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On cardinalities and compact closures

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Abstract

We show that there exists a Hausdorff topology on the set $\mathbb{R}$ of real numbers such that a subset A of $\mathbb{R}$ has compact closure if and only if A is countable. More generally, given any set X and any infinite set S, we prove that there exists a Hausdorff topology on X such that a subset A of X has compact closure if and only if the cardinality of A is less than or equal to that of S. When we attempt to replace "less than or equal to" in the preceding statement with "strictly less than," the situation is more delicate; we show that the theorem extends to this case when S has regular cardinality but can fail when it does not. This counterexample shows that not every bornology is a bornology of compact closure. These results lie in the intersection of analysis, general topology, and set theory.

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1. Introduction

A bornology on a set X is a covering $\mathcal{B}$ of X such that (i) if $A, B \in \mathcal{B}$, then $A \cup B \in \mathcal{B}$, and (ii) if $B \in \mathcal{B}$ and $A \subset B$, then $B \in \mathcal{B}$. Bornologies are objects of much study in analysis—see, for example, [2]. The prototypical example of a bornology is the collection of all bounded sets in a metric space. Another standard example, for a Hausdorff space X, is the collection of all subsets of X with compact closure. The latter construction is rather general, and one may well wonder: Given a bornology on X, does there necessarily exist a topology

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on X with respect to which $\mathcal{B}$ is precisely the bornology of sets with compact closure? In this paper, we answer that question in the negative by constructing a set Y for which the bornology of subsets of Y with cardinality strictly less than that of Y cannot be a bornology of compact closure—see Example 3.10.

This example leads to the following general question. Given two sets X and S, take the bornology $\mathcal{B}$ of subsets of X with cardinality less than or equal to that of S. Is $\mathcal{B}$ a bornology of compact closure? One can also ask this question for the bornology of subsets of X with cardinality strictly less than that of S. For the first question, we show that the answer is always yes. For the second question, our counterexample Y mentioned above shows that the answer is not always yes; however, we prove that it is whenever S has regular cardinality.

Considerations in general topology may lead one to ask the same questions without reference to bornologies. We now re-introduce this topic from this new point of view. When a set X is endowed with the discrete topology, a subset A of X is compact if and only if A is finite. One may wonder next, does there necessarily exist a topology on X such that a subset A of X is compact if and only if A is countable? One quickly realizes that unless X is finite, no such topology can be Hausdorff. For if so, then let $A = \{a_n \mid n \in \mathbb{N}\}$ be a countably infinite subset of X with $a_n \neq a_m$ whenever $n \neq m$. (Here $\mathbb{N}$ denotes the set of natural numbers.) Note that each set $A_k := \{a_n \mid n \geq k\}$ is countable, hence compact, hence closed because the topology is Hausdorff. But then $\{A_k\}$ is a nested collection of nonempty closed subsets of the compact set A, yet it has empty intersection, which is a contradiction.

Hausdorff being a typical property to impose a topological space, we therefore modify the question slightly: Does there exist a Hausdorff topology on X in which a set has compact closure if and only if it is countable? In particular, what about the case $X = \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers?

If we assume both the continuum hypothesis (CH) and the Axiom of Choice (AC), then the answer to this last question is an immediate yes, for the following reason. Recall that CH states that no uncountable set has cardinality strictly less than that of $\mathbb{R}$. Let Ω be the least uncountable ordinal, that is, an uncountable well-ordered set such that every subset of the form $\{y \in \Omega \mid y \leq x\}$ for $x \in \Omega$ is countable. It follows from CH that the cardinality of $\mathbb{R}$ equals that of Ω. We may then identify $\mathbb{R}$ with Ω and give it the topology induced by the order on Ω. A straightforward exercise shows that with this topology, the closure of a set A in $\mathbb{R}$ is compact if and only if A is countable.

