A quantitative approach to weak compactnesss in Fréchet spaces and spaces $C(X)$ [☆]

C. Angosto[a], J. Kąkol[b], M. López-Pellicer[c]

[a] *Depto. de Matemática Aplicada y Estadistica: Universidad Politécnica de Cartagena, 30203 Cartagena, Spain*
[b] *Faculty of Mathematics and Informatics. A. Mickiewicz University, 61-614 Poznań, Poland*
[c] *Depto. de Matemática Aplicada and IMPA. Universitat Politècnica de València, E-46022 Valencia, Spain*

Abstract

Let E be a Fréchet space, i.e. a metrizable and complete locally convex space (lcs), E'' its strong second dual with a defining sequence of seminorms $\|\cdot\|_n$ induced by a decreasing basis of absolutely convex neighbourhoods of zero U_n, and let $H \subset E$ be a bounded set. Let $ck(H) := \sup\{d(clust_{E''}(\varphi), E) : \varphi \in H^{\mathbb{N}}\}$ be the "worst" distance of the set of $weak^*$-cluster points in E'' of sequences in H to E, and $k(H) := \sup\{d(h, E) : h \in \overline{H}\}$ the worst distance of $\overline{H}$ the $weak^*$-closure in the bidual of H to E, where d means the natural metric of E''. Let $\gamma_n(H) := \sup\left\{|\lim_p \lim_m u_p(h_m) - \lim_m \lim_p u_p(h_m)| : (u_p) \subset U_n^0,\ (h_m) \subset H\right\}$, provided the involved limits exist. We extend a recent result of Angosto-Cascales to Fréchet spaces by showing that: If $x^{**} \in \overline{H}$, there is a sequence $(x_p)_p$ in H such that $d_n(x^{**}, y^{**}) \le \gamma_n(H)$ for each $\sigma(E'', E')$-cluster point y^{**} of $(x_p)_p$ and $n \in \mathbb{N}$. Moreover, $k(H) = 0$ iff $ck(H) = 0$. This provides a quantitative version of the weak angelicity in a Fréchet space. We show that $ck(H) \le \hat{d}(\overline{H}, C(X, Z)) \le 17ck(H)$, where $H \subset Z^X$ is relatively compact and $C(X, Z)$ is the space of Z-valued continuous functions for a web-compact space X and a separable metric space Z. If X is web-compact and normal and $Z := \mathbb{R}$, we show that $ck(H) \le \hat{d}(\overline{H}, C(X)) \le 12ck(H)$. A corresponding result for strongly web-compact spaces X is also obtained with sharper constants. This yields a quantitative version of Orihuela's angelic theorem for spaces $C_p(X, Z)$ and applies also to show: If X is the $weak^*$-dual of a (DF)-space or an (LF)-space and $H \subset \mathbb{R}^X$ is bounded, then $ck(H) \le \hat{d}(\overline{H}, C(X)) \le 5ck(H)$.

1. Introduction

Many classical results about compactness in functional analysis can be deduced from suitable inequalities about distances to spaces of continuous functions. This line of

[☆] The research was supported for the first named author by the project MTM2008-05396 of the Spanish Ministry of Science and Innovation and by Fundación Séneca (CARM), grant 08848/PI/08, for the second named author by National Center of Science, Poland, grant no. N N201 605340 and for the second and third authors by the project MTM2008-01502 of the Spanish Ministry of Science and Innovation.

Email addresses: `carlos.angosto@upct.es` (C. Angosto), `kakol@amu.edu.pl` (J. Kąkol), `mlopezpe@mat.upv.es` (M. López-Pellicer)

research motivates a number of specialists to study several quantitative counterparts of some classical results. We refer to works [9], [2], [3], [4], [11], [15], [16] also as a good source of references. Especially results from [9] and [2], yielding several characterizations of weak compactness of bounded sets in a Banach space, motivated our present paper. Papers cited above provided some tools which have been used for new quantitative versions of Gantmacher's theorem about weak compactness of adjoint operators in Banach spaces, Eberlein–Grothendieck's theorem, Grothendieck's characterization of weak compactenss in real Banach spaces $C(K) := C(K, \mathbb{R})$, and the classical Krein-Smulyan's theorem.

Theorem 1 below deals with the following non-negative functions defined on the family of bounded sets H in a Banach space E, see [2, Definiton 1]:

(i) $\gamma(H) := \sup\{|\lim_m \lim_n f_m(x_n) - \lim_n \lim_m f_m(x_n)| : (f_m)_m \subset B_{E'}, (x_n)_n \subset H\}$ assuming that the iterated limit exist,

(ii) $ck(H) := \sup\{d(clust_{E''}(\phi), E), \phi \in H^{\mathbb{N}}\}$,

(iii) $k(H) := \hat{d}(\overline{H}^{\omega^*}, E) = \sup\{d(x^{**}, E), x^{**} \in \overline{H}^{\omega^*}\}$,

where d is the usual inf distance for sets associated to the natural norm in E''.

Let X be a completely regular Hausdorff space. If $H \subset C(X) \subset \mathbb{R}^{\mathbb{X}}$ is pointwise bounded, the closure $\overline{H}^{\mathbb{R}^{\mathbb{X}}}$ is compact in the topology τ_p of pointwise convergence in $\mathbb{R}^{\mathbb{X}}$.

If $\hat{d}(\overline{H}^{\mathbb{R}^{\mathbb{X}}}, C(X)) := \sup\{d(f, C(X)) : f \in \overline{H}^{\mathbb{R}^{\mathbb{X}}}\}$, where d is the standard supremum metric, then $\hat{d} = 0$ iff $\overline{H}^{\mathbb{R}^{\mathbb{X}}} \subset C(X)$ iff H is τ_p-relatively compact in $C(X)$. Therefore $\hat{d} > 0$ provides a measure of non-τ_p-compactness for H in $C(X)$.

The following interesting result [2, Theorem 2.3] motivated our work.

