$(n+1)$-tensor norms of Lapresté's type

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Abstract

We study an $(n+1)$-tensor norm $\alpha_{\mathbf{r}}$ extending to $(n+1)$-fold tensor products the classical one of Lapresté in the case $n = 1$. We characterize the maps of the minimal and the maximal multilinear operator ideals related to $\alpha_{\mathbf{r}}$ in the sense of Defant and Floret. As an application we give a complete description of the reflexivity of the $\alpha_{\mathbf{r}}$-tensor product $(\otimes_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}})$.

1 Introduction

In [14] Pietsch proposed building a systematic theory of ideals of multilinear mappings between Banach spaces, similar to the already well-developed one regarding linear maps, as a first step to study ideals of more general non linear operators. Since then several classes of multilinear operators more or less related to classical absolutely p-summing operators has been studied although without to deal with aspects derived from a general organized theory.

Having in mind the close connection existing in linear case between problems of this kind and tensor products (see [2] for a systematic survey of the actual state of the art), in the present setting it is expected an analogous connection with multiple tensor products. However a systematic study of this approach has not been initiated until the works [4] and [5] of Floret, mainly motivated by the potential applications of the new theory to infinite holomorphy. In this way, classical notions of maximal operator ideals and its associated α-tensor norm, dual tensor norm α' and the related α-nuclear and α-integral operators can be extended to the framework of multilinear operator ideals and multiple tensor products.

However, there are few concrete examples of multi-tensor norms to whose the general concepts of the theory have been applied and checked. The purpose of this paper is to study an $(n+1)$-tensor norm $\alpha_{\mathbf{r}}$ on tensor products $\bigotimes_{j=1}^{n+1} E_j,\ 1 \leq n,$

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of $n+1$ Banach spaces E_j, extending the classical one of Lapresté for $n=1$, as well its associated $\alpha_{\mathbf{r}}$-nuclear and $\alpha_{\mathbf{r}}$-integral multilinear operators. Knowledge of such operators allows us to characterize the reflexivity of the corresponding tensor product $\big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\big)$ of spaces ℓ^{u_j}.

The paper is organized as follows. First we introduce the notation and some general facts to be used. In section 2 we define the $(n+1)$-fold tensor product $\bigotimes_{\alpha_{\mathbf{r}}}\big(E_1, E_2, ..., E_n, F\big)$, $n \in \mathbb{N}$ of type $\alpha_{\mathbf{r}}$ of Banach spaces $E_j, 1 \leq j \leq n$ and F. We find its topological dual introducing the so called $\mathbf{r}$-dominated maps and we obtain multilinear extensions of the classical theorems of Grothendieck-Pietsch and Kwapien (theorem 3). The latter one is the key to approximate $\mathbf{r}$-dominated maps by multilinear maps of finite rank in many usual cases (theorem 7) and to compare different tensor norms $\alpha_{\mathbf{r}}$, a tool which will be very useful in our applications in the final section of the paper.

The elements of a completed $\alpha_{\mathbf{r}}$-tensor product canonically lead to multilinear $\mathbf{r}$-nuclear operators from $\prod_{j=1}^{n} E_j$ into F, which are considered in section 3 and characterized by means of suitable factorizations in theorem 9. According the pattern of the general theory of multi-tensor norms, the next step must be the study of the so called $\mathbf{r}$-integral multilinear maps, i. e. the maps in the ideal associated to the $\alpha_{\mathbf{r}}$-tensor norm in the sense of Defant-Floret [2]. To do this we need a technical result about the structure of some ultraproducts which follows easily from the work of Raynaud [15]. It will be presented in section 4 just before its use.

In section 4 we characterize the $\mathbf{r}$-integral operators, obtaining as main result the "continuous" version of the previous factorizations of $\mathbf{r}$-nuclear operators. Finally in section 5 we apply the characterizations of sections 3 and 4 to study the reflexivity of $\alpha_{\mathbf{r}}$-tensor products and, more particulary, to characterize the reflexivity of $\alpha_{\mathbf{r}}$-tensor products of ℓ^u spaces, a result that, as far as we know, is new indeed for classical Lapresté's tensor norms.

We shall deal always with vector spaces defined over the field $\mathbb{R}$ of real numbers. Notation of the paper is standard in general. Some not so usual notations are settled now.