Although this logic no longer holds when we do not assume CH, it suggests an approach. Begin by taking a well-ordering of $\mathbb{R}$. Recursively define a topology on $\mathbb{R}$ by constructing a neighborhood basis at each point, assuming one has been constructed at each previous point. We carefully select this neighborhood basis so that the sets with compact closure are precisely the countable sets. Indeed, we may generalize this reasoning considerably. The details are carried out in Section 3, where we prove our two main theorems. The first (Theorem 3.1) states that given any set X and any infinite set S, there exists a Hausdorff topology on X such that the sets with compact closure are precisely those whose cardinality is less than or equal to that of S. We obtain the first sentence in the abstract by taking $X = \mathbb{R}$ and $S = \mathbb{N}$. In Section 2, we discuss some set-theoretic preliminaries, including the definitions of "regular" and "singular" cardinals. The second main theorem (Theorem 3.9) states that when the cardinality of S is regular, the phrase "less than or equal to" in Theorem 3.1 can be replaced by the phrase "strictly less than." We conclude with an example to show that the regularity condition in Theorem 3.9 cannot be eliminated.

2. Background from set theory

Throughout this paper, we work within the Zermelo-Fraenkel axiom system (ZF).

Recall that a linear ordering on a set X is said to be a *well-ordering* if every nonempty subset of X has a smallest element.

Theorem 2.1 (Well-Ordering Principle)**.** *Every set admits a well-ordering.*

It is well-known that the well-ordering principle is equivalent to the Axiom of Choice (AC). The first step in our proofs will be to well-order the set X, so the proofs depend on the well-ordering principle, and hence AC, right from the git-go.

It is also well-known that a countable union of countable sets is countable. More generally, a union over a set no bigger than X of sets no bigger than X is no bigger than X. More precisely, we have the following theorem, where the notation $|B| \leq |C|$ means that the cardinality of B is less than or equal to that of C. (Likewise, we will later use the notation $|B| < |C|$ to indicate that the cardinality of B is strictly less than that of C.)

Theorem 2.2. *Let X be an infinite set, and let I be a set with $|I| \leq |X|$. For each $i \in I$, let A_i be a set with $|A_i| \leq |X|$. Then $|\bigcup_{i\in I} A_i| \leq |X|$.*

Theorem 2.2 is proved in [1]. The proof depends on AC.

Our third and final use of AC comes as we define the terms *regular* and *singular* for cardinalities. Roughly speaking, we say that the cardinality of a set is regular if the set cannot be written as a smaller union of smaller sets, and that it is singular otherwise. We now make this concept more precise.

Definition 2.3. Let X be a set. We say that X has *singular cardinality* if there exists a set I with $|I| < |X|$ such that for each $i \in I$ there exists a set A_i with $|A_i| < |X|$, and that $X = \bigcup_{i\in I} A_i$. We say that X has *regular cardinality* if X does not have singular cardinality.

We say that this definition relies on AC because although it is not the standard definition, it is equivalent to the standard definition under the assumption of AC. We refer to [1] for details.

There is no purpose to Definition 2.3 unless both regular and singular cardinals exist. As the name suggests, regular cardinals are not hard to find. For instance, $\aleph_0 := |\mathbb{N}|$ is regular, because $\mathbb{N}$ does not equal a finite union of finite sets. Producing a singular cardinal requires a deliberate construction, such as the following.

Example 2.4. Let $X_1 = \mathbb{N}$. For each $n \in \mathbb{N}$, define X_{n+1} to be the power set of X_n, i.e., the set of all subsets of X_n. By Cantor's theorem, $|X_n| < |X_{n+1}|$. Hence $Y := \bigcup_{n\in\mathbb{N}} X_n$ has singular cardinality.

3. Main theorems

Throughout this section, fix a nonempty set X and an infinite set S. If A is a subset of a topological space, then we denote its closure by $\overline{A}$. Our first objective in this section is to prove the following theorem.

Theorem 3.1. *There exists a Hausdorff topology on X so that if $A \subset X$, then $\overline{A}$ is compact if and only if $|A| \leq |S|$.*

Choose a well-ordering $\leq$ on X. Assume that with respect to this ordering, X has a maximal element M. (If not, then create a new ordering by reversing all inequalities involving the minimal element.) To prove Theorem 3.1, we begin by defining a topology. The definition is recursive and depends on knowing that what has been defined so far already forms a topology, a fact that in turn requires proof. So we must simultaneously make a recursive definition and an inductive proof. For $y \in X$, we define the closed ray $(-\infty, y] := \{x \in X \mid x \leq y\}$ and the open ray $(-\infty, y) := \{x \in X \mid x < y\}$. Observe that $X = (-\infty, M]$.

