Duals of variable exponent Hörmander spaces $(0 < p^- \le p^+ \le 1)$ and some applications

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Dedicated to the memory of Nigel J. Kalton

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Abstract In this paper we characterize the dual $\left(\mathscr{B}^c_{p(\cdot)}(\Omega)\right)'$ of the variable exponent Hörmander space $\mathscr{B}^c_{p(\cdot)}(\Omega)$ when the exponent $p(\cdot)$ satisfies the conditions $0 < p^- \le p^+ \le 1$, the Hardy-Littlewood maximal operator M is bounded on $L_{p(\cdot)/p_0}$ for some $0 < p_0 < p^-$ and Ω is an open set in $\mathbb{R}^n$. It is shown that the dual $\left(\mathscr{B}^c_{p(\cdot)}(\Omega)\right)'$ is isomorphic to the Hörmander space $\mathscr{B}^{\mathrm{loc}}_\infty(\Omega)$ (this is the $p^+ \le 1$ counterpart of the isomorphism $\left(\mathscr{B}^c_{p(\cdot)}(\Omega)\right)' \simeq \mathscr{B}^{\mathrm{loc}}_{\widetilde{p'(\cdot)}}(\Omega)$, $1 < p^- \le p^+ < \infty$, recently proved by the authors) and hence the representation theorem $\left(\mathscr{B}^c_{p(\cdot)}(\Omega)\right)' \simeq l^{\mathbb{N}}_\infty$ is obtained. Our proof relies heavily on the properties of the Banach envelopes of the steps of $\mathscr{B}^c_{p(\cdot)}(\Omega)$ and on the extrapolation theorems in the variable Lebesgue spaces of entire analytic functions obtained in a precedent paper. Other results for $p(\cdot) \equiv p$, $0 < p < 1$, are also given (e.g. $\mathscr{B}^c_p(\Omega)$ does not contain any infinite-dimensional q-Banach subspace with $p < q \le 1$ or the quasi-Banach space $\mathscr{B}_p \cap \mathscr{E}'(Q)$ contains a copy of l_p when Q is a cube in $\mathbb{R}^n$). Finally, a question on complex interpolation (in the sense of Kalton) of variable exponent Hörmander spaces is proposed.

Keywords Variable exponent · Hardy-Littlewood maximal operator · Banach envelope · $L_{p(\cdot)}$-spaces of entire analytic functions, Hörmander spaces

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1 Introduction

Interest has increased recently in the variable exponent Lebesgue, Sobolev, Bessel Potential, Besov and Triebel-Lizorkin spaces (and in the harmonic analysis on the variable Lebesgue spaces) because of their applications to PDE of non-standard growth, modelling electrorheological fluids and quasi-Newtonian fluids, magnetostatics and image restoration (see e.g. [1, 2] and the books of Diening et al. [8] and Cruz-Uribe and Fiorenza [6]). In [17] we studied the properties of the (non-weighted) variable exponent Hörmander spaces $\mathscr{B}_{p(\cdot)}$, $\mathscr{B}^c_{p(\cdot)}(\Omega)$ and $\mathscr{B}^{\mathrm{loc}}_{p(\cdot)}(\Omega)$ (recall that the classical Hörmander spaces $\mathscr{B}_{p,k}$, $\mathscr{B}^c_{p,k}(\Omega)$ and $\mathscr{B}^{\mathrm{loc}}_{p,k}(\Omega)$ play a crucial role in the theory of linear partial differential operators (see e.g. [9])). In particular, extending a Hörmander's result [9, Chapter XV] to our context, we showed that if $p^- > 1$ and the Hardy-Littlewood maximal operator M is bounded on $L_{p(\cdot)}$ then $\left(\mathscr{B}^c_{p(\cdot)}(\Omega)\right)'$ is isomorphic to $\mathscr{B}^{\mathrm{loc}}_{\widetilde{p'(\cdot)}}(\Omega)$. In the present paper we extend this duality to exponents $p(\cdot)$ satisfying the conditions $0 < p^- \leq p^+ \leq 1$ and such that the Hardy-Littlewood maximal operator M is bounded on $L_{p(\cdot)/p_0}$ for some $0 < p_0 < p^-$. The techniques used are different from those used in [17] since if $p^+ < 1$ then the dual of $L_{p(\cdot)}$ is trivial and the steps $\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K)$ are quasi-Banach spaces instead of Banach spaces. A number of applications of this duality are also given. Firstly we prove that the steps of $\mathscr{B}^c_{p(\cdot)}(\Omega)$ are quasi-Banach spaces whose duals separate points. Then we introduce and study an important locally convex topology on $\mathscr{B}^c_{p(\cdot)}(\Omega)$ (considering the Banach envelopes of those steps) and we show that the space $\mathscr{B}^{\mathrm{loc}}_{\infty}(\Omega)$ is isomorphic to $\left(\mathscr{B}^c_{p(\cdot)}(\Omega)\right)'$ (this is the main result of the paper). The estimates obtained in [16, Theorem 3.5] play an essential role in the proof of this isomorphism. As a consequence of this result, we obtain a sequence space representation of the dual $\left(\mathscr{B}^c_{p(\cdot)}(\Omega)\right)'$ improving a result of [17] (the corresponding results for $p(\cdot) \equiv p$, $0 < p < 1$, are also new). Other results for $p(\cdot) \equiv p$, $0 < p < 1$, are also obtained (for instance, $\mathscr{B}^c_p(\Omega)$ does not contain any infinite-dimensional q-Banach subspace with $p < q \leq 1$ and the quasi-Banach space $\mathscr{B}_p \cap \mathscr{E}'(Q)$ contains a copy of l_p when Q is a cube in $\mathbb{R}^n$). Finally, two related questions on complex interpolation (in the sense of Kalton [13, Section 3]) of variable exponent Hörmander spaces are proposed.

1.1 Notation

1. Let E and F be topological linear spaces over $\mathbb{C}$. If E and F are (topologically) isomorphic we put $E \simeq F$. The (topological) dual of E is denoted by E' and is given the topology of uniform convergence on all the bounded subsets of E. We put $E \hookrightarrow F$ if E is a linear subspace of F and the canonical injection is continuous. If E is a Banach space, $E^{\mathbb{N}}$ (resp. $E^{(\mathbb{N})}$) is the topological product (resp. the locally convex direct sum) of a countable number of copies of E. If $\{E_i\}_{i=1}^{\infty}$ is a sequence of topological linear spaces such that $E_i \hookrightarrow E_{i+1}$ for each i, then their inductive limit is denoted by $\mathrm{ind}_i E_i$ (see [15]).
2. If $f \in L_1(\mathbb{R}^n)$ the Fourier transform of f, $\hat{f}$ or $\mathscr{F} f$, is defined by $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i\xi x}\, dx$. If f is a function on $\mathbb{R}^n$, then $\tilde{f}(x) = f(-x)$ for $x \in \mathbb{R}^n$. B_r is the closed Euclidean ball $\{x : |x| \leq r\}$ in $\mathbb{R}^n$. $C_0^\infty(\mathbb{R}^n)$, $C_0^\infty(\Omega)$ and $S(\mathbb{R}^n)$ are the usual Schwartz spaces (in the last space the norms $\max_{|\alpha|\leq m} \sup_{x\in\mathbb{R}^n} (1+|x|^2)^m |\partial^\alpha \varphi(x)|$, $m = 0, 1, 2, \ldots$, are denoted by $|\varphi|_m$). $\mathscr{D}'(\mathbb{R}^n)$, $\mathscr{D}'(\Omega)$ and $S'(\mathbb{R}^n)$ are their corresponding duals. $\mathscr{E}'(K)$ (K compact in $\mathbb{R}^n$) is the set of distributions on $\mathbb{R}^n$ with support contained in K. The Fourier transform in $S'(\mathbb{R}^n)$ is also denoted by $^\wedge$ (or $\mathscr{F}$). If $u \in S'(\mathbb{R}^n)$, $\tilde{u}$ is defined by $\langle \varphi, \tilde{u} \rangle = \langle \tilde{\varphi}, u \rangle$ for

all $\varphi \in S(\mathbb{R}^n)$; thus $^\sim$ coincides with the operator $(2\pi)^{-n}\mathscr{F}^2$. When we consider function spaces (or distribution spaces) defined on the whole Euclidean space $\mathbb{R}^n$, we shall omit the "$\mathbb{R}^n$" of their notation. The letter C will always denote a positive constant, not necessarily the same at each ocurrence.