Given a normed space E, we shall denote by B_E its closed unit ball and $J_E : E \longrightarrow E''$ will be the canonical isometric inclusion of E into the bidual space E''. $B_{E'}$ will be considered as a compact topological space $(B_{E'}, \sigma(E', E))$ when provided with the topology induced by the weak*-topology $\sigma(E', E)$. For every $x \in E$, we shall denote by f_x the continuous function defined on $(B_{E'}, \sigma(E', E))$ as $f_x(x') = \langle x, x' \rangle$ for every $x' \in B_{E'}$. The symbol $E \approx F$ will mean that E and F are isomorphic normed spaces. The closed linear span in a Banach space E of a sequence $\{x_m\}_{m=1}^{\infty} \subset E$ (respectively of a single vector x) will be represented by $\big[x_n\big]_{m=1}^{\infty}$ (resp. $\big[x\big]$).

As usual, $\mathbf{e}_k$ denotes the k-th standard unit vector in every ℓ^p, $1 \leq p \leq \infty$.

$\ell_h^p,\ h \in \mathbb{N}$ will be the ℓ^p-space defined over the set $\{1,2,..,h\}$ with the standard measure.

Given a normed space E, a sequence $\{x_m\}_{m=1}^k \subset E, k \in \mathbb{N}\cup\{\infty\}$, and $1 \le p \le \infty$, we define in the case $p < \infty$

$$\pi_p\big((x_m)_{j=1}^k\big) := \left(\sum_{m=1}^k \|x_m\|^p\right)^{\frac{1}{p}}, \quad \varepsilon_p\big((x_m)_{m=1}^k\big) := \sup_{x'\in B_{E'}} \left(\sum_{m=1}^k \Big|\Big\langle x_m, x'\Big\rangle\Big|^p\right)^{\frac{1}{p}}$$

and when $p = \infty$

$$\pi_\infty\big((x_m)_{m=1}^k\big) := \varepsilon_\infty\big((x_m)_{m=1}^k\big) = \sup_{1\le m\le k} \|x_m\|.$$

A sequence $\{x_m\}_{m=1}^\infty \subset E$ is called weakly p-absolutely summable, notation $(x_m)_{m=1}^\infty \in \ell^p(E)$, (resp. p-absolutely summable), if $\varepsilon_p\big((x_m)_{m=1}^\infty\big) < \infty$ (resp. $\pi_p\big((x_m)_{m=1}^\infty\big) < \infty$). Given Banach spaces E and F, an operator or linear map $T \in \mathcal{L}(E,F)$ is said to be p-absolutely summing if there exists $C \ge 0$ such that

$$(x_m)_{m=1}^\infty \in \ell^p(E) \implies \pi_p\Big(\big(T(x_m)\big)_{m=1}^\infty\Big) \le C\ \varepsilon_p\big((x_m)_{m=1}^\infty\big). \tag{1}$$

The linear space $\mathfrak{P}_p(E,F)$ of all p-absolutely summing operators from E into F becomes a Banach space under the norm $\mathbf{P}_p(T) := \inf\big\{C \ge 0 \ \big|\ (1) \text{ holds}\big\}$ for every $T \in \mathfrak{P}_p(E,F)$.

We consider always a finite cartesian product $\prod_{m=1}^h E_m$ of normed spaces $E_m, 1 \le m \le h \in \mathbb{N}$ as a normed space provided with the ℓ^∞-norm $\big\|(x_m)_{m=1}^h\big\| = \sup_{m=1}^h \big\|x_m\big\|$. If F is a Banach space we shall denote by $\mathcal{L}^h\big(\prod_{m=1}^h E_m, F\big)$ the Banach space of all h-linear continuous maps from $\prod_{m=1}^h E_m$ into F. Given $T \in \mathcal{L}^h\big(\prod_{m=1}^h E_m, F\big)$ we can define in a natural way the transposed *linear* map $T' : F' \longrightarrow \mathcal{L}^h\Big(\prod_{m=1}^h E_m, \mathbb{R}\Big)$ putting

$$\forall\ y' \in F' \quad \forall\ (x_m)_{m=1}^h \in \prod_{m=1}^h E_m \quad \Big\langle T'(y'), (x_m)_{m=1}^h\Big\rangle = \Big\langle T\big((x_m)_{m=1}^h\big), y'\Big\rangle.$$

Given maps $A_j \in \mathcal{L}(E_j, F_j)$ between normed spaces E_j and $F_j, 1 \le j \le n$ we write

$$(A_j)_{j=1}^n := (A_1, A_2, ..., A_n) : \prod_{j=1}^n E_j \longrightarrow \prod_{j=1}^n F_j$$

to denote the *continuous linear* map defined by

$$\forall\ (x_j)_{j=1}^n \in \prod_{j=1}^n E_j \quad (A_j)_{j=1}^n\big((x_1, x_2, ..., x_n)\big) = \big(A_1(x_1), A_2(x_2), ..., A_n(x_n)\big).$$