Theorem 1. *For any bounded set H in a Banach space E we have $ck(H) \leq k(H) \leq \gamma(H) \leq 2ck(H) \leq 2k(H)$. If $x^{**} \in \overline{H}^{\omega^*}$, there exists a sequence $(x_n)_n$ in H such that $\|x^{**} - y^{**}\| \leq \gamma(H)$ for any cluster point y^{**} of $(x_n)_n$ in E''. H is weakly relatively compact in E iff one (equivalently all) of $ck(H)$, $k(H)$, $\gamma(H)$ is zero.*

In the first part we show that some techniques from above cited papers can be used also in the frame of Fréchet spaces, i.e., metrizable and complete lcs.

We provide quantitative characterizations of weak-compactness in a Fréchet space. The approximation Theorem 7 (extending [2, Theorem 3.2] to Fréchet spaces) is the quantitative version of the weak angelicity of a Fréchet space.

Theorem 14 and Corollary 15 provide a quantitative version of Orihuela's angelic theorem [18, Theorem 3] showing that $ck(H) \leq \hat{d}(\overline{H}, C(X, Z)) \leq 17ck(H)$, where $H \subset Z^X$ is relatively compact and $C(X, Z)$ is the space of Z-valued continuous functions for web-compact spaces X and separable metric space Z. If X is web-compact and normal and $Z := \mathbb{R}$, then $ck(H) \leq \hat{d}(\overline{H}, C(X)) \leq 12ck(H)$. A corresponding result for strongly web-compact spaces X is also obtained with a more sharper constants using the same proofs that [5, Theorem 3.1 and Theorem 3.2], results for web-compact spaces require extra work. This yields also a quantitative approach to the weak angelicity of any lcs in the class $\mathfrak{G}$.

Notation and terminology: Let E be a Fréchet space and let $(U_n)_n$ be a decreasing basis of absolutely convex neighbourhoods of zero. By $(E', \beta(E', E))$ and

$(E'', \beta(E'', E'))$ we mean the strong dual of E and $(E', \beta(E', E))$, respectively. In $(E'', \beta(E'', E'))$ the sequence of bipolars $(U_n^{00})_n$ is a decreasing basis of absolutely convex neighbourhoods of zero. By $\|h\|_n = \sup\{|h(u)| : u \in U_n^0\}$ we denote the seminorm in E'' associated with U_n^0 and d_n means the pseudometric defined by $\|.\|_n$. The restriction of $\|.\|_n$ to E, also denoted by $\|.\|_n$, is the seminorm defined by U_n. The topology of E can be defined by the F-norm $d(x,y) := \sum_n 2^{-n}\|x-y\|_n(1+\|x-y\|_n)^{-1}$ for $x, y \in E$. The topology of the space $(E'', \beta(E'', E'))$ is defined by the F-norm $d(x^{**}, y^{**}) := \sum_n 2^{-n}\|x^{**} - y^{**}\|_n(1+\|x^{**}-y^{**}\|_n)^{-1}$ for all $x^{**}, y^{**} \in E''$. Additionally without loss of generality, we assume in this paper that $2U_{n+1} \subset U_n$ for $n \in \mathbb{N}$; and this clearly implies that $2\|x^{**}\|_n \leq \|x^{**}\|_{n+1}$ for $n \in \mathbb{N}$ and each $x^{**} \in E''$.

If H is a bounded subset of E then H^0 is a neighbourhood of zero in $(E', \beta(E', E))$ and the bypolar H^{00} is a compact subset of $(E'', \sigma(E'', E'))$. Therefore an E-bounded subset H is weakly relatively compact if and only if $\overline{H}^{\sigma(E'', E')}$ is contained in E.

Next concepts are the natural extensions of the given above:

$$\gamma_n(H) := \sup\left\{\left|\lim_p \lim_m u_p(h_m) - \lim_m \lim_p u_p(h_m)\right| : (u_p) \subset U_n^0, (h_m) \subset H\right\}$$

assuming the involved limits exist. Let

$$ck_n(H) := \sup\left\{d_n\left(clust_{E''}(\varphi), E\right) : \varphi \in H^{\mathbb{N}}\right\}$$

and

$$ck(H) := \sup\left\{d\left(clust_{E''}(\varphi), E\right) : \varphi \in H^{\mathbb{N}}\right\}$$

where $clust_{E''}(\varphi) := \bigcap_p \overline{\{\varphi(m) : m > p\}}^{\sigma(E'', E')}$ is the set of all cluster points in E'' of the sequence $\varphi \in H^{\mathbb{N}}$ and $d_n(A, B) = \inf\{d_n(a, b) : a \in A, b \in B\}$. Also define

$$k_n(H) := \sup\left\{d_n(h, E) : h \in \overline{H}^{\sigma(E'', E')}\right\},$$

and

$$k(H) := \sup\left\{d(h, E) : h \in \overline{H}^{\sigma(E'', E')}\right\}.$$

We say that H ε-interchanges limits with a subset B of E' if

$$\sup\left\{\left|\lim_p \lim_m u_p(h_m) - \lim_m \lim_p u_p(h_m)\right| : (u_p) \subset B, (h_m) \subset H\right\} \leq \varepsilon$$

where $\varepsilon \geq 0$ and the involved limits exist. For $\varepsilon = 0$ we say H interchanges limits with B, see [14]. $\gamma_n(H) \leq \varepsilon$ ($\gamma_n(H) = 0$) means: H ε-interchanges (interchanges) limits with U_n^0. Note that

$$2\gamma_n(H) \leq \gamma_{n+1}(H), \;\; 2ck_n(H) \leq ck_{n+1}(H), \;\; 2k_n(H) \leq k_{n+1}(H).$$

Hence $\sup_n \gamma_n(H) < \infty$, $\sup_n ck_n(H) < \infty$, $\sup_n k_n(H) < \infty$ iff $\gamma_n(H) = 0$, $ck_n(H) = 0$, $k_n(H) = 0$, $n \in \mathbb{N}$, respectively.

A space X is *angelic* if every relatively countably compact set A in X is relatively compact and for each $x \in \overline{A}$ there is a sequence $(x_n)_n$ in A converging to x, see [13].