3. Throughout this paper all vector spaces are assumed complex. By definition, a quasi-normed space is a vector space X with a quasi-norm $x \to \|x\|$ satisfying: (i) $\|x\| > 0$, $x \neq 0$, (ii) $\|\alpha x\| = |\alpha|\,\|x\|$, (iii) $\|x+y\| \leq C\left(\|x\| + \|y\|\right)$, $x, y \in X$, for some C independent of x, y. If X is complete, we say it is a quasi-Banach space. The quasi-norm is p-subadditive for some $p > 0$ if $\|x+y\|^p \leq \|x\|^p + \|y\|^p$, $x, y \in X$; in this case, if X is complete, we say it is a p-Banach space. Recall that if a quasi-normed space $(X, \|\cdot\|)$ is locally convex then it becomes a normed space: Let $B_X = \{x : \|x\| < 1\}$ be and let U be a balanced convex open neighborhood of 0 such that $U \subset B_X$. If $\varepsilon > 0$ is such that $\varepsilon B_X \subset U$ then the Minkowski functional of U, $\|\cdot\|_U$ ($\|\cdot\|_U = \inf\{\lambda > 0 : x \in \lambda U\}$), is a norm equivalent to $\|\cdot\|$ since
$$\varepsilon\,\|x\|_U \leq \|x\| \leq \|x\|_U$$
holds for all $x \in X$. (See [11, Chapter 1] and [14, Chapter 25].)

4. $\mathscr{P}^0$ is the set of all measurable functions $p(\cdot)$ on $\mathbb{R}^n$ with range in $(0,\infty)$ such that $p^- = \operatorname{ess\,inf}_{x\in\mathbb{R}^n} p(x) > 0$ and $p^+ = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x) < \infty$. $L_{p(\cdot)}$ denotes the set of all complex-valued measurable functions on $\mathbb{R}^n$ such that for some $\lambda > 0$, $\int_{\mathbb{R}^n}\left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty$. With the norm (quasi-norm if $p^- < 1$) defined by $\|f\|_{p(\cdot)} := \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n}\left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \leq 1\right\}$, $L_{p(\cdot)}$ becomes a Banach (quasi-Banach if $p^- < 1$) space. If $p^- < 1$ we can also define $L_{p(\cdot)}$ as the set of all measurable functions f such that $|f|^{p_0} \in L_{q(\cdot)}$, where $0 < p_0 \leq p^-$ and $q(x) = \frac{p(x)}{p_0}$. In this case we have $\|f\|_{p(\cdot)} = \left\||f|^{p_0}\right\|_{q(\cdot)}^{1/p_0}$. (See [7], [8] and [6].)

5. If K is a compact subset of $\mathbb{R}^n$ and $0 < p \leq \infty$, then $L_p^K := \{f \in S' :\ \operatorname{supp}\hat{f} \subset K,\ f \in L_p\}$. $\left(L_p^K, \|\cdot\|_p\right)$ is a quasi-Banach (Banach if $p \geq 1$) space (see [19, Chapters 1, 2]). If $p(\cdot) \in \mathscr{P}^0$ then
$$L_{p(\cdot)}^K := \{f \in S' :\ \operatorname{supp}\hat{f} \subset K,\ \|f\|_{p(\cdot)} < \infty\}.$$
$\left(L_{p(\cdot)}^K, \|\cdot\|_{p(\cdot)}\right)$ is a quasinormed space (normed if $p^- \geq 1$) linear space. From the Paley-Wiener-Schwartz theorem it follows that the elements of $L_{p(\cdot)}^K$ are entire analytic functions of exponential type. When $p(\cdot) \equiv p$, a constant, then $L_{p(\cdot)}^K = L_p^K$ with equality of quasi-norms (resp. norms). We shall denote by S^K the collection of all $f \in S$ such that $\operatorname{supp}\hat{f} \subset K$; obviously $S^K \subset L_{p(\cdot)}^K$. The spaces $L_{p(\cdot)}^K$ have been introduced and studied in [16].

6. Let $p(\cdot) \in \mathscr{P}^0$ be and let Ω be an open set in $\mathbb{R}^n$. Then $\mathscr{B}_{p(\cdot)} := \{u \in S' :\ \hat{u} \in L_{p(\cdot)}\}$. If $u \in \mathscr{B}_{p(\cdot)}$ we put $\|u\|_{\mathscr{B}_{p(\cdot)}} := \|\hat{u}\|_{p(\cdot)}$. $\left(\mathscr{B}_{p(\cdot)}, \|\cdot\|_{\mathscr{B}_{p(\cdot)}}\right)$ is a quasi-normed space isomorphic to $\left(L_{p(\cdot)} \cap S', \|\cdot\|_{p(\cdot)}\right)$ (a Banach space isomorphic to $L_{p(\cdot)}$ if $p^- \geq 1$). Now consider the space
$$\mathscr{B}_{p(\cdot)}^c(\Omega) := \bigcup\left\{\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K) :\ K \text{ compact in } \Omega\right\}.$$
If every $\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K)$ is equipped with the topology induced by $\mathscr{B}_{p(\cdot)}$, then $\mathscr{B}_{p(\cdot)}^c(\Omega)$ (endowed with the corresponding inductive linear topology) becomes a strict inductive limit
$$\mathscr{B}_{p(\cdot)}^c(\Omega) := \operatorname{ind}_K\left[\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K)\right].$$

Finally,

$$\mathscr{B}^{\mathrm{loc}}_{p(\cdot)}(\Omega) := \left\{u \in \mathscr{D}'(\Omega):\ \varphi u \in \mathscr{B}_{p(\cdot)} \text{ for all } \varphi \in C_0^\infty(\Omega)\right\}.$$

The topology of this space is generated by the seminorms (semiquasi-norms when $p^- < 1$) $u \to \|u\|_{p(\cdot),\varphi} := \|\varphi u\|_{\mathscr{B}_{p(\cdot)}}$, $\varphi \in C_0^\infty(\Omega)$.

The spaces $\mathscr{B}_{p(\cdot)}$, $\mathscr{B}^c_{p(\cdot)}(\Omega)$ and $\mathscr{B}^{\mathrm{loc}}_{p(\cdot)}(\Omega)$ are called variable exponent Hörmander spaces and have been introduced in [17]. If $p(\cdot) \equiv p$ and $p \geq 1$, these spaces coincide with the Hörmander spaces $\mathscr{B}_{p,1}$, $\mathscr{B}^{\mathrm{loc}}_{p,1}(\Omega)$ and $\mathscr{B}^{\mathrm{loc}}_{p,1}(\Omega)$ respectively (see [9, Chapter X]). Throughout this paper, $\mathscr{B}^{\mathrm{loc}}_{\infty}(\Omega)$ will denote the Hörmander space $\mathscr{B}^{\mathrm{loc}}_{\infty,1}(\Omega)$ (see again [9, Chapter X]).