Some times we will write (A_j) instead of $(A_j)_{j=1}^n$. Concerning $(n+1)$-tensor norms, $n \geq 1$ (or multi-tensor norms) we refer the reader to the pioneer works [4] and [5]. If it is needed to emphasize, $\alpha\Big(z; \bigotimes_{j=1}^{n+1} M_j\Big)$ or similar notations will denote the value of the multi-tensor norm α of $z \in \otimes_{j=1}^{n+1} M_j$.

As customary, for $p \in [1, \infty]$, p' will be the conjugate extended real number such that $1/p + 1/p' = 1$. Given $n \geq 1$, in all the paper we denote by $\mathbf{r}$ an $(n+2)$-pla of extended real numbers $\mathbf{r} = (r_0, r_1, r_2, ..., r_n, r_{n+1})$ such that $1 < r_0 \leq \infty$, $1 < r_j < \infty$, $1 \leq j \leq n+1$, and

$$1 = \frac{1}{r_0} + \frac{1}{r_1'} + \frac{1}{r_2'} + ... + \frac{1}{r_{n+1}'}. \tag{2}$$

Such $\mathbf{r}$ will be called an admissible $(n+2)$-pla. Moreover, we define w such that

$$\frac{1}{w} := \frac{1}{r_1'} + \frac{1}{r_2'} + ... + \frac{1}{r_n'} \tag{3}$$

which gives the equality

$$n = \frac{1}{w} + \sum_{j=1}^{n} \frac{1}{r_j}. \tag{4}$$

For later use we note that (2) implies

$$1 = \frac{r_0'}{r_1'} + \frac{r_0'}{r_2'} + ... + \frac{r_0'}{r_n'} + \frac{r_0'}{r_{n+1}'} \quad \text{and} \quad \frac{1}{r_{n+1}} = \frac{1}{r_0} + \frac{1}{r_1'} + \frac{1}{r_2'} + ... + \frac{1}{r_n'} \tag{5}$$

as well

$$\frac{1}{w} = \frac{1}{r_0'} - \frac{1}{r_{n+1}'} = \frac{1}{r_{n+1}} - \frac{1}{r_0} \quad \Longrightarrow 1 = \frac{1}{w} + \frac{1}{r_0} + \frac{1}{r_{n+1}'}. \tag{6}$$

and moreover,

$$\forall\, 1 \leq j \leq n \qquad r_{n+1} < w < r_j', \tag{7}$$

and

$$\forall\, 1 \leq j \leq n+1 \quad r_j < r_0. \tag{8}$$

To finish this introduction we consider the following construction which will be of fundamental importance in all the paper. Given any measure space $(\Omega, \mathcal{A}, \mu)$ and an admissible $(n+2)$-pla $\mathbf{r}$, as a direct consequence of generalized Hölder's inequality and (2), we have a canonical $(n+1)$-linear map $\mathfrak{M}_\mu : L^{r_0}(\Omega, \mathcal{A}, \mu) \times \prod_{j=1}^n L^{r_j'}(\Omega, \mathcal{A}, \mu) \longrightarrow L^{r_{n+1}}(\Omega, \mathcal{A}, \mu)$ defined by the rule

$$\forall\, (f_j)_{j=0}^n \in L^{r_0}(\Omega, \mu) \times \prod_{j=1}^n L^{r_j'}(\Omega, \mu) \qquad \mathfrak{M}_\mu\big((f_j)\big) = \prod_{j=0}^n f_j$$

verifying $\left\|\mathfrak{M}_\mu\big((f_j)\big)\right\| \le \|g\|_{L^{r_0}(\Omega)} \prod_{j=1}^n \|f_j\|_{L^{r'_j}(\Omega)}$. If $(\Omega, \mathcal{A}, \mu)$ is $\mathbb{N}$ with the counting measure we will write simply $\mathfrak{M}$ instead of $\mathfrak{M}_\mu$. Moreover, given $g \in L^{r_0}(\Omega, \mu)$ we shall write D_g to denote the n-linear map from $\prod_{j=1}^n L^{r'_j}(\Omega, \mu)$ into $L^{r_{n+1}}(\Omega, \mu)$ such that

$$\forall\ (f_j)_{j=1}^n \in \prod_{j=1}^n L^{r'_j}(\Omega, \mu) \qquad D_g\big((f_j)_{j=1}^n\big) = \mathfrak{M}_\mu\big((g, f_1, ..., f_n)\big). \tag{9}$$