2. First observations and remarks

For $x^{**} \in E''$ we have $d(x^{**}, E) = 0$ iff $x^{**} \in E$ iff $d_n(x^{**}, E) = 0$ for $n \in \mathbb{N}$. Hence

Proposition 2. *For a bounded subset H of a Fréchet space E the set H is weakly relatively compact iff $k(H) = 0$ iff $k_n(H) = 0$ for all $n \in \mathbb{N}$.*

Moreover, from the definitions it follows easily that $ck_n(H) \leq k_n(H)$. To prove more we need the following two additional lemmas.

Lemma 3. *Let H be a bounded subset of a Fréchet space E and let $h \in \overline{H}^{\sigma(E'',E')}$. Then for each $n \in \mathbb{N}$ there exists a net $(u_\beta)_\beta$ in U_n^0 that $\sigma(E', E)$-converges to 0 and such that for each net $(h_\alpha)_\alpha$ in H that $\sigma(E'', E')$-converges to h we have $d_n(h, E) = \lim_\beta \lim_\alpha u_\beta(h_\alpha)$. Consequently, there exist sequences $(h_m)_m$ in H and $(u_p)_p$ in U_n^0 such that $d_n(h, E) = \lim_p \lim_m u_p(h_m)$ and $\lim_m \lim_p u_p(h_m) = 0$. Hence $k_n(H) \leq \gamma_n(H)$.*

Proof. The linear functional u defined on the linear hull of E and h by $u(e + \lambda h) = \lambda d_n(h, E)$ for $e \in E$ verifies $|u(e + \lambda h)| = |\lambda| \, d_n(h, E) = d_n(\lambda h, E) = d_n(e + \lambda h, E) \leq \|e + \lambda h\|_n$. By the Hahn-Banach theorem u admits a linear extension to E'', also named u, such that

$$|u(x^{**})| \leq \|x^{**}\|_n$$

for each $x^{**} \in E''$. Clearly $u \in \left(U_n^{00}\right)^0 = (U_n^0)^{00}$ and we obtain a net $(u_\beta)_\beta$ in U_n^0 such that

$$u(x^{**}) = \lim_\beta u_\beta(x^{**})$$

for each $x^{**} \in E''$. In particular

$$d_n(h, E) = u(h) = \lim_\beta u_\beta(h), \;\; 0 = d(e, E) = u(e) = \lim_\beta u_\beta(e)$$

for each $e \in E$ so $(u_\beta)_\beta$ $\sigma(E', E)$-converges to 0. If$(h_\alpha)_\alpha$ is a net in H that $\sigma(E'', E')$-converges to h, then each $u_\beta(h)$ is the limit of the net $(u_\beta(h_\alpha))_\alpha$ and

$$d_n(h, E) = u(h) = \lim_\beta \lim_\alpha u_\beta(h_\alpha), \;\; 0 = \lim_\alpha u(h_\alpha) = \lim_\alpha \lim_\beta u_\beta(h_\alpha).$$

By Lemma 2.1 of [8] there exist sequences $(h_m)_m$ in H and $(u_p)_p$ in U_n^0 such that

$$d_n(h, E) = \lim_p \lim_m u_p(h_m), \;\; 0 = \lim_m \lim_p u_p(h_m).$$

By $d_n(h, E) = \lim_p \lim_m u_p(h_m) - \lim_m \lim_p u_p(h_m)$ we have $k_n(H) \leq \gamma_n(H)$.

Lemma 4. *Let $(h_\alpha)_\alpha$ be a net in a bounded subset H of a Fréchet space E. Let h be a $\sigma(E'', E')$-cluster point of $(h_\alpha)_\alpha$. If $(v_\beta)_\beta$ is a net in U_n^0 such that the involved limits $\lim_\beta \lim_\alpha v_\beta(h_\alpha)$ and $\lim_\alpha \lim_\beta v_\beta(h_\alpha)$ exist, then*

$$\left| \lim_\beta \lim_\alpha v_\beta(h_\alpha) - \lim_\alpha \lim_\beta v_\beta(h_\alpha) \right| \leq 2 d_n(h, E).$$

Hence $\gamma_n(H) \leq 2ck_n(H)$ for each $n \in \mathbb{N}$.

Proof. If $(u_\beta)_\beta$ is a net in U_n^0 that $\sigma(E',E)$-converges to 0 and the involved limits in $\lim_\beta \lim_\alpha u_\beta(h_\alpha)$ exist, then

$$\left|\lim_\beta \lim_\alpha u_\beta(h_\alpha)\right| \leq d_n(h,E). \tag{1}$$

Indeed, for each $\varepsilon > 0$ let $h_\varepsilon \in E$ be such that

$$d_n(h,h_\varepsilon) < d_n(h,E) + \varepsilon.$$

By the hypothesis $\lim_\alpha u_\beta(h_\alpha) = u_\beta(h)$ and $\lim_\beta u_\beta(h_\varepsilon) = 0$. Then

$$\left|\lim_\beta \lim_\alpha u_\beta(h_\alpha)\right| = \left|\lim_\beta u_\beta(h)\right| = \left|\lim_\beta u_\beta(h-h_\varepsilon)\right| \leq d_n(h,h_\varepsilon) < d_n(h,E)+\varepsilon.$$

