Now, take $x_1, \ldots, x_N \in X$, define $f \in H^1_p(\mathbb{T}^\infty, X)$ by

$$f(e^j) = \begin{cases} x_j & \text{if } 1 \leq j \leq N, \\ 0 & \text{if } j > N \end{cases}$$

and extend it by linearity. By the previous inequality and Proposition 2.5 we have

$$\left( \sum_{j=1}^N \|x_j\|_X^{r'} \right)^{1/r'} \leq M \left( \int_{\mathbb{T}^\infty} \left| \sum_{j=1}^N x_j z_j \right|^{r'} d\mu \right)^{1/r'}.$$  

By Kahane’s inequality, $X$ has cotype $r'$, which means that $r' > \text{Cot} X$ or, equivalently, $r < \frac{\text{Cot} X}{\text{Cot} X - 1}$. Since $r < M_p(X)$ was arbitrary, we obtain (16).

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BOHR’S ABSOLUTE CONVERGENCE PROBLEM FOR $\mathcal{H}_p$-DIRICHLET SERIES IN BANACH SPACES


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