2 The dual of $\mathscr{B}^c_{p(\cdot)}(\Omega)$ $(0 < p^- \leq p^+ \leq 1)$ and some applications

In [9], the isomorphism $\mathscr{B}^c_{2,k}(\Omega)' \simeq \mathscr{B}^{\mathrm{loc}}_{2,1/\check{k}}(\Omega)$ is shown (being Ω an open convex set in $\mathbb{R}^n$ and k a weight satisfying the estimate $k(x+y) \leq (1+C|x|)^N k(y)$, $x,y \in \mathbb{R}^n$, C and N positive constants). In Theorem 4.3 of [17] this isomorphism is extended to variable exponent Hörmander spaces with $1 < p^- \leq p^+ < \infty$: $\left(\mathscr{B}^c_{p(\cdot)}(\Omega)\right)' \simeq \mathscr{B}^{\mathrm{loc}}_{\widetilde{p'(\cdot)}}(\Omega)$. The technique used in [17] depends crucially on the condition $p^- > 1$. In this section we show that $\left(\mathscr{B}^c_{p(\cdot)}(\Omega)\right)'$ is isomorphic to $\mathscr{B}^{\mathrm{loc}}_{\infty}(\Omega)$ when the exponent $p(\cdot)$ satisfies $0 < p^- \leq p^+ \leq 1$. Our proof is based on the results of [16, 17], in particular on the extrapolation theorem [17, Theorem 3.5], and on the properties of the Banach envelopes of the steps $\left(\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K), \|\cdot\|_{\mathscr{B}_{p(\cdot)}}\right)$ of $\mathscr{B}^c_{p(\cdot)}(\Omega)$. Furthermore, we obtain a sequence space representation of the dual $\mathscr{B}^c_{p(\cdot)}(\Omega)'$ for $0 < p^- \leq p^+ \leq 1$. We also show that if Ω is an open cube with side length 1 and $0 < p < 1$, then $\mathscr{B}^c_{p(\cdot)}(\Omega)$ does not contain any infinite-dimensional q-Banach subspace with $p < q \leq 1$. As a consequence of this result we prove that $\left(\mathscr{B}_p \cap \mathscr{E}'(K), \|\cdot\|_{\mathscr{B}_p}\right)$ ($K = [-R,R]^n$ with $R < 1/2$) contains a copy of l_p and that if $0 < p_1, p_2 \leq 1$ then $\mathscr{B}^c_{p_1}(\Omega) \simeq \mathscr{B}^c_{p_2}(\Omega)$ if and only if $p_1 = p_2$.

Throughout this section, $p(\cdot)$ is a variable exponent in $\mathscr{P}^0$ such that $0 < p^- \leq p^+ \leq 1$ and the Hardy-Littlewood maximal operator M is bounded on $L_{p(\cdot)/p_0}$ for some $0 < p_0 < p^-$, Ω denotes an open set in $\mathbb{R}^n$, $\{\theta_j\}_{j=1}^\infty$ denotes a $C_0^\infty(\Omega)$-partition of unity on Ω and $\{K_j\}_{j=1}^\infty$ is a fundamental sequence of compact subsets of Ω such that $K_j = \overline{\mathring{K}_j}$, $\mathring{K}_j$ has the segment property and $\operatorname{supp}\theta_j \subset K_j$ for each j.

We start recalling some basic facts about the Banach envelope of a quasi-normed space. Let $(X, \|\cdot\|_X)$ be a quasi-normed space whose dual X' separates the points of X and let B_X be the unit ball of X. Then X' is a Banach space under the norm $\|x'\| = \sup\{|\langle x, x'\rangle| : x \in B_X\}$. The Banach envelope X_c of $(X, \|\cdot\|_X)$ is the completion of X in the norm $\|\cdot\|_c$ defined by

$$\|x\|_c := \sup\left\{|\langle x, x'\rangle| : \|x'\| \leq 1\right\}.$$

$\|\cdot\|_c$ coincides with the Minkowski functional of the convex hull of B_X, $\|\cdot\|_c \leq \|\cdot\|_X$ and the inclusion $X \hookrightarrow X_c$ is continuous with dense range. X_c has the property that any bounded linear operator $L : X \to Y$ into a Banach space extends with preservation of norm to a bounded linear operator $L : X_c \to Y$, thus $(X_c)'$ (and $(X, \|\cdot\|_c)'$) becomes linearly isometric to X' (see [11, pp. 27, 28], [12, Introduction]).

Next we prove two results on the space $\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K)$.

Proposition 2.1 *Let K be a compact subset of $\mathbb{R}^n$. If $K = \overline{O}$ and O is an open set with the segment property, then $\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K)$ (equipped with the quasi-norm $\|\cdot\|_{\mathscr{B}_{p(\cdot)}}$) is a quasi-Banach space whose dual separates points.*

Proof Since $\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K)$ is isomorphic (via the Fourier transform) to $L^{-K}_{p(\cdot)}$, it suffices to apply Theorem 3.5 of [16]. □

Proposition 2.2 *Let $0 < p < 1$ and let $K = [-R,R]^n$ with $0 < R < 1/2$. Then $\mathscr{B}_p \cap \mathscr{E}'(K)$ (equipped with $\|\cdot\|_{\mathscr{B}_p}$) is non-locally convex.*

Proof Since $\mathscr{B}_p \cap \mathscr{E}'(K)$ and L^K_p are isomorphic it suffices to see that L^K_p is non-locally convex. It is a well-known fact that the mapping

$$D : L^K_p \to l_p(\mathbb{Z}^n) : f \to \big(f(m)\big)_{m\in\mathbb{Z}^n}$$

is an isomorphic embedding (see [4, pp. 101, 197] for $n = 1$ and [5, Lema 1.8, p. 17] for $n \ge 1$). If L^K_p were locally convex (i.e. a Banach space, see Notation 3) then the operator D would be a compact operator by virtue of a result of Stiles [18, Theorem 4] and thus L^K_p would be finite-dimensional. The proof is complete since that L^K_p is infinite-dimensional (e.g. $S^K \subset L^K_p$). □

Let $\mathscr{B}^c_{p(\cdot)}(\Omega)[\mathscr{T}]$ the topological inductive limit of the sequence of quasi-Banach spaces $\big\{\big(\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_j), \|\cdot\|_{\mathscr{B}_{p(\cdot)}}\big) : j \ge 1\big\}$. Let $\mathscr{B}^c_{p(\cdot)}(\Omega)[\mathscr{T}_c]$ be the topological inductive limit of the sequence of normed spaces $\big\{\big(\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_j), \|\cdot\|_j\big) : j \ge 1\big\}$ where $\|\cdot\|_j$ is the Minkowski functional of the convex hull of the unit ball of the quasi-Banach space $\big(\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_j), \|\cdot\|_{\mathscr{B}_{p(\cdot)}}\big)$. Then we have

Proposition 2.3

1. *$\mathscr{T}_c \subset \mathscr{T}$ and $\big(\mathscr{B}^c_{p(\cdot)}(\Omega)[\mathscr{T}]\big)' = \big(\mathscr{B}^c_{p(\cdot)}(\Omega)[\mathscr{T}_c]\big)'$.*
2. *$\mathscr{T}_c$ is generated by the system of norms $\big\{q_{(C_i)}(\cdot) := \sum_{i=1}^{\infty} C_i \|\theta_i \cdot\|_i : (C_i)_{i=1}^{\infty} \in (\mathbb{R}_+)^{\mathbb{N}}\big\}$.*

Proof Firstly let us recall that for any compact subset K of Ω, $\theta u \in \mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K \cap \operatorname{supp}\theta)$ for all $\theta \in C^\infty_0(\Omega)$ and for all $u \in \mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K)$ and that, for every $u \in \mathscr{B}^c_{p(\cdot)}(\Omega)$, $\theta_i u = 0$ for all i large enough (see [17, Theorem 3.5/4, Remark 3.6/2]).