The inequality $|\lim_\beta \lim_\alpha u_\beta(h_\alpha)| < d_n(h,E)+\varepsilon$ is true for each positive number ε, so we have

$$\left|\lim_\beta \lim_\alpha u_\beta(h_\alpha)\right| \leq d_n(h,E).$$

To prove the main inequality pick v a $\sigma(E',E)$-cluster point of $(v_\beta)_\beta$. By hypothesis, $\lim_\alpha \lim_\beta v_\beta(h_\alpha)$ exists and $v(h_\alpha)$ is a cluster point of $(v_\beta(h_\alpha))_\beta$ so $\lim_\beta v_\beta(h_\alpha) = v(h_\alpha)$. Then $\lim_\alpha v(h_\alpha)$ exists and $v(h)$ is a cluster point of $(v(h_\alpha))_\alpha$ so $\lim_\alpha v(h_\alpha) = v(h)$. Therefore $u_\beta := 2^{-1}(v_\beta - v)$ is a net in U_n^0 that $\sigma(E',E)$ converges to 0 and such that the involved limits in $\lim_\beta \lim_\alpha u_\beta(h_\alpha)$ exists, because by hypothesis the limits in $\lim_\beta \lim_\alpha v_\beta(h_\alpha)$ exist and $\lim_\beta \lim_\alpha v(h_\alpha) = \lim_\beta v(h) = v(h)$. Then

$$\begin{aligned}&\left|\lim_\beta \lim_\alpha v_\beta(h_\alpha) - \lim_\alpha \lim_\beta v_\beta(h_\alpha)\right| = \left|\lim_\beta \lim_\alpha v_\beta(h_\alpha) - \lim_\alpha v(h_\alpha)\right| = \\ &= \left|\lim_\beta \lim_\alpha v_\beta(h_\alpha) - \lim_\beta \lim_\alpha v(h_\alpha)\right| = 2\left|\lim_\beta \lim_\alpha 2^{-1}\left(v_\beta - v\right)(h_\alpha)\right| \leq 2d_n(h,E),\end{aligned}$$

where the last inequality follows from (1). Hence $\gamma_n(H) \leq 2ck_n(H)$ for each $n \in \mathbb{N}$.

Proposition 5. *$ck_n(H) \leq k_n(H) \leq \gamma_n(H) \leq 2ck_n(H)$ for a bounded subset H of a Fréchet space E and each $n \in \mathbb{N}$. Then $ck(H) = 0$ iff $k(H) = 0$.*

Proof. The second and third inequalities follow from previous lemmas. The first inequality is obvious.

Proposition 6. *If H is a bounded subset of a Fréchet space E, then the following conditions are equivalent:*

(i) ck(H)=0,
(ii) k(H)=0,
(iii) H is weakly relatively countably compact,
(iv) H is weakly relatively compact.

Proof. It is clear that (ii)⇒(iv)⇒(iii)⇒(i). The implication (i)⇒(ii) follows from Proposition 5.

3. Approximation by sequences and angelicity

The following theorem extends the last part of [2, Theorem 2.3] to Fréchet spaces.

Theorem 7. *Let H be a bounded subset of a Fréchet space E defined by the sequence of increasing seminorms $(\|.\|_n)_n$. Let $x^{**} \in \overline{H}^{\sigma(E'',E')}$. There exists a sequence $(x_m)_m$ in H such that if y^{**} is a $\sigma(E'',E')$-cluster point of $(x_m)_m$, then $\|x^{**} - y^{**}\|_n \leq \gamma_n(H)$ for each $n \in \mathbb{N}$.*

Proof. The proof is based on the following two observations.

Claim 1. Let S be a finite subset of $\overline{H}^{\sigma(E'',E')}$ and let n and m be two natural numbers. There exists a finite subset $L_n(m) \subset U_n^0$ such that for each $x^* \in U_n^0$ there exists $y^* \in L_n(m)$ satisfying

$$\sup\{|s(x^* - y^*)| : s \in S\} < m^{-1}.$$

Indeed, let p be the cardinal number of S. Then the claim follows from the fact that $\left\{(x^{**}(x^*))_{x^{**}\in S} : x^* \in U_n^0\right\}$ is a bounded subset of a product of p-copies of the scalar field (of real or complex numbers).

Claim 2. There exists a sequence $(x_m)_m$ in H and for each $m \in \mathbb{N}$ there exist finite sets $L_n(m) \subset U_n^0$ for $n = 1, 2, \cdots, m$, such that if $x^* \in U_n^0$, there is $y^* \in L_n(m)$ verifying

$$\sup\{|s(x^* - y^*)| : s \in \{x^{**}, x_1, x_2, \cdots, x_{m-1}\}\} < m^{-1}, \tag{2}$$

and

$$\sup\{|(x^{**} - x_m)(z^*)| : z^* \in L_m\} < m^{-1} \tag{3}$$

for $L_m = \bigcup\{L_n(q) : 1 \leq n \leq q \leq m\}$.

Indeed, applying Claim 1 with $S = \{x^{**}\}$ and $n = m = 1$ we provide a finite set $L_1(1) \subset U_1^0$ such that for each $x^* \in U_1^0$ there exists $y^* \in L_1(1)$ such that

$$\sup\{|s(x^* - y^*)| : s \in \{x^{**}\}\} < 1^{-1}.$$

Then for $x^{**} \in \overline{H}^{\sigma(E'',E')}$ there exists $x_1 \in H$ such that for $L_1 = L_1(1)$ we have

$$\sup\{|(x^{**} - x_1)(z^*)| : z^* \in L_1\} < 1^{-1}.$$

Assume that Claim 2 has been checked for a fixed $m \in \mathbb{N}$. To complete the proof it is enough to apply $m+1$ times the Claim 1 with $S = \{x^{**}, x_1, x_2, \cdots, x_m\}$ and we obtain the $m+1$ finite sets $L_n(m+1) \subset U_n^0$, $n = 1, 2, \cdots, m+1$, such that for each n and each $x^* \in U_n^0$ there exists $y^* \in L_n(m+1)$ satisfying

$$\sup\{|s(x^* - y^*)| : s \in \{x^{**}, x_1, x_2, \cdots, x_m\}\} < (m+1)^{-1},$$

and for $x^{**} \in \overline{H}^{\sigma(E'',E')}$ there exists $x_{m+1} \in H$ such that

$$\sup\{|(x^{**} - x_{m+1})(z^*)| : z^* \in L_{m+1}\} < (m+1)^{-1},$$

where $L_{m+1} = \bigcup\{L_n(q) : 1 \leq n \leq q \leq m+1\}$.