Lemma/Definition 3.2. *For any given $x \in X$, define $\mathcal{N}_x$, $\mathcal{B}_x$, $\mathcal{T}_x$, and $\mathcal{W}_x$ according to (1)–(6) below with $y = x$, assuming that (1)–(6) are true for all $y < x$.*

(1) *We define $\mathcal{N}_y$ to be the collection of all sets of the form $(-\infty, y] \setminus K$ such that K is $\mathcal{T}_y$-closed in $(-\infty, y)$ and such that if C is a $\mathcal{T}_y$-closed subset of K with $|C| \leq |S|$, then C is $\mathcal{T}_y$-compact. Here $\mathcal{T}_y$ is defined as in (3).*
(2) *We define $\mathcal{B}_y := \bigcup_{z<y} \mathcal{N}_z$.*
(3) *We have that $\mathcal{B}_y$ is a basis for a topology $\mathcal{T}_y$ on $(-\infty, y)$.*
(4) *We have that $\mathcal{N}_y \cup \mathcal{B}_y$ is a basis for a topology $\mathcal{W}_y$ on $(-\infty, y]$.*
(5) *For all $z \leq y$, if K is a $\mathcal{T}_y$-closed subset of $(-\infty, y)$, then $K \cap (-\infty, z)$ is a $\mathcal{T}_z$-closed subset of $(-\infty, z)$.*
(6) *For all $z \leq y$, we have that $(-\infty, z]$ is a $\mathcal{W}_y$-closed subspace of $(-\infty, y]$.*

Proof. Note that in (1), (5) and (6) above, as well as in the proof below, we always take $\mathcal{W}_p$ as the topology on any closed ray $(-\infty, p]$ and $\mathcal{T}_p$ as the topology on any open ray $(-\infty, p)$. This should be assumed when not explicitly stated.

Let $x \in X$, and assume that (1)–(6) have been established for all $y < x$. Items (1) and (2) are definitions and so do not require proof. Hence it suffices to show that (3), (4), (5),and (6) hold when $y = x$.

Proof of (3): We must show that $\mathcal{B}_x$ is a basis for a topology on $(-\infty, x)$. From (1), we have that $(-\infty, y] \in \mathcal{N}_y$ for all $y < x$ (take $K = \varnothing$), and so $\mathcal{B}_x$ covers $(-\infty, x)$. Next let $U, V \in \mathcal{B}_x$, and let $p \in U \cap V$. We will show that $p \in E \subset U \cap V$ for some $E \in \mathcal{B}_x$. By definition of $\mathcal{B}_x$, we have that $U \in \mathcal{N}_q$ and $V \in \mathcal{N}_z$ for some $q < x$ and $z < x$. So by (4), we have that $K_1 := (-\infty, q] \setminus U$ is closed in $(-\infty, q]$ and $K_2 := (-\infty, z] \setminus V$ is closed in $(-\infty, z]$. By (6), we have that $L_1 := K_1 \cap (-\infty, p]$ and $L_2 := K_2 \cap (-\infty, p]$ are closed in $(-\infty, p]$. Then $L_1 \cup L_2$ is closed in $(-\infty, p]$, so $E_1 := (-\infty, p] \setminus (L_1 \cup L_2)$ is open in $(-\infty, p]$. So by (4), E_1 is a union of members of $\mathcal{N}_p \cup \mathcal{B}_p$. Because $p \in E_1$ and members of $\mathcal{B}_p$ are subsets of $(-\infty, p)$, we must have $p \in E \in \mathcal{N}_p$ for some E. Then $p \in E \subset E_1 \subset U \cap V$, and $E \in \mathcal{B}_x$.