1. For all j we have $\|\cdot\|_j \le \|\cdot\|_{\mathscr{B}_{p(\cdot)}}$. This proves that the identity $id : \mathscr{B}^c_{p(\cdot)}(\Omega)[\mathscr{T}] \to \mathscr{B}^c_{p(\cdot)}(\Omega)[\mathscr{T}_c]$ is continuous, i.e. that $\mathscr{T}_c \subset \mathscr{T}$. On the other hand, the duals of the spaces $\mathscr{B}^c_{p(\cdot)}(\Omega)[\mathscr{T}]$ and $\mathscr{B}^c_{p(\cdot)}(\Omega)[\mathscr{T}_c]$ coincide since the corresponding steps have linearly isometric duals (see Proposition 2.1 and previous remarks to this proposition).

2. Taking into account that for every $u \in \mathscr{B}^c_{p(\cdot)}(\Omega)$ there exists a positive integer m such that $u = \sum_{i=1}^{m} \theta_i u$ and that every $\|\cdot\|_i$ is a norm, it is immediate to verify that the $q_{(C_i)}$ are norms. Let $\mathscr{T}'$ be the topology generated by this system of norms. Let us see that the identity

$$id : \mathscr{B}^c_{p(\cdot)}(\Omega)[\mathscr{T}'] \to \mathscr{B}^c_{p(\cdot)}(\Omega)[\mathscr{T}_c]$$

is continuous. Let $\|\cdot\|$ be a seminorm on $\mathscr{B}^c_{p(\cdot)}(\Omega)$ such that its restriction to each step $\big(\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_j), \|\cdot\|_j\big)$ is continuous (these seminorms generate the topology $\mathscr{T}_c$). Then there exist constants $C_j > 0$ such that $\|u\| \le C_j \|u\|_j$ for all $u \in \mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_j)$, $j = 1, 2, \dots$

Let $u \in \mathscr{B}^c_{p(\cdot)}(\Omega)$. We know that there is a positive integer m such that $\theta_i u = 0$ for all $i > m$ and that $u = \sum_{i=1}^m \theta_i u$. Then, since each $\theta_i u$ is in $\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_i)$, we get

$$\|u\| = \Big\|\sum_{i=1}^m \theta_i u\Big\| \le \sum_{i=1}^m \|\theta_i u\| \le \sum_{i=1}^m C_i \|\theta_i u\|_i = \sum_{i=1}^\infty C_i \|\theta_i u\|_i = q_{(C_i)}(u)$$

and this proves the required continuity. Thus $\mathscr{T}_c$ is coarser than $\mathscr{T}'$. Next we shall show that $\mathscr{T}' \subset \mathscr{T}$. It will be sufficient to see that every canonical injection

$$\big(\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_j), \|\cdot\|_{\mathscr{B}_{p(\cdot)}}\big) \hookrightarrow \mathscr{B}^c_{p(\cdot)}(\Omega)[\mathscr{T}']$$

is continuous. Given $q_{(C_i)}$, the Theorem 3.5/2 of [16] and the continuity of the Fourier transform show that there are a positive integer k and a positive constant C such that

$$\begin{aligned} q_{(C_i)}(u) &= \sum_{i=1}^m C_i \|\theta_i u\|_i \le \sum_{i=1}^m C_i \|\theta_i u\|_{\mathscr{B}_{p(\cdot)}} = \sum_{i=1}^m C_i \|\widehat{\theta_i u}\|_{p(\cdot)} \\ &= (2\pi)^{-n} \sum_{i=1}^m C_i \|\hat{\theta}_i * \hat{u}\|_{p(\cdot)} \le C \sum_{i=1}^m C_i |\theta_i|_k \|\hat{u}\|_{p(\cdot)} = C\Big(\sum_{i=1}^m C_i |\theta_i|_k\Big) \|u\|_{\mathscr{B}_{p(\cdot)}} \end{aligned}$$

holds for all $u \in \mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_j)$ (m is independent of u). Thus $\mathscr{T}' \subset \mathscr{T}$. Then, taking into account 1. and the inclusions $\mathscr{T}_c \subset \mathscr{T}' \subset \mathscr{T}$, we get $\big(\mathscr{B}^c_{p(\cdot)}(\Omega)[\mathscr{T}']\big)' = \big(\mathscr{B}^c_{p(\cdot)}(\Omega)[\mathscr{T}_c]\big)'$. But $\mathscr{B}^c_{p(\cdot)}(\Omega)[\mathscr{T}_c]$ is an inductive limit of normed spaces which implies that $\mathscr{T}_c$ is the finest locally convex topology on $\mathscr{B}^c_{p(\cdot)}(\Omega)$ which has $\big(\mathscr{B}^c_{p(\cdot)}(\Omega)[T_c]\big)'$ as dual space (see [15, § 21, p. 260 & § 28, p. 379]), therefore necessarily $\mathscr{T}'$ is coarser than $\mathscr{T}_c$. Thus, $\mathscr{T}_c = \mathscr{T}'$ and the proof of proposition is complete. □

Remark

1. In general, the topology $\mathscr{T}_c$ is strictly coarser than the topology $\mathscr{T}$: Let us assume $\Omega =]-\frac{1}{2}, \frac{1}{2}[^n$ and $0 < p < 1$. Then, since $\big(\mathscr{B}_p \cap \mathscr{E}'([-R,R]^n), \|\cdot\|_{\mathscr{B}_p}\big)$ with $0 < R < 1/2$ is a topological linear subspace of $\mathscr{B}^c_p(\Omega)[\mathscr{T}]$ (see [17, Theorem 3.5/3]), the Proposition 2.2 shows that $\mathscr{B}^c_p(\Omega)[\mathscr{T}]$ is non-locally convex. Since $\mathscr{T}_c$ is locally convex, we obtain the required conclusion.
2. It is easy to prove that the inductive limit topology $\mathscr{T}$ is also generated by the system of p_0-norms
$$\Big\{\Big(\sum_{i=1}^\infty C_i \|\theta_i \cdot\|^{p_0}_{\mathscr{B}_{p(\cdot)}}\Big)^{1/p_0} : (C_i)_{i=1}^\infty \in (\mathbb{R}_+)^{\mathbb{N}}\Big\}.$$

Proposition 2.4 $\big(\mathscr{B}^c_{p(\cdot)}(\Omega)[\mathscr{T}]\big)'$ *is a Fréchet space.*

Proof Since the topology of $\big(\mathscr{B}^c_{p(\cdot)}(\Omega)[\mathscr{T}]\big)'$ (i.e. the topology of the uniform convergence on bounded subsets of $\mathscr{B}^c_{p(\cdot)}(\Omega)[\mathscr{T}]$) is metrizable by [17, Theorem 3.5/3], the proof of the proposition follows by standard arguments. □

Now we can show the $p^+ \le 1$ counterpart of Theorem 4.3 of [17] ($\big(\mathscr{B}^c_{p(\cdot)}(\Omega)\big)' \simeq \mathscr{B}^{\text{loc}}_{\widetilde{p'(\cdot)}}(\Omega)$ for $1 < p^- \le p^+ < \infty$). We will need the spaces $l_1(C_i, X_i)$ and $l_\infty(C_i, X_i)$: If $(C_i) \in (\mathbb{R}_+)^{\mathbb{N}}$ and (X_i) is a sequence of normed spaces then $l_1(C_i, X_i)$ (resp. $l_\infty(C_i, X_i)$) is the set of all sequences $(x_i) \in \prod_{i=1}^{\infty} X_i$ such that $\|(x_i)\|_1 = \sum_{i=1}^{\infty} C_i \|x_i\|_{X_i} < \infty$ (resp. $\|(x_i)\|_\infty = \sup_i C_i \|x_i\|_{X_i} < \infty$). It is well known that the Banach spaces $\left(l_\infty\left(\frac{1}{C_i}, X_i'\right), \|\cdot\|_\infty\right)$ and $\left(l_1(C_i, X_i), \|\cdot\|_1\right)'$ are linearly isometric via the mapping A defined by $(x_i') \to \langle (x_i), A\big((x_i')\big)\rangle := \sum_{i=1}^{\infty} \langle x_i, x_i'\rangle$.