Finally we show that the sequence $(x_m)_m$ from Claim 2 is as required. Fix $x^* \in U_n^0$. From (2) it follows that for each $q > n$ there exists $y_q^* \in L_n(q)$ such that $\left|x^{**}(x^* - y_q^*)\right| < q^{-1}$ and $\left|x_j(x^* - y_q^*)\right| < q^{-1}$ for each $j < q$. Therefore

$$x^{**}(x^*) = \lim_q x^{**}(y_q^*) \tag{4}$$

and

$$x_j(x^*) = \lim_q x_j(y_q^*). \tag{5}$$

By (3) we have $x^{**}(z^*) = \lim_m x_m(z^*)$ for each $z^* \in \bigcup_m L_m$. In particular $x^{**}(y_q^*) = \lim_m x_m(y_q^*)$ for each y_q^*. This and (4) imply

$$x^{**}(x^*) = \lim_q \lim_m x_m(y_q^*). \tag{6}$$

Let y^{**} be a $\sigma(E'', E)$ cluster point of $(x_m)_m$. For the previously fixed $x^* \in U_n^0$ there exists a subsequence $(x_{m_r})_r$ such that $y^{**}(x^*) = \lim_r x_{m_r}(x^*)$, and then by (5) we have

$$y^{**}(x^*) = \lim_r \lim_q x_{m_r}(y_q^*). \tag{7}$$

Since $y_q^* \in L_n(q) \subset U_n^0$, we apply (6) and (7) to show $|x^{**}(x^*) - y^{**}(x^*)| \le \gamma_n(H)$. Since this holds for each $x^* \in U_n^0$, we conclude $\|x^{**} - y^{**}\|_n \le \gamma_n(H)$.

Corollary 8 ([13, 3.10 (1)]). *Every Fréchet space E is $\sigma(E, E')$-angelic.*

PROOF. Let H be a relatively countably compact subset of $(E, \sigma(E, E'))$. Then $ck(H) = 0$ so by Proposition 6, H is relatively compact in $(E, \sigma(E, E'))$. Theorem 7 implies that, if $x \in \overline{H}^{\sigma(E,E')}$ there exists a sequence $(x_p)_p$ in H such that x is the unique $\sigma(E, E')$-cluster point of the sequence $(x_p)_p$ in $(E, \sigma(E, E'))$. This and $\sigma(E, E')$-compactness of $\overline{H}^{\sigma(E,E')}$ implies that the sequence $(x_p)_p$ converges to x in $(E, \sigma(E, E'))$ and then angelicity of $(E, \sigma(E, E'))$ follows.

4. Approximation by sequences in $C_p(X)$

For a topological space X, a metric space (Z, d), we consider in Z^X the *standar supremum metric*, that we also denoted be d that we allow to take the value $+\infty$, i.e.,

$$d(f, g) = \sup\{d(f(x), g(x)) : x \in X\}.$$

For a relatively compact set $H \subset (Z^X, \tau_p)$ define

$$ck(H) := \sup_{\varphi \in H^{\mathbb{N}}} d(clust_{Z^X}(\varphi), C(X, Z)),$$

where $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. Clearly each relatively countably compact set in (Z^X, τ_p) is relatively compact. Let $\hat{d}(A, B) := \sup\{d(a, B) : a \in A\}$. Recall the following concepts.

(**a**) X is a *Lindelöf Σ-space* if there is an upper semi-continuous map from a (nonempty) subset $\Omega \subset \mathbb{N}^{\mathbb{N}}$ with compact values in X whose union is X, where the set of integers $\mathbb{N}$

is discrete and $\mathbb{N}^{\mathbb{N}}$ has the product topology, see [1]. If the same holds for $\Omega = \mathbb{N}^{\mathbb{N}}$, then X is called *K-analytic*.

(**b**) X is *quasi-Suslin* if there exists a set-valued map T from $\mathbb{N}^{\mathbb{N}}$ into X covering X such that if $\alpha_n \to \alpha$ and $x_n \in T(\alpha_n)$, then $(x_n)_n$ has a cluster point in $T(\alpha)$, see [19].

(**c**) X is *web-compact* [18] if there exists a nonempty subset $\Sigma \subset \mathbb{N}^{\mathbb{N}}$ and a family $\{A_\alpha : \alpha \in \Sigma\}$ in X whose union D is dense in X and, if

$$C_{n_1,\ldots,n_k} := \bigcup\{A_\beta : \beta = (m_k) \in \Sigma, m_j = n_j, j = 1, \ldots, k\}$$

for $\alpha = (n_k) \in \Sigma$ with $x_k \in C_{n_1,n_2,\ldots,n_k}$, then $(x_k)_k$ has a cluster point in X. If X is web-compact with $D = X$, we call X *strongly web-compact*. All *quasi-Suslin* spaces are strongly web-compact. By [18, Theorem 3] the space $C_p(X)$ is angelic if X is web-compact.

We need the following two technical facts from [5, Lemma 1] and [5, Lemma 3].

Lemma 9. *Let X be a topological space, Z a metric space and $H \subset Z^X$ a τ_p-relatively compact set. Then H 2ϵ-interchanges limits with relatively countably compact sets of X, where $\epsilon := ck(H) + \hat{d}(H, C(X,Z))$.*

Lemma 10. *Let (Z,d) be a separable metric space, X be a set and $H \subset Z^X$ with the pointwise topology τ_p and $\epsilon \geq 0$. Assume*

(i) $X = \bigcup\{A_\alpha : \alpha \in \Sigma\}$ for some family of sets $\{A_\alpha : \alpha \in \Sigma\}$.

(ii) For each $\alpha = (n_k) \in \Sigma$ the set H ϵ-interchanges limits in Z with every sequence $(x_n)_n$ in X that is eventually in each $C_{n_1\ldots n_k}$ for $k \in \mathbb{N}$.

Then for each $f \in \overline{H}$ (the closure in Z^X) there exists a sequence $(f_n)_n$ in H such that $\sup_{x\in X} d(f(x), g(x)) \leq \epsilon$ for any cluster point g of $(f_n)_n$ in Z^X.

We prove an extension of [5, Theorem 3.1] which yields to the quantitative version of Orihuela's angelic theorem [18, Theorem 3], see Corollary 17. The first Theorem 11 works for strongly web-compact spaces X. The other one Theorem 14 deals just with web-compact spaces X.