It will be important for later applications to remark that $\mathfrak{M}_\mu$ induces a linearization map $\widetilde{\mathfrak{M}}_\mu : \big(L^{r_0}(\Omega, \mu)\widehat{\bigotimes}\big(\widehat{\bigotimes}_{j=1}^n L^{r'_j}(\Omega, \mu)\big), \pi\big) \longrightarrow L^{r_{n+1}}(\Omega, \mu)$ and a canonical map

$$\widehat{\mathfrak{M}}_\mu : \big(L^{r_0}(\Omega, \mu)\widehat{\bigotimes}\big(\widehat{\bigotimes}_{j=1}^n L^{r'_j}(\Omega, \mu)\big), \pi\big)/Ker(\widetilde{\mathfrak{M}}_\mu) \longrightarrow L^{r_{n+1}}(\Omega, \mu)$$

such that $\left\|\widehat{\mathfrak{M}}_\mu\right\| \le 1$. Moreover, by (5) we obtain $f = f^{\frac{r_{n+1}}{r_0}} \prod_{j=1}^n f^{\frac{r_{n+1}}{r'_j}}$ for every $f \ge 0$ in $L^{r_{n+1}}(\Omega, \mu)$. As $f = f^+ - f^-$ for every $f \in L^{r_{n+1}}(\Omega, \mu)$ it turns out that $\widetilde{\mathfrak{M}}_\mu$ is a surjective map and $\widehat{\mathfrak{M}}_\mu$ becomes an isomorphism such that $\left\|\widehat{\mathfrak{M}}_\mu^{-1}\right\| \le 2$.

2 $\alpha_{\mathbf{r}}$-tensor products and r-dominated multilinear maps

Let $E_j, 1 \le j \le n+1$ be normed spaces. Using classical methods we can show that

$$\alpha_{\mathbf{r}}\left(z; \bigotimes_{j=1}^{n+1} E_j\right) := \inf \pi_{r_0}\Big((\lambda_m)_{m=1}^h\Big) \prod_{j=1}^{n+1} \varepsilon_{r'_j}\Big((x_m^j)_{m=1}^h\Big), \tag{10}$$

taking the infimum over all representations of z of type

$$z = \sum_{m=1}^h \lambda_m \left(\otimes_{j=1}^{n+1} x_{jm}\right), \quad x_{jm} \in E_j \ \ 1 \le j \le n+1,\ 1 \le m \le h,\ \ h \in \mathbb{N},$$

is a norm on $\bigotimes_{j=1}^{n+1} E_j$ which defines an $(n+1)$-tensor norm in the class of normed spaces. It is interesting to note that if $n = 1$ we obtain the classical tensor norm $\alpha_{r_2 r_1}$ of Lapresté (see [[2]] for details).

The just defined normed tensor product space will be denoted by $\left(\bigotimes_{j=1}^{n+1} E_j, \alpha_{\mathbf{r}}\right)$ or $\bigotimes_{\alpha_{\mathbf{r}}}\big(E_1, E_2, ..., E_{n+1}\big)$ and its completion by $\widehat{\bigotimes}_{\alpha_{\mathbf{r}}}\big(E_1, E_2, ..., E_{n+1}\big)$. It is clear that for every permutation σ on the set $\{1, 2, ..., n+1\}$ the map

$$I_\sigma : \sum_{i=1}^m \lambda_m \otimes_{j=1}^{n+1} x_{jm} \in \left(\otimes_{j=1}^{n+1} E_j, \alpha_r\right) \longrightarrow \sum_{i=1}^m \lambda_m \otimes_{j=1}^{n+1} x_{\sigma(j)m} \in \left(\bigotimes_{j=1}^{n+1} E_{\sigma(j)}, \alpha_{\mathbf{s}}\right),$$

where $\mathbf{s}$ is the admissible $(n+2)$-pla $s_0 := r_0$ and $s_j = r_{\sigma(j)}$, $1 \le j \le n+1$, is an isometry from $\left(\bigotimes_{j=1}^{n+1} E_j, \alpha_{\mathbf{r}}\right)$ onto $\left(\bigotimes_{j=1}^{n+1} E_j, \alpha_{\mathbf{s}}\right)$. We shall use this type of isomorphism in section 5 in the particular case of transpositions σ simply indicating the transposed indexes $\sigma(j_0) = j_1, \sigma(j_1) = j_0$ in the way $j_0 \to j_1, j_1 \to j_0$.