Theorem 2.1 *$\left(\mathscr{B}^c_{p(\cdot)}(\Omega)[\mathscr{T}]\right)'$ is isomorphic to $\mathscr{B}^{\mathrm{loc}}_\infty(\Omega)$ when $0 < p^+ \le 1$. In particular, $\left(\mathscr{B}^c_p(\Omega)[\mathscr{T}]\right)' \simeq \mathscr{B}^{\mathrm{loc}}_\infty(\Omega)$ for $0 < p \le 1$.*

Proof Let L be a continuous linear functional on $\mathscr{B}^c_{p(\cdot)}(\Omega)[\mathscr{T}]$. By Proposition 2.3/1, L is also a continuous linear functional on $\mathscr{B}^c_{p(\cdot)}(\Omega)[\mathscr{T}_c]$ and so, by Proposition 2.3/2, there exists an element (C_i) in $(\mathbb{R}_+)^{\mathbb{N}}$ such that

$$|\langle u, L\rangle| \le \sum_{i=1}^{\infty} C_i \|\theta_i u\|_i, \quad u \in \mathscr{B}^c_{p(\cdot)}(\Omega).$$

Then the mapping $Z : \mathscr{B}^c_{p(\cdot)}(\Omega)[\mathscr{T}_c] \to l_1\left(C_i, (\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_i), \|\cdot\|_i)\right) : u \to (\theta_i u)$, is well defined and is linear, injective and continuous (see the proof of Proposition 2.3). Since the linear functional $L \circ Z^{-1}$ satisfies $|\langle (\theta_i u), L \circ Z^{-1}\rangle| \le \|(\theta_i u)\|_1$, $u \in \mathscr{B}^c_{p(\cdot)}(\Omega)$, the Hahn-Banach theorem shows the existence of a linear functional $(L \circ Z^{-1})^- \in \Big(l_1\big(C_i, (\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_i), \|\cdot\|_i)\big)\Big)'$ of norm at most 1 which extends $L \circ Z^{-1}$. Then, by the isometric isomorphism

$$A : l_\infty\Big(\frac{1}{C_i}, \big(\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_i), \|\cdot\|_i\big)'\Big) \to \Big(l_1\big(C_i, (\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_i), \|\cdot\|_i)\big)\Big)'$$

defined by $\langle (u_i), A\big((\sigma_i)\big)\rangle = \sum_{i=1}^{\infty} \langle u_i, \sigma_i\rangle$, we can find $(\xi_i) \in l_\infty\big(\frac{1}{C_i}, (\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_i), \|\cdot\|_i)'\big)$ such that $A\big((\xi_i)\big) = (L \circ Z^{-1})^-$, i.e. such that $\sum_{i=1}^{\infty} \langle u_i, \xi_i\rangle = \langle (u_i), (L \circ Z^{-1})^-\rangle$ for all $(u_i) \in l_1\big(C_i, (\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_i), \|\cdot\|_i)\big)$. In particular, we get the following representation of L

$$\langle u, L\rangle = \langle Z(u), (L \circ Z^{-1})^-\rangle = \sum_{i=1}^{\infty} \langle \theta_i u, \xi_i\rangle, \quad u \in \mathscr{B}^c_{p(\cdot)}(\Omega).$$

Next, we shall prove that the mapping

$$\Phi : \left(\mathscr{B}^c_{p(\cdot)}(\Omega)[\mathscr{T}]\right)' \to \mathscr{B}^{\mathrm{loc}}_\infty(\Omega)$$

defined by $\Phi(L) = \sum_{i=1}^{\infty} [\theta_i \xi_i]$, where (ξ_i) is the sequence which represents to L and $[\theta_i \xi_i]$ is the distribution on Ω defined by $\langle \varphi, [\theta_i \xi_i]\rangle = \langle \theta_i \varphi, \xi_i\rangle$ for $\varphi \in C_0^\infty(\Omega)$, is an isomorphism. Firstly let us see that Φ is well defined:

(i) We claim that each $[\theta_i \xi_i] \in \mathscr{B}^{\mathrm{loc}}_\infty(\Omega)$. For every $\varphi \in C_0^\infty(\Omega)$, $\theta_i \varphi \in \mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_i)$ and so $\langle \theta_i \varphi, \xi_i\rangle$ makes sense. Furthermore, if $\varphi_\nu \to 0$ in $C_0^\infty(K)$ then also $\theta_i \varphi_\nu \to 0$ in $C_0^\infty(K)$ and this implies that $\theta_i \varphi_\nu \to 0$ in S, i.e. $\widehat{\theta_i \varphi_\nu} \to 0$ in S. This shows that $\theta_i \varphi_\nu \to 0$ in $\left(\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_i), \|\cdot\|_{\mathscr{B}_{p(\cdot)}}\right)$ and thus in $\left(\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_i), \|\cdot\|_i\right)$. Therefore, $\langle \varphi_\nu, [\theta_i \xi_i]\rangle = \langle \theta_i \varphi_\nu, \xi_i\rangle \to 0$ and $[\theta_i \xi_i]$ becomes a distribution on Ω. To establish the claim, it remains to prove that $\varphi[\theta_i \xi_i] \in \mathscr{B}_\infty$, i.e. $(\varphi[\theta_i \xi_i])^\wedge \in L_\infty$, for each $\varphi \in C_0^\infty(\Omega)$. Given such a φ, it is easily seen that $\varphi[\theta_i \xi_i]$ is a distribution on $\mathbb{R}^n$ whose support is contained in K_i. Thus $\left(\varphi[\theta_i \xi_i]\right)^\wedge$ coincides with the Fourier-Laplace transform of $\varphi[\theta_i \xi_i]$ (see [10, Theorem 7.1.14]) defined by

$$(\varphi[\theta_i \xi_i])^\wedge(x) = \langle e^{-i(\cdot)x} \chi, \varphi[\theta_i \xi_i]\rangle, \quad x \in \mathbb{R}^n,$$

where $\chi \in C_0^\infty(\Omega)$ and $\chi = 1$ in a neighborhood of K_i. Since $\theta_i \chi = \theta_i$, we obtain

$$\big(\varphi[\theta_i \xi_i]\rangle\big)^\wedge(x) = \langle \theta_i \varphi e^{-i(\cdot)x}, \xi_i \rangle$$

and so

$$\begin{aligned} \big|(\varphi[\theta_i\xi_i])^\wedge(x)\big| &\le \|\xi_i\| \, \big\|\theta_i \varphi e^{-i(\cdot)x}\big\|_i \le \|\xi_i\| \, \big\|\theta_i \varphi e^{-i(\cdot)x}\big\|_{\mathscr{B}_{p(\cdot)}} \\ &= \|\xi_i\| \, \big\|(\theta_i \varphi e^{-i(\cdot)x})^\wedge\big\|_{p(\cdot)} = \|\xi_i\| \, \big\|\widehat{\theta_i\varphi}\big((\cdot)+x\big)\big\|_{p(\cdot)} \end{aligned}$$