Theorem 11. *Let X be a web-compact space with a representation $D = \bigcup\{A_\alpha : \alpha \in \Sigma\}$ with $X = \overline{D}$. Let (Z,d) be a separable metric space and $H \subset Z^X$ a τ_p-relatively compact set. Then for each $f \in \overline{H}$ (the closure in Z^X) there exists a sequence $(f_n)_n$ in H such that*

$$\sup_{x\in D} d(f(x), g(x)) \leq 2ck(H) + 2\hat{d}(H, C(X,Z)) \leq 4ck(H)$$

for any cluster point g of $(f_n)_n$ in Z^X.

Proof. Let $\epsilon := ck(H) + \hat{d}(H, C(X,Z))$ and let $\tilde{H} = \{f|_D : f \in H\}$. We prove that condition (ii) in Lemma 10 holds for D and $\tilde{H}$. Take $\alpha = (n_k) \in \Sigma$ and let $(x_n)_n$ be a sequence in D that is eventually in each $C_{n_1\ldots n_k}$ for $k \in \mathbb{N}$. Note that each subsequence of $(x_n)_n$ admits a subsequence $(y_k)_k$ such that $y_k \in C_{n_1\ldots n_k}$ for $k \in \mathbb{N}$. Indeed, for n_1 there is $m_1 \in \mathbb{N}$ such that $x_n \in C_{n_1}$ for all $n \geq m_1$. Set $y_1 := x_{m_1}$. By induction we obtain a subsequence $(y_k)_k$ of $(x_n)_n$ such that $y_k \in C_{n_1\ldots n_k}$ for each $k \in \mathbb{N}$. The same procedure holds for any subsequence of $(x_n)_n$. Since X is web-compact, every such sequence $(y_k)_k$ has a cluster point in X. This means that the set $\{x_n : n \in \mathbb{N}\}$ is relatively countably compact in X. Now Lemma 9 applies to deduce that H 2ϵ-interchanges limits with every sequence $(x_n)_n$ so $\tilde{H}$ 2ϵ-interchanges limits as claimed. Condition (ii) has been checked.

Now let $f \in \overline{H}$ (the closure in Z^X). Since $f|_D \in \overline{\tilde{H}}$ (the closure in Z^D), by Lemma 10 there exists a sequence $(g_n)_n$ in $\tilde{H}$ such that $\sup_{x\in D} d(f(x), h(x)) \leq 2\epsilon$ for each cluster point h of $(g_n)_n$ in Z^X. For each g_n there exists $f_n \in H$ such that $f_n|_D = g_n$. If g is a cluster point of $(f_n)_n$ then $g|_D$ is cluster point of $(g_n)_n$ so $\sup_{x\in D} d(f(x), g(x)) \leq 2\epsilon$. This yields the first inequality. The other one follows like in the proof of [5, Theorem 3.1]: For $f \in H$ set $\varphi(n) := f$ for all $n \in \mathbb{N}$. Then $clust_{Z^X}(\varphi) = \{f\}$ and hence $\hat{d}(H, C(X, Z)) \leq ck(H)$.

Corollary 12. *Let X be a strongly web-compact space, (Z, d) a separable metric space, and let $H \subset Z^X$ be a τ_p-relatively compact set. Then*

$$ck(H) \leq \hat{d}(\overline{H}, C(X, Z)) \leq 3ck(H) + 2\hat{d}(H, C(X, Z))) \leq 5ck(H).$$

Proof. The first inequality follows from the definition. The last one follows from the proof of Theorem 11 with $D = X$. Now we show the middle one: We may assume that $ck(H) < \infty$. Fix $t \in \mathbb{R}$ with $ck(H) < t$ and $f \in \overline{H}$. Fix

$$\epsilon := ck(H) + \hat{d}(H, C(X, Z)).$$

By Theorem 11 there exists a sequence $(f_n)_n$ in H such that $\sup_{x\in X} d(f(x), g(x)) \leq 2\epsilon$ for any cluster point g of $(f_n)_n$ in Z^X. Since $ck(H) < t$, for this sequence $(f_n)_n$ there exists a cluster point g of $(f_n)_n$ such that $d(g, C(X, Z)) < t$. This yields the inequality.

Proposition 13. *Let X be a web-compact space with a representation $D = \bigcup\{A_\alpha : \alpha \in \Sigma\}$ with $X = \overline{D}$. Let (Z, d) be a separable metric space, let $H \subset Z^X$ be a τ_p-relatively compact set and let $f \in \overline{H}$ (the closure in Z^X). Then for each $\delta > 0$ and $x \in X$ there exists $U \subset X$ a neighbourhood of x such that*

$$d(f(x), f(d)) < 4ck(H) + 2\hat{d}(H, C(X, Z)) + \delta$$

for every $d \in U \cap D$.

Proof. Fik $x \in X$ and define $H' = \{j \in H : d(j(x), f(x)) < 4^{-1}\delta\}$. Since H' is the intersection of H and an open neigbourhood of f in Z^X, then $f \in \overline{H'}$. By Theorem 11 there exists a sequence $(f_n)_n$ in $H' \subset H$ such that

$$\sup_{d\in D} d(f(d), g(d)) \leq 2ck(H') + 2\hat{d}(H', C(X, Z)) \leq 2ck(H) + 2\hat{d}(H, C(X, Z)) \tag{8}$$

for any cluster point g of $(f_n)_n$ in Z^X. Observe also that if g is a cluster point of $(f_n)_n$, then

$$d(f(x), g(x)) \leq 4^{-1}\delta. \tag{9}$$

By definition of $ck(H)$ we can choose a cluster point g of $(f_n)_n$ such that $d(g, C(X, Z)) < ck(H) + 4^{-1}\delta$. Choose $h \in C(X, Z)$ such that

$$d(g(z), h(z)) < ck(H) + 4^{-1}\delta \tag{10}$$

for all $z \in X$. Since h is a continuous function, there exists $U \subset X$ a neighbourhood of x such that

$$d(h(x), h(z)) < 4^{-1}\delta \tag{11}$$

if $z \in U$. If $d \in U \cap D$ then

$$\begin{aligned}
d(f(x),&f(d)) \leq \\
&\leq d(f(x), g(x)) + d(g(x), h(x)) + d(h(x), h(d)) + d(h(d), g(d)) + d(g(d), f(d)) \\
&\leq 4^{-1}\delta + d(g(x), h(x)) + d(h(x), h(d)) + d(h(d), g(d)) + 2ck(H) + 2\hat{d}(H, C(X, Z)) \\
&< 4ck(H) + 2\hat{d}(H, C(X, Z)) + \delta
\end{aligned}$$

where we have applied (8) and (9) in the second inequality and (10) and (11) in the last inequality.