Proof of (4): Next, we prove that $\mathcal{N}_x \cup \mathcal{B}_x$ is a basis for a topology on $(-\infty, x]$. As in our proof of (3), we have that $\mathcal{N}_x \cup \mathcal{B}_x$ covers $(-\infty, x]$. Let $U, V \in \mathcal{N}_x \cup \mathcal{B}_x$, and let $p \in U \cap V$. We will show that $p \in M \subset U \cap V$ for some $M \in \mathcal{N}_x \cup \mathcal{B}_x$. Case 1: $U, V \in \mathcal{N}_x$. Then $U = (-\infty, x] \setminus K_1$ and $V = (-\infty, x] \setminus K_2$ for some K_1, K_2 of the form specified by (1). We will show that $K_1 \cup K_2$ also has this form. Let C be a $\mathcal{T}_x$-closed subset of $K_1 \cup K_2$ with $|C| \leq |S|$. Then $C \cap K_1$ is a $\mathcal{T}_x$-closed subset of K_1 with $|C \cap K_1| \leq |S|$, so $C \cap K_1$ is $\mathcal{T}_x$-compact. Similarly, $C \cap K_2$ is $\mathcal{T}_x$-compact. Hence we may take $M = U \cap V = (-\infty, x] \setminus (K_1 \cup K_2)$. Case 2: $U \in \mathcal{B}_x, V \in \mathcal{N}_x$. Then for some $y < x$ we have that $U = (-\infty, y] \setminus K_1$ for some K_1 closed in $(-\infty, y)$ such that every $\mathcal{T}_y$-closed subset C of K_1 with $|C| \leq |S|$ is $\mathcal{T}_y$-compact, and $V = (-\infty, x] \setminus K_2$ for some K_2 closed in $(-\infty, x)$ with the same property, appropriately modified. Let $p \in U \cap V$. We'll show that there exists $E \in \mathcal{B}_x$ such that $p \in E \in U \cap V$. Let $L_1 = K_1 \cap (-\infty, p]$ and $L_2 = K_2 \cap (-\infty, p]$. Our argument from the proof of (3) will go through provided that $(-\infty, p]$ is closed in $(-\infty, x)$ and in $(-\infty, y)$. But this follows from (6), because if $p < z < x$, then $(-\infty, z] \setminus (-\infty, p]$ is open in $(-\infty, z]$, hence a union of members of $\mathcal{N}_z \cup \mathcal{B}_z$, hence open in $(-\infty, x)$. Similarly for $(-\infty, y)$. Case 3: $U \in \mathcal{N}_x, V \in \mathcal{B}_x$. Similar to Case 2. Case 4: $U, V \in \mathcal{B}_x$. The proof is identical to that of (3).

Proof of (5): If $z = x$ then (5) is tautological, so assume that $z < x$. Let K be a $\mathcal{T}_x$-closed subset of $(-\infty, x)$. Let $L = K \cap (-\infty, z)$, and suppose that $p \in (-\infty, z) \setminus L$. We will show that there exists $E \in \mathcal{T}_z$ such that $p \in E$ and $E \cap L = \varnothing$. We know that there exists $Q \in \mathcal{B}_x$ such that $p \in Q$ and $Q \cap K = \varnothing$. Then $Q \in \mathcal{N}_y$ for some $y < x$, so Q is open in $(-\infty, y]$. So by (6), we know that $Q \cap (-\infty, z]$ is open in $(-\infty, z]$. Hence, by (4), we have that $Q \cap (-\infty, z]$ is a union of members of $\mathcal{N}_z \cup \mathcal{B}_z$. If $U \in \mathcal{N}_z$, then $U \cap (-\infty, z)$ is open in $(-\infty, z)$, by (1). If $U \in \mathcal{B}_z$, then $U \cap (-\infty, z) = U$ is open in $(-\infty, z)$, by (3). So $E := (Q \cap (-\infty, z]) \cap (-\infty, z) = Q \cap (-\infty, z)$ is a union of $\mathcal{T}_z$-open sets and is therefore open in $(-\infty, z)$. Moreover, $p \in E$ and $E \cap L = \varnothing$.

Proof of (6): If $z = x$ then (6) is tautological, so assume that $z < x$. First we show that $(-\infty, z]$ is closed in $(-\infty, x]$. The proof of (4), Case 2, shows that $(-\infty, z]$ is $\mathcal{T}_x$-closed. Let C be a $\mathcal{T}_x$-closed subset of $(-\infty, z]$ with $|C| \leq |S|$. We will show that C is $\mathcal{T}_x$-compact. From this it will follow that $(-\infty, x] \setminus (-\infty, z] \in \mathcal{N}_x$ and therefore that $(-\infty, z]$ is closed in $(-\infty, x]$. To show that C is $\mathcal{T}_x$-compact, it suffices to show that $C \cup \{z\}$ is $\mathcal{T}_x$-compact, because C is $\mathcal{T}_x$-closed. Let $\{U_\alpha\}$ be an open cover of $C \cup \{z\}$ by members of $\mathcal{B}_x$. Choose a set U_{α_0} from this cover with $z \in U_{\alpha_0}$. Then $U_{\alpha_0} \in \mathcal{N}_y$ for some y with $z \leq y < x$. So $U_{\alpha_0} = (-\infty, y] \setminus K$ for some K closed in $(-\infty, y)$ such that if D is a $\mathcal{T}_y$-closed subset of K with $|D| \leq |S|$, then D is $\mathcal{T}_y$-compact. By (5), we know that $C \cap K$ is $\mathcal{T}_y$-closed. Also, $|C \cap K| \leq |C| \leq |S|$. Hence $C \cap K$ is $\mathcal{T}_y$-compact. It follows from (5) that $U_\alpha \cap (-\infty, y)$ is $\mathcal{T}_y$-open for all α. Hence finitely many sets $U_{\alpha_1} \cap (-\infty, y), \ldots, U_{\alpha_n} \cap (-\infty, y)$ cover $C \cap K$. Therefore $U_{\alpha_0}, U_{\alpha_1}, \ldots, U_{\alpha_n}$ is our desired finite subcover of $C \cup \{z\}$.