where $\|\xi_i\|$ is the norm of the functional ξ_i. Now we show that $\big\|\widehat{\theta_i\varphi}\big((\cdot)+x\big)\big\|_{p(\cdot)} \le C$ with C independent of $x \in \mathbb{R}^n$. Indeed, if $q(\cdot) = p(\cdot)/p_0$ we have, by using [8, Lemma 3.2.5],

$$\begin{aligned} \big\|\widehat{\theta_i\varphi}\big((\cdot)+x\big)\big\|_{p(\cdot)} &:= \big\||\widehat{\theta_i\varphi}((\cdot)+x)|^{p_0}\big\|_{q(\cdot)}^{1/p_0} \\ &\le \max\Big\{ \Big(\int_{\mathbb{R}^n} |\widehat{\theta_i\varphi}(y+x)|^{p(y)} dy\Big)^{1/p^-}, \Big(\int_{\mathbb{R}^n} |\widehat{\theta_i\varphi}(y+x)|^{p(y)} dy\Big)^{1/p^+} \Big\} \\ &\le 2^{1/p^- -1} \max\Big\{ \|\widehat{\theta_i\varphi}\|_{p^-} + \|\widehat{\theta_i\varphi}\|_{p^+}^{p^+/p^-}, \|\widehat{\theta_i\varphi}\|_{p^+} + \|\widehat{\theta_i\varphi}\|_{p^-}^{p^-/p^+} \Big\} \end{aligned}$$

and this bound is independent of $x \in \mathbb{R}^n$. Therefore $\varphi[\theta_i\xi_i] \in \mathscr{B}_\infty$ and $[\theta_i\xi_i] \in \mathscr{B}_\infty^{\mathrm{loc}}(\Omega)$.

(ii) The series $\sum_{i=1}^\infty [\theta_i\xi_i]$ converges in $\mathscr{B}_\infty^{\mathrm{loc}}(\Omega)$ since this space is a Fréchet space and for all $\varphi \in C_0^\infty(\Omega)$ we have $\sum_{i=1}^\infty \big\|[\theta_i\xi_i]\big\|_{\infty,\varphi} = \sum_{i=1}^\infty \big\|\varphi[\theta_i\xi_i]\big\|_{\mathscr{B}_\infty} < \infty$ (take into account that $\theta_i\varphi = 0$, and thus $\varphi[\theta_i\xi_i] = 0$, for all i large enough since $\operatorname{supp}\varphi$ is a compact subset of Ω).

(iii) If $(L \circ Z^{-1})^{=} \in \Big(l_1\big(C_i, (\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_i), \|\cdot\|_i)\big)\Big)'$ is another extension of $L \circ Z^{-1}$ and $(\eta_i) \in l_\infty\Big(\frac{1}{C_i}, \big(\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_i), \|\cdot\|_i\big)'\Big)$ is such that $\langle u, L\rangle = \sum_{i=1}^\infty \langle \theta_i u, \eta_i\rangle$ for all $u \in \mathscr{B}_{p(\cdot)}^c(\Omega)$, then $\sum_{i=1}^\infty [\theta_i\xi_i] = \sum_{i=1}^\infty [\theta_i\eta_i]$ (using the embedding $\mathscr{B}_\infty^{\mathrm{loc}}(\Omega) \hookrightarrow \mathscr{D}'(\Omega)$ [9, Theorem 10.1.26] we have $\langle \varphi, \sum_{i=1}^\infty [\theta_i\xi_i]\rangle = \sum_{i=1}^\infty \langle \varphi, [\theta_i\xi_i]\rangle = \sum_{i=1}^\infty \langle \theta_i\varphi, \xi_i\rangle = \langle \varphi, L\rangle = \cdots = \langle \varphi, \sum_{i=1}^\infty [\theta_i\eta_i]\rangle$ for any $\varphi \in C_0^\infty(\Omega)$).

(iv) Let $(C_i^1) \in (R_+)^{\mathbb{N}}$ be another sequence such that $|\langle u, L\rangle| \le \sum_{i=1}^\infty C_i^1 \|\theta_i u\|_i$ for all $u \in \mathscr{B}_{p(\cdot)}^c(\Omega)$. Let Z^1 be the corresponding operator, let $\big(L \circ (Z^1)^{-1}\big)^-$ be an extension of $L \circ (Z^1)^{-1}$ to $l_1\big(C_i^1, (\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_i), \|\cdot\|_i)\big)$ and let $(\xi_i^1) \in l_\infty\Big(\frac{1}{C_i^1}, \big(\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_i), \|\cdot\|_i\big)'\Big)$ be the sequence which represents this extension. Then $\langle u, L\rangle = \sum_{i=1}^\infty \langle \theta_i u, \xi_i^1\rangle$ in $\mathscr{B}_{p(\cdot)}^c(\Omega)$ and, reasoning as in *(iii)*, we see that $\sum_{i=1}^\infty [\theta_i\xi_i] = \sum_{i=1}^\infty [\theta_i\xi_i^1]$.

All this shows that Φ is well defined. The simple proof of the linearity of Φ will be omitted. If $\Phi(L) = 0$ then $0 = \langle \varphi, \Phi(L)\rangle = \sum_{i=1}^\infty \langle \theta_i\varphi, \xi_i\rangle = \langle \varphi, L\rangle$ for any $\varphi \in C_0^\infty(\Omega)$, and since $C_0^\infty(\Omega)$ is dense in $\mathscr{B}_{p(\cdot)}^c(\Omega)$ [17, Theorem 3.5] we obtain $L = 0$. Therefore, Φ is injective. Let us see that Φ is surjective: Let (χ_i) be a sequence in $C_0^\infty(\Omega)$ such that $\chi_i = 1$ in K_i and $\operatorname{supp}\chi_i \subset \overset{\circ}{K}_{i+1}$. Let ν be an element of $\mathscr{B}_\infty^{\mathrm{loc}}(\Omega)$. For each $\varphi \in C_0^\infty(\Omega)$, $\sum_{i=1}^\infty \|\theta_i\nu\|_{\infty,\varphi} = \sum_{i=1}^\infty \|(\theta_i\varphi)\nu\|_{\mathscr{B}_\infty} < \infty$ ($\theta_i\varphi = 0$ for all i large enough) and so the series $\sum_{i=1}^\infty \theta_i\nu$ converges in $\mathscr{B}_\infty^{\mathrm{loc}}(\Omega)$. Then we have the decomposition (recall that (θ_i) is a $C_0^\infty(\Omega)$-partition of unity on Ω)

$$\nu = \sum_{i=1}^\infty \theta_i\nu = \sum_{i=1}^\infty (\theta_i\chi_i)\nu = \sum_{i=1}^\infty \theta_i(\chi_i\nu) = \sum_{i=1}^\infty \theta_i\nu_i \tag{2.1}$$

where $\nu_i = \chi_i \nu$. We now define the functional

$$\langle u, L\rangle = (2\pi)^{-n} \sum_{i=1}^{\infty} \int_{\mathbb{R}^n} \widehat{\theta_i u}(x)\, \hat{\nu}_i(x)\, dx\,, \quad u \in \mathscr{B}^c_{p(\cdot)}(\Omega)\,,$$