Theorem 14. *Let X be a web-compact space, (Z, d) be a separable metric space and $H \subset Z^X$ a τ_p-relatively compact set. Then for each $f \in \overline{H}$ (the closure in Z^X) there exists a sequence $(f_n)_n$ in H such that*

$$\sup_{x \in X} d(f(x), g(x)) \leq 10ck(H) + 6\hat{d}(H, C(X, Z)) \leq 16ck(H)$$

for any cluster point g of $(f_n)_n$ in Z^X.

Proof. Since X is a web-compact space, there is a representation $D = \bigcup\{A_\alpha : \alpha \in \Sigma\}$ with $X = \overline{D}$ that satisfies the definition of web-compact space. By Theorem 11, there exists a sequence $(f_n)_n$ in H such that

$$\sup_{d \in D} d(f(d), g(d)) \leq 2ck(H) + 2\hat{d}(H, C(X, Z)) \tag{12}$$

for any cluster point g of $(f_n)_n$ in Z^X. Fix $x \in X$, g a cluster point of $(f_n)_n$ and $\delta > 0$. Since $f, g \in \overline{H}$ by Proposition 13, there exist $U, V \subset X$ neighbourhoods of x such that

$$d(f(x), f(d)) < 4ck(H) + 2\hat{d}(H, C(X, Z))) + \delta$$

for every $d \in U \cap D$ and

$$d(g(x), g(d)) < 4ck(H) + 2\hat{d}(H, C(X, Z))) + \delta$$

for every $d \in V \cap D$. Pick $d \in D \cap U \cap V$, then

$$\begin{aligned}
d(f(x), g(x)) &\leq d(f(x), f(d)) + d(f(d), g(d)) + d(g(d), g(x)) \\
&\leq d(f(x), f(d)) + 2ck(H) + 2\hat{d}(H, C(X, Z)) + d(g(d), g(x)) \\
&< 10ck(H) + 6\hat{d}(H, C(X, Z)) + 2\delta.
\end{aligned}$$

Since $\delta > 0$ is arbitrary, the proof is over.

The following corollary follows from Theorem 14 like Corollary 12 from Theorem 11.

Corollary 15. *Let X be a web-compact space, (Z,d) a separable metric space and let $H \subset Z^X$ be a τ_p-relatively compact set. Then*

$$ck(H) \le \hat{d}(\overline{H}, C(X,Z)) \le 11ck(H) + 6\hat{d}(H, C(X,Z))) \le 17ck(H).$$

Corollary 16. *Let X be a web-compact space, (Z,d) a separable metric space and let $H \subset C(X,Z)$ be a τ_p-relatively compact set in X^Z. The following conditions are equivalent:*

(i) $ck(H) = 0$,

(ii) H *is a relatively countably compact subset of* $C(X,Z)$,

(iii) H *is a relatively compact subset of* $C(X,Z)$.

Proof. Clearly (iii)⇒(ii)⇒(i). The implication (i)⇒(iii) follows from Corollary 15.

Corollary 17 (Orihuela [18]). $C_p(X,Z)$ *is angelic for metric* Z *and web-compact* X.

Proof. By Fremlin's [13, Theorem 3.5] the space $C_P(X) := C_p(X,\mathbb{R})$ is angelic iff $C_p(X,Z)$ is angelic for any metric space X, so we prove the case for $C_p(X)$. Hence we need only to show the angelicity of $C_p(X)$ for web-compact. Now Corollary 17 follows from Theorem 14 (similarly as we did in the proof of Corollary 8).

Theorem 11 applies to spaces in the class $\mathfrak{G}$, i.e. lcs E for which its topological dual E' is covered by a family $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of subsets such that $A_\alpha \subset A_\beta$ if $\alpha \le \beta$ and in each A_α sequences are equicontinuous [8]. All (LM)-spaces (hence metrizable lcs), dual metric spaces (hence (DF)-spaces), etc., belong to class $\mathfrak{G}$, [8]. The space $X := (E', \sigma(E',E))$ is quasi-Suslin (hence strongly web-compact) for $E \in \mathfrak{G}$, see [12].

By [2, Proposition 2.6] for each Banach space E with the Corson property (C) we have $\hat{d}(\overline{H}, E) = ck(H)$, so this holds for any reflexive Banach space. Is the same true for any separable metrizable lcs ? The first part of Corollary 18 is due to Cascales-Orihuela [8].

Corollary 18. *Every lcs $E \in \mathfrak{G}$ is weakly angelic. If E is separable and metrizable, then $\hat{d}(\overline{H}, C(X)) = ck(H)$ for bounded $H \subset \mathbb{R}^X$ with $X := (E', \sigma(E',E))$.*

Proof. Corollary 17 yields the first claim. For the other one it is enough to show that X has countable tightness, since then we apply [5, Corollary 3.6]. Let $(U_n)_n$ be a basis of neighbourhoods of zero in E. Then $E' = \bigcup_n U_n^0$. Since E is separable, each set U_n° is metrizable and X is Lindelöf. Since X is hereditarily separable, it has countable tightness: Take a subset $V \subset X$. Let $D \subset V$ be a countable set such that $V \subset \overline{D}$. Then $\overline{V} \subset \overline{D}$. Therefore, any point in the closure of V is a limit of a countable subset.

We present also a particular case of Corollary 15 with better constants. First we note the following

Proposition 19. *Let X be a web-compact space, (Z,d) be a separable metric space, $H \subset Z^X$ a τ_p-relatively compact set and $f \in \overline{H}$ (the closure in Z^X). Then for each $\delta > 0$ and $x \in X$ there exists $U \subset X$ a neighbourhood of x such that*

$$d(f(x), f(y)) < 8ck(H) + 4\hat{d}(H, C(X,Z))) + \delta$$

for every $y \in U$.