Next, we show that if K is a $\mathcal{W}_z$-closed subset of $(-\infty, z]$, then K is $\mathcal{W}_x$-closed. By (4), we have that $(-\infty, z] \setminus K$ is a union of members of $\mathcal{N}_z \cup \mathcal{B}_z \subset \mathcal{B}_x$, so $(-\infty, z] \setminus K$ is open in $(-\infty, x]$. Therefore $(-\infty, x] \setminus K = ((-\infty, z] \setminus K) \cup ((-\infty, x] \setminus (-\infty, z])$ is $\mathcal{W}_x$-open.

Finally, we show that if K is a $\mathcal{W}_x$-closed subset of $(-\infty, x]$, then $K \cap (-\infty, z]$ is $\mathcal{W}_z$-closed. Let $p \in (-\infty, z] \setminus K$. Then $p \in U \subset (-\infty, x] \setminus K$ for some $U \in \mathcal{N}_x \cup \mathcal{B}_x$. It suffices to show that $U \cap (-\infty, z] \in \mathcal{W}_z$. Case 1: $U \in \mathcal{B}_x$. Then $U \in \mathcal{N}_y$ for some $y < x$. So U is open in $(-\infty, y]$. Therefore $U \cap (-\infty, z] \in \mathcal{W}_z$ by (6), inductively. Case 2: $U \in \mathcal{N}_x$. Then $U = (-\infty, x] \setminus K_1$ for some K_1 closed in $(-\infty, x)$, by (1). So U equals a union of sets $W_\beta \in \mathcal{B}_x$. Arguing as in Case 1, we see that for all β, we have that $W_\beta \cap (-\infty, z] \in \mathcal{W}_z$. Therefore $U \cap (-\infty, z] = \bigcup (W_\beta \cap (-\infty, z]) \in \mathcal{W}_z$. □

Regrettably, the statement and proof of Lemma/Definition 3.2 contain many unavoidable technicalities. The idea, though, is fairly straightforward. Recall that if A is a Hausdorff topological space and p is a point not in A, then we define the *co-compact topology* on $A \cup \{p\}$ by choosing the neighborhood basis at p to consist of all complements of compact subsets of A. The definition of $\mathcal{N}_y$ is somewhat similar to this; we take the neighborhood basis at y to be the collection of all complements of closed sets, every "small" closed subset of which is compact. By "small" here we mean, "of cardinality no greater than that of S."

Example 3.3. Let $X = \mathbb{N} \cup \{M\}$, where $M \notin \mathbb{N}$. We order X by taking the usual order on $\mathbb{N}$ and declaring that $n < M$ for all $n \in \mathbb{N}$. Let $S = \mathbb{N}$. For any $n \in \mathbb{N}$, the topology $\mathcal{W}_n$ on $(-\infty, n]$ defined by Lemma/Definition 3.2 is the discrete topology. Then the topology $\mathcal{T}_M$ on $\mathbb{N} = (-\infty, M)$ is also discrete. The compact subsets of $(-\infty, M)$ are precisely the finite subsets. It follows that the topology $\mathcal{W}_M$ on $X = (-\infty, M]$ coincides with the order topology.