and we show that is $\mathscr{T}$-continuous. Fix $\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_j)$. Take $u \in \mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_j)$. Every $\theta_i u$ is in $\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_j)$ and every $\nu_i \in \mathscr{B}_\infty$, thus $\widehat{\theta_i u} \in L^{-K_j}_{p(\cdot)}$ and $\hat{\nu}_i \in L_\infty$. Furthermore, since $L^{-K_j}_{p(\cdot)} \hookrightarrow L^{-K_j}_1$ (see [16, Theorem 3.5/5]), there is a constant $C > 0$ such that

$$\|\widehat{\theta_i u}\|_1 \le C\, \|\widehat{\theta_i u}\|_{p(\cdot)} = C\, \|\theta_i u\|_{\mathscr{B}_{p(\cdot)}}$$

holds for all i. We also know that there is a positive integer m such that $\theta_i u = 0$ for all $i > m$ (C and m only depend on j). Then we have

$$\begin{aligned}|\langle u, L\rangle| \le (2\pi)^{-n} \sum_{i=1}^{m} \int_{\mathbb{R}^n} \big|\widehat{\theta_i u}(x)\big|\, |\hat{\nu}_i(x)|\, dx \le C \sum_{i=1}^{m} \|\widehat{\theta_i u}\|_1 \|\hat{\nu}_i\|_\infty \\ \le C \sum_{i=1}^{m} \|\theta_i u\|_{\mathscr{B}_{p(\cdot)}} \|\nu_i\|_{\mathscr{B}_\infty}\,.\end{aligned}$$

Reasoning now as in Proposition 2.3/2 we can find a positive integer k and a constant C such that $\|\theta_i u\|_{\mathscr{B}_{p(\cdot)}} \le C\, |\theta_i|_k \|u\|_{\mathscr{B}_{p(\cdot)}}$ for $1 \le i \le m$ and so we obtain

$$|\langle u, L\rangle| \le C \left(\sum_{i=1}^{m} |\theta_i|_k \|\nu_i\|_{\mathscr{B}_\infty} \right) \|u\|_{\mathscr{B}_{p(\cdot)}}\,. \tag{2.2}$$

Thus L is continuous on $\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_j)$ (actually for all j) and we conclude that $L \in \big(\mathscr{B}^c_{p(\cdot)}(\Omega)[\mathscr{T}]\big)'$. We shall show that $\Phi(L) = \nu$. By Proposition 2.3/1, the former dual coincides with $\big(\mathscr{B}^c_{p(\cdot)}(\Omega)[\mathscr{T}_c]\big)'$. Then, by Proposition 2.3/2, there exists $(C_i) \in (\mathbb{R}_+)^{\mathbb{N}}$ such that

$$|\langle u, L\rangle| \le \sum_{i=1}^{\infty} C_i\, \|\theta_i u\|_i$$

holds for all $u \in \mathscr{B}^c_{p(\cdot)}(\Omega)$. Let $(\xi_i) \in l_\infty\Big(\frac{1}{C_i}, \big(\mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(K_i), \|\cdot\|_i\big)'\Big)$ such that $\langle u, L\rangle = \sum_{i=1}^{\infty} \langle \theta_i u, \xi_i\rangle$ for all $u \in \mathscr{B}^c_{p(\cdot)}(\Omega)$. Then $\Phi(L) = \sum_{i=1}^{\infty} [\theta_i \xi_i]$ and, for any $\varphi \in C^\infty_0(\Omega)$, we have

$$\begin{aligned}\langle \varphi, \Phi(L)\rangle = \langle \varphi, L\rangle = (2\pi)^{-n} \sum_{i=1}^{\infty} \int_{\mathbb{R}^n} \widehat{\theta_i \varphi}\, \hat{\nu}_i\, dx = (2\pi)^{-n} \sum_{i=1}^{\infty} \langle \widehat{\widehat{\theta_i \varphi}}, \hat{\nu}_i\rangle \\ = \sum_{i=1}^{\infty} \langle \theta_i \varphi, \nu_i\rangle = \sum_{i=1}^{\infty} \langle \varphi, \theta_i \nu_i\rangle = \langle \varphi, \sum_{i=1}^{\infty} \theta_i \nu_i\rangle = \langle \varphi, \nu\rangle\,,\end{aligned}$$

and so $\Phi(L) = \nu$ and Φ is surjective. Summarizing, Φ is an algebraic isomorphism.

Finally, we prove that Φ is a (topological) isomorphism. We first show the continuity of Φ^{-1}: Let A a bounded subset of $\mathscr{B}^c_{p(\cdot)}(\Omega)[\mathscr{T}]$. By [17, Theorem 3.5/3], there is a j such that A is a bounded subset of $\left(\mathscr{B}_{p(\cdot)}\cap\mathscr{E}'(K_j),\|\cdot\|_{\mathscr{B}_{p(\cdot)}}\right)$. Then, taking into account the decomposition (2.1), the estimate (2.2) and the inequalities $\|v\|_{\infty,\chi_i}\le(2\pi)^{-n}\|\hat{\chi}_i\|_1\|v\|_{\infty,\chi_{i+1}}$, we get

$$\begin{aligned}p_A(\Phi^{-1}(v)) &= \sup\left\{|\langle u,\Phi^{-1}(v)\rangle| : u\in A\right\}\\ &= \sup\left\{|\langle u,L\rangle| : u\in A\right\}\le\sup\left\{C\left(\sum_{i=1}^{m}|\theta_i|_k\|v_i\|_{\mathscr{B}_\infty}\right)\|u\|_{\mathscr{B}_{p(\cdot)}} : u\in A\right\}\\ &\le C\left(\sum_{i=1}^{m}|\theta_i|_k\|v_i\|_{\mathscr{B}_\infty}\right)=C\left(\sum_{i=1}^{m}|\theta_i|_k\|v\|_{\infty,\chi_i}\right)\le C\|v\|_{\infty,\chi_m}\end{aligned}$$

for all $v\in\mathscr{B}_\infty^{\mathrm{loc}}(\Omega)$ and thus Φ^{-1} is continuous. Then Φ becomes a (topological) isomorphism by the open mapping theorem (by Proposition 2.4 $\left(\mathscr{B}^c_{p(\cdot)}[\mathscr{T}]\right)'$ is also a Fréchet space).

Lastly, if $p(\cdot)\equiv p$ and $0<p\le 1$ then the Hardy-Littlewood maximal operator M is bounded on L_{p/p_0} for each $p_0\in]0,p[$ and so we also have the isomorphism $\left(\mathscr{B}^c_p(\Omega)[\mathscr{T}]\right)'\simeq\mathscr{B}_\infty^{\mathrm{loc}}(\Omega)$. □

Remark If $p(\cdot)$ is a variable exponent such that $1<p^-\le p^+<\infty$, it is possible to prove the isomorphism $\left(\mathscr{B}^c_{p(\cdot)}(\Omega)\right)'\simeq\mathscr{B}^{\mathrm{loc}}_{\widetilde{p'(\cdot)}}(\Omega)$ (obtained in [17, Theorem 4.3]) following step by step the proof of the preceding theorem and using Remark 3.6/2 of [17] instead of the Proposition 2.3 (the topologies $\mathscr{T}$ and $\mathscr{T}_c$ coincide in this case): In fact, using the notations of Theorem 2.1 and sustituing in the proof $\mathscr{B}_\infty^{\mathrm{loc}}(\Omega)$ by $\mathscr{B}^{\mathrm{loc}}_{\widetilde{p'(\cdot)}}(\Omega)$, it suffices to notice that $\varphi[\theta_i\xi_i]\in\mathscr{B}_{\widetilde{p'(\cdot)}}$, i.e. $\left(\varphi[\theta_i\xi_i]\right)^\wedge\in L_{\widetilde{p'(\cdot)}}$, for each $\varphi\in C_0^\infty(\Omega)$ (use Lemma 4.1 of [17]), and that in the proof of the surjectivity of Φ, when one needs to show that the functional

$$\langle u,L\rangle=(2\pi)^{-n}\sum_{i=1}^{\infty}\int_{\mathbb{R}^n}\widehat{\theta_i u}(x)\,\widehat{\tilde{v}_i}(x)\,dx,\quad u\in\mathscr{B}^c_{p(\cdot)}(\Omega),$$

is $\mathscr{T}$ continuous, one must use the generalized inequality of Hölder.