PROOF. By Proposition 13 we know that x has a neigbourhood U such that

$$d(f(x), f(d)) < 4ck(H) + 2\hat{d}(H, C(X, Z))) + 2^{-1}\delta$$

for all $d \in U \cap D$. If $y \in U$, we can apply again Proposition 13 to get a neigbourhood V of y such that

$$d(f(y), f(d)) < 4ck(H) + 2\hat{d}(H, C(X, Z))) + 2^{-1}\delta$$

for all $d \in V \cap D$. Since D is a dense set we can choose $d \in D \cap U \cap V$ and then $d(f(x), f(y) \leq d(f(x), f(d)) + d(f(d), f(y)) < 8ck(H) + 4\hat{d}(H, C(X, Z))) + \delta$.

The *oscillation* (denoted by $osc(f, x)$) and *semi-oscillation* (denoted by $\mathrm{osc}^*(f, x)$) of a function $f \in Z^X$ at the point $x \in X$ are defined by

$$osc(f, x) = \inf_{U} \sup_{y,z \in U} d(f(y), f(z))$$

$$\mathrm{osc}^*(f, x) = \inf_{U} \sup_{y \in U} d(f(x), f(y))$$

where the infimum is taken over the neigbourhoods U of x in X.

In the following, we recall the relationship between the oscillation of a function and its distance from the continuous ones. In the cited reference, the theorem is stated under more restricted conditions: X is paracompact and f is uniformly bounded on X. The proof in the reference has two parts, (i) and (ii). For the first part (i), one can find an outline of the proof for our case when X is normal in Engelking [10, Exercise 1.7.5 (b)]. The second part (ii) does not require f to be uniformly bounded.

Theorem 20 ([7, Proposition 1.18]). *Let X be a normal space. If $f \in \mathbb{R}^X$, then*

$$d(f, C(X)) = \frac{1}{2} \sup_{x \in X} osc(f, x).$$

Combining this theorem with Proposition 19 we note the following version of Corollary 15 for the case $Z := \mathbb{R}$.

Proposition 21. *Let X be a web-compact and normal space, and $H \subset \mathbb{R}^X$ a τ_p-relatively compact set. Then*

$$ck(H) \leq \hat{d}(\overline{H}, C(X)) \leq 8ck(H) + 4\hat{d}(H, C(X))) \leq 12ck(H).$$

PROOF. We only have to prove the second inequality. For this, take $f \in \overline{H}$. By Proposition 19 we have

$$\mathrm{osc}^*(f, x) \leq 8ck(H) + 4\hat{d}(H, C(X)).$$

Since $osc(f, x) \leq 2\, \mathrm{osc}^*(f, x)$ then

$$osc(f, x) \leq 16ck(H) + 8\hat{d}(H, C(X)),$$

so by Theorem 20

$$d(f, C(X)) \leq 8ck(H) + 4\hat{d}(H, C(X)),$$

and the proof is over.

References

[1] A. V. Arkhangel'skii, *Topological function spaces*, Math. and its Applications **78**. Kluwer Academic Publishers. Dordrecht Boston London (1992).

[2] C. Angosto and B. Cascales. *Measures of weak noncompactness in Banach spaces.* Topology Appl. **156** (2009), 1412–1421.

[3] C. Angosto, *Distance to spaces of functions*, PhD thesis, Universidad de Murcia (2007).

[4] C. Angosto, B. Cascales, *A new look at compactness via distances to functions spaces*, World Sc. Pub. Co. (2008).

[5] C. Angosto, B. Cascales, *The quantitative difference between countable compactness and compactness* J. Math. Anal. Appl. **87** (2008), 1–13.

[6] C. Angosto, B. Cascales, I. Namioka, *Distances to spaces of Baire one functions*, Math. Z. **263** (2009), 103–124.

[7] Y. Benyamini, J. Lindenstrauus, *Geometric nonlinear functional analysis. Vol. 1*, American Mathematical Society, Providence, RI (2000).

[8] B. Cascales, J. Orihuela, *On Compactness in Locally Convex Spaces*, Math. Z. **195** (1987), 365-381.

[9] B. Cascales, W. Marciszesky and M. Raja. *Distance to spaces of continuous functions.* Topology Appl. **153** (2006), 2303 - 2319.

[10] R. Engelking,*General topology*, PWN−Polish Scientific Publishers, Warsaw, 1977, Translated from the Polish by the author, Monografie Matematyczne, Tom 60. [Mathematical Monographs, Vol. 60]. MR 58 ♯18316b.

[11] M. Fabian, P. Hájek, V. Montesinos, V. Zizler, *A quantitative version of Krein's theorem*, Rev. Mat. Iberoam. **21** (2005), 237–248.

[12] J. C. Ferrando, J. Kąkol, M. López Pellicer, S. A. Saxon, *Quasi-Suslin weak duals*, J. Math. Anal. Appl. **339** (2008), 1253-1263.

[13] K. Floret, *Weakly compact sets*, Lecture Notes in Math. **801**, Springer, Berlin (1980).

[14] A. Grothendieck. *Criteres de compacité dans les spaces fonctionnelles généraux.* Amer. J. Math. **74** (1952), 168–186.

[15] A. S. Granero, *An extension of the Krein-Smulian theorem*, Rev. Mat. Iberoam. **22** (2005), 93–100.

[16] A. S. Granero, M. Sánchez, *The class of universally Krein-Šmulian Banach spaces*, Bull. Lond. Math. Soc. **39** (2007), 529–540.

[17] G. Köthe, *Topological Vector Spaces I*, Springer Verlag, Berlin, (1969).

[18] J. Orihuela, *Pointwise compactness in spaces of continuous functions*, J. London Math. Soc. **36** (1987), 143–152.

[19] M. Valdivia, *Topics in Locally Convex Spaces*, North-Holland, Amsterdam (1982).