Example 3.4. Let Ω be the least uncountable ordinal, and let $X = \Omega \cup \{M\}$, where $M \notin \Omega$. We order X by taking the usual order on Ω and declaring that $x < M$ for all $x \in \Omega$. Let $S = \mathbb{N}$. We claim that the topology $\mathcal{W}_M$ on $X = (-\infty, M]$ cannot coincide with the order topology on X in this case. For suppose otherwise. Then $K := (-\infty, M)$ is a closed subspace of $(-\infty, M)$, and moreover every closed subset C of K with $|C| \leq |S|$ is compact. So $\{M\} = (-\infty, M] \setminus K$ is open in X. But in the order topology on X, the set $\{M\}$ is not open.

From this point on, we endow $X = (-\infty, M]$ with the topology $\mathcal{W}_M$ as in Lemma/Definition 3.2. The following four lemmas establish that $\mathcal{W}_M$ possesses the property required by Theorem 3.1.

Lemma 3.5. *X is Hausdorff.*

Proof. Let a, b be distinct points in X. Without loss of generality, assume that $a < b$. We have that $(-\infty, a] \in \mathcal{N}_a$ and $(-\infty, b] \in \mathcal{N}_b$, so $(-\infty, a]$ and $(-\infty, b]$ are open in X. By (6) in Lemma/Definition 3.2, we have that $(-\infty, a]$ is closed in X. So $(-\infty, a]$ and $(-\infty, b] \setminus (-\infty, a]$ are disjoint open neighborhoods of a, b, respectively. □

Lemma 3.6. *If A is a closed subset of X with $|A| \leq |S|$, then A is compact.*

Proof. It suffices to show that $A \cup \{M\}$ is compact. Let $\{U_\alpha\}$ be a cover of $A \cup \{M\}$ by members of $\mathcal{N}_M \cup \mathcal{B}_M$. Then $M \in U_{\alpha_0}$ for some member U_{α_0} of the cover with $U_{\alpha_0} \in \mathcal{N}_M$. Then $L = X \setminus U_{\alpha_0}$ is closed in $(-\infty, M)$ and every $\mathcal{T}_M$-closed subset C of L with $|C| \leq |S|$ is $\mathcal{T}_M$-compact. From Lemma/Definition 3.2, we see that $A \cap (-\infty, M)$ is $\mathcal{T}_M$-closed. Hence $L \cap A \cap (-\infty, M)$ is $\mathcal{T}_M$-closed as well. Also, $|L \cap A \cap (-\infty, M)| \leq |S|$. Therefore $L \cap A \cap (-\infty, M)$ is $\mathcal{T}_M$-compact. For all α, we have that $U_\alpha \cap (-\infty, M) \in \mathcal{T}_M$. So finitely many sets $U_{\alpha_1}, \ldots, U_{\alpha_n}$ cover $L \cap A \cap (-\infty, M)$. Then $U_{\alpha_0}, U_{\alpha_1} \ldots, U_{\alpha_n}$ give us the desired subcover. □

Lemma 3.7. *If $A \subset X$ and $|A| \leq |S|$, then $|\overline{A}| \leq |S|$.*

Proof. Suppose that $|\overline{A}| > |S|$. Let m be the smallest element of X such that $|(-\infty, m] \cap \overline{A}| > |S|$. Let ℓ be the least upper bound of $A \cap (-\infty, m)$. (The fact that X is well-ordered guarantees that m and ℓ exist.) Let $Q = \{a \in A \mid a < m\}$. Let $C = \bigcup_{a \in Q}((-\infty, a] \cap \overline{A})$. By Theorem 2.2, $|C| \leq |S|$. Observe that $(-\infty, m] \setminus (-\infty, \ell]$ is an open set disjoint from A; therefore it is disjoint from $\overline{A}$. So $(-\infty, m] \cap \overline{A}$ contains C and at most two other points, namely ℓ and m. Becasue S is infinite, therefore $|(-\infty, m] \cap \overline{A}| \leq |S|$, a contradiction. □

Lemma 3.8. *If $A \subset X$ and $|A| > |S|$, then $\overline{A}$ is not compact.*

Proof. Temporarily assume that $\overline{A}$ is compact. Observe that $|\overline{A}| > |S|$. Let m be the smallest element of X such that $|(-\infty, m] \cap \overline{A}| > |S|$. By Lemma/Definition 3.2, we have that $(-\infty, m]$ is a closed subspace of $X = (-\infty, M]$, so $(-\infty, m] \cap \overline{A}$ is $\mathcal{W}_m$-compact. Let $K = (-\infty, m) \cap \overline{A}$. Note that $|K| > |S|$, by definition of m. Let C be any $\mathcal{T}_m$-closed subset of K such that $|C| \leq |S|$. We will show that C is $\mathcal{T}_m$-compact. By definition of $\mathcal{W}_m$, this will show that K is a $\mathcal{W}_m$-closed subset of the $\mathcal{W}_m$-compact set $(-\infty, m] \cap \overline{A}$ and therefore that K is $\mathcal{W}_m$-compact.