In [17, Remark 4.4] it is shown that if Ω is an open interval of $\mathbb{R}$ and $0<p<1$ then $(\mathscr{B}^c_p(\Omega)[\mathscr{T}])'\simeq\mathrm{proj}_j E_j$ where the Banach spaces E_j are isomorphic to l_∞. The next corollary is a sequence space representation of the dual $(\mathscr{B}^c_{p(\cdot)}(\Omega)[\mathscr{T}])'$ which improves that result.

Corollary 2.1 *$\left(\mathscr{B}^c_{p(\cdot)}(\Omega)[\mathscr{T}]\right)'$ is isomorphic to $(l_\infty)^{\mathbb{N}}$ if $0<p^+\le 1$.*

Proof By a result of Vogt [20] we know that $\mathscr{B}^c_1(\Omega)[\mathscr{T}]\simeq(l_1)^{(\mathbb{N})}$. By using this isomorphism and Theorem 2.1, we have

$$\left(\mathscr{B}^c_{p(\cdot)}(\Omega)[\mathscr{T}]\right)'\simeq\mathscr{B}_\infty^{\mathrm{loc}}(\Omega)\simeq(\mathscr{B}^c_1(\Omega)[\mathscr{T}])'\simeq\left((l_1)^{(\mathbb{N})}\right)'\simeq(l_\infty)^{\mathbb{N}}$$

(for the last isomorphism see, e.g. [15, p. 287]). □

We finish with a result which extends Proposition 2.2.

Theorem 2.2 *Let Ω be a cube of $\mathbb{R}^n$ with side length* 1.

1. *If $0 < p < 1$, then $\mathscr{B}^c_p(\Omega)[\mathscr{T}]$ does not contain any infinite-dimensional q-Banach subspace with $p < q \le 1$.*
2. *If $0 < p_1, p_2 \le 1$, then $\mathscr{B}^c_{p_1}(\Omega)[\mathscr{T}] \simeq \mathscr{B}^c_{p_2}(\Omega)[\mathscr{T}]$ if and only if $p_1 = p_2$.*

Proof 1. Without loss of generality we can suppose $\Omega = \left]-\frac{1}{2}, \frac{1}{2}\right[^n$. Then we have $\mathscr{B}^c_p(\Omega)[\mathscr{T}]$ $= \mathrm{ind}_i \left[\mathscr{B}_p \cap \mathscr{E}'(Q_i)\right]$ where $Q_i = [-R_i, R_i]^n$ and $R_i \nearrow 1/2$. Assume that $\mathscr{B}^c_p(\Omega)[\mathscr{T}]$ contains an infinite-dimensional q-Banach subspace X. By [17, Theorem 3.5/3], X becomes a subspace of a step $\mathscr{B}_p \cap \mathscr{E}'(Q_j)$. Then we have the following diagram

$$X \xrightarrow{\ j\ } \mathscr{B}_p \cap \mathscr{E}'(Q_j) \xrightarrow{\ \mathscr{F}\ } L_p^{Q_j} \xrightarrow{\ D\ } l_p(\mathbb{Z}^n)$$

where j is the canonical injection, $\mathscr{F}$ is the Fourier transform operator and D is the sampling operator (see the proof of Proposition 2.2). Since $p < q$, a result of Stiles ([18, p. 118], [11, p. 25]) proves that the bounded operator $A = D \circ \mathscr{F} \circ j$ is compact. But $\mathscr{F}$ is a topological isomorphism and D is an isomorphic embedding, thus ImA and, consequently, X are finite-dimensional. This contradiction finishes the proof of 1.

2. Since the steps $\mathscr{B}_{p_1} \cap \mathscr{E}'(Q_i)$ (resp. $\mathscr{B}_{p_2} \cap \mathscr{E}'(Q_i)$) are infinite-dimensional p_1-Banach (resp. p_2-Banach) subspaces of $\mathscr{B}^c_{p_1}(\Omega)[\mathscr{T}]$ (resp. $\mathscr{B}^c_{p_2}(\Omega)[\mathscr{T}]$) the result is a consequence of 1. □

Remark Observe that, applying a result of Bastero [3, Corollary 5], it is easily seen that each step $\mathscr{B}_p \cap \mathscr{E}'(Q_i)$ contains a subspace isomorphic to l_p. In fact, since $L_p^{Q_i}$ ($\simeq \mathscr{B}_p \cap \mathscr{E}'(Q_i)$) is a closed subspace of L_p, $L_p^{Q_i}$ contains a subspace isomorphic to l_r for some $p \le r \le 2$ (use [3, Corollary 5]). Then, applying Theorem 2.2/1, we conclude that $r = p$.

Questions

1. In [17] we have posed a question on complex interpolation between the Banach spaces $\mathscr{B}_{p_i(\cdot)} \cap \mathscr{E}'(Q)$ when $1 \le p_i^- \le p_i^+ < \infty$, $i = 0, 1$. In [13, Section 3] Kalton elaborated a method of complex interpolation for compatible pairs (X_0, X_1) of quasi-Banach spaces such that $X_0 \cap X_1$ is dense in X_i, $i = 0, 1$, and the quasi-Banach space $X_0 + X_1$ is analytically convex (i.e. there is a constant C such that for every polynomial $P : \mathbb{C} \to X_0 + X_1$ we have $\|P(0)\|_{X_0+X_1} \le C \max_{|z|=1} \|P(z)\|_{X_0+X_1}$). In that context we pose the following related questions:
   (a) If $0 < p_i^- \le p_i^+ \le 1$, $i = 0, 1$, and $Q = [-R, R]^n$, is the quasi-Banach space
   $$\mathscr{B}_{p_0(\cdot)} \cap \mathscr{E}'(Q) + \mathscr{B}_{p_1(\cdot)} \cap \mathscr{E}'(Q)$$
   (equivalently, the quasi-Banach space $L^Q_{p_0(\cdot)} + L^Q_{p_1(\cdot)}$) analytically convex?
   (b) If the answer to 1. is affirmative, is the complex interpolation formula
   $$\left[\mathscr{B}_{p_0(\cdot)} \cap \mathscr{E}'(Q), \mathscr{B}_{p_1(\cdot)} \cap \mathscr{E}'(Q)\right]_\theta = \mathscr{B}_{p(\cdot)} \cap \mathscr{E}'(Q)$$
   (equivalently, $\left[L^Q_{p_0(\cdot)}, L^Q_{p_1(\cdot)}\right]_\theta = L^Q_{p(\cdot)}$) valid?. The former formula is understood in the sense of equivalence of quasi-norms and $0 < \theta < 1$, $\frac{1}{p(x)} = \frac{1-\theta}{p_0(x)} + \frac{\theta}{p_1(x)}$ and $[\cdot,\cdot]_\theta$ is the interpolation functor in the sense of Kalton [13, Section 3].
2. Calculate the dual of the space $\mathscr{B}^c_{p(\cdot)}(\Omega)$ when the variable exponent $p(\cdot) \in \mathscr{P}^0$, $p^- \le 1 < p^+$, and the Hardy-Littlewood maximal operator M is bounded in $L_{p(\cdot)/p_0}$ for some $0 < p_0 < p^-$.

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