Let ℓ be the least upper bound of C in X. We must have that $\ell < m$, for otherwise, by Theorem 2.2, we would have that

$$|K| = \left| \bigcup_{c \in C} ((-\infty, c] \cap \overline{A}) \right| \leq |S|.$$

So $C \subset (-\infty, \ell]$. It follows then from Lemma/Definition 3.2 that because C is $\mathcal{T}_m$-closed, therefore C is $\mathcal{W}_\ell$-closed, and therefore C is $\mathcal{W}_m$-closed. But C is a subset of the $\mathcal{W}_m$-compact set $\overline{A}$, so C is $\mathcal{W}_m$-compact, therefore $\mathcal{T}_m$-compact.

Note that $\{(-\infty, k] : k \in K\}$ is a $\mathcal{W}_m$-open cover of K. Hence

$$K \subset (-\infty, k_1] \cup \cdots \cup (-\infty, k_n]$$

for some $k_1, \ldots, k_n$. So

$$K \subset \bigcup_{j=1}^{n} ((-\infty, k_j] \cap \overline{A}).$$

But each $k_j < m$, so $|(-\infty, k_j] \cap \overline{A}| \leq |S|$, by definition of m. But then $|K| \leq |S|$, because S is infinite. This contradicts the fact that $|K| > |S|$. ☐

Theorem 3.1 follows at once from Lemmas 3.5, 3.6, 3.7, and 3.8.

Theorem 3.9. *Let X be a set, and let S be an infinite set with regular cardinality. Then there exists a Hausdorff topology on X so that if $A \subset X$, then $\overline{A}$ is compact if and only if $|A| < |S|$.*

Proof. The proof is identical to that of Theorem 3.1 with two small modifications. One must replace every instance of "$\leq |S|$" with "$< |S|$." Also, one must use Definition 2.3 in place of Theorem 2.2 whenever the latter is invoked. ☐

The following example illustrates how Theorem 3.9 can fail when S has singular cardinality.

Example 3.10. Consider the sets defined in Example 2.4. We will show that there does not exist a topology on Y such that $A \subset Y$ has compact closure if and only if $|A| < |Y|$.

Suppose otherwise. The fact that Y does not have strictly smaller cardinality than itself implies that $\overline{Y} = Y$ is not compact. Let $\{U_\alpha\}$ be an open cover of Y with no finite subcover. We know that $|X_n| < |Y|$ for all n, so $\overline{X_n}$ is compact. Cover $\overline{X_n}$ with finitely many sets $U_{\alpha_{n,1}}, \ldots, U_{\alpha_{n,j_n}}$ from the collection $\{U_\alpha\}$, and let $V_n = \bigcup_{\ell=1}^{j_n} U_{\alpha_{n,\ell}}$. By our assumption on $\{U_\alpha\}$, there exists a point $b_n \in Y$ such that $b_n \notin W_n$, where $W_n = \bigcup_{k=1}^{n} V_k$. Note that the sets W_n form an increasing chain of open sets. Also note that $X_n \subset \overline{X_n} \subset V_n \subset W_n$.

Let $S = \{b_n \mid n \in \mathbb{N}\}$. Then $|S| < |Y|$. We will show that $\overline{S}$ is not compact, thereby producing the desired contradiction. For each $\ell \in \mathbb{N}$, let $S_\ell = \{b_n \mid n \geq \ell\}$. Then $\{\overline{S_\ell}\}$ is a collection of closed subsets of $\overline{S}$ with the finite intersection property. It suffices to show that $\bigcap_{\ell=1}^{\infty} \overline{S_\ell} = \varnothing$. Let $y \in Y$. Then $y \in X_m$ for some m. Hence y is in the open set W_m. Observe that $W_m \cap S_m = \varnothing$, because of how we chose the points b_n. Therefore $y \notin \overline{S_m}$. Therefore $\{\overline{S_\ell}\}$ has empty intersection, and so $\overline{S}$ cannot be compact.

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