To compute the topological dual of an $\alpha_{\mathbf{r}}$-tensor product we set a new definition:

Definition 1 *Let F and $E_j, 1 \le j \le n$ be normed spaces. A map $T \in \mathcal{L}^n\left(\prod_{j=1}^n E_j, F\right)$ is said to be $\mathbf{r}$-dominated if there is $C \ge 0$ such that for every $h \in \mathbb{N}$ and every set of finite sequences $\{x_{jk}\}_{k=1}^h \subset E_j$, $1 \le j \le n$ and $\{y'_k\}_{k=1}^h \subset F'$ the inequality*

$$\pi_{r'_0}\left(\left(\left|\left\langle T(x_{1k}, x_{2k}, ..., x_{nk}), y'_k\right\rangle\right|\right)_{k=1}^m\right) \le C \left(\prod_{j=1}^n \varepsilon_{r'_j}\left(\left(x_{jk}\right)_{k=1}^m\right)\right) \varepsilon_{r'_{n+1}}\left(\left(y'_k\right)_{k=1}^h\right) \tag{11}$$

holds.

It is easy to see that the linear space $\mathfrak{P}_{\mathbf{r}}\left(\prod_{j=1}^n E_j, F\right)$ of $\mathbf{r}$-dominated n-linear maps from $\prod_{j=1}^n E_j$ into F is normed setting $\mathbf{P}_{\mathbf{r}}(T) := \inf\left\{C \ge 0 \;\middle|\; (11) \text{ holds}\right\}$ for every $T \in \mathfrak{P}_{\mathbf{r}}\left(\prod_{j=1}^n E_j, F\right)$, becoming a Banach space when F does. The interest on $\mathbf{r}$-dominated multilinear maps follows from the next result:

Theorem 2 $\left(\bigotimes_{\alpha_{\mathbf{r}}}(E_1, E_2, ..., E_n, F)\right)' = \mathfrak{P}_{\mathbf{r}}\left(\prod_{j=1}^n E_j, F'\right)$ *for all normed spaces F and E_j, $1 \le j \le n$.*

Proof. 1). Given $T \in \mathfrak{P}_{\mathbf{r}}\left(\prod_{j=1}^n E_j, F'\right)$ and $z = \sum_{k=1}^h \lambda_k \left(\otimes_{j=1}^n x_{jk}\right) \otimes y_k$ in $\left(\bigotimes_{j=1}^n E_j\right) \bigotimes F$ we define $\varphi_T(z) = \sum_{k=1}^h \lambda_k \left\langle T\left((x_{1k}, x_{2k}, ..., x_{nk})\right), y_k\right\rangle$. It follows directly from Hölder's inequality, definition 1 and (10)

$$\left|\varphi_T(z)\right| \le \mathbf{P}_{\mathbf{r}}(T)\ \alpha_{\mathbf{r}}(z) \implies \left\|\varphi_T\right\| \le \mathbf{P}_{\mathbf{r}}(T). \tag{12}$$

2) Conversely, let $\psi \in \left(\bigotimes_{\alpha_{\mathbf{r}}}(E_1, E_2, ..., E_n, F)\right)'$. We define $T_\psi \in \mathcal{L}^n(\prod_{j=1}^n E_j, F')$ as

$$\forall\ (x_j)_{j=1}^n \in \prod_{j=1}^n E_j,\ \forall\ y \in F \quad \left\langle T_\psi\left((x_j)_{j=1}^n\right), y\right\rangle = \psi\left(x_1 \otimes x_2 \otimes ... x_n \otimes y\right).$$

Given $\{x_{jk}\}_{k=1}^h \subset E_j$, $1 \le j \le n$ and $\{y_k\}_{k=1}^h \subset F$, $h \in \mathbb{N}$ we have

$$\pi_{r'_0}\left(\left(\left\langle T_\psi\left(\left(x_{jk}\right)_{j=1}^n\right), y_k\right\rangle\right)_{k=1}^h\right) = \sup_{(\alpha_k) \in B_{\ell_h^{r_0}}} \left|\sum_{k=1}^h \alpha_k\ \psi\left(\left(\otimes_{j=1}^n x_{jk}\right) \otimes y_k\right)\right| =$$

$$= \sup_{(\alpha_k)\in B_{\ell_h^{r_0}}} \left| \psi\left(\sum_{k=1}^{h} \alpha_k \left(\otimes_{j=1}^{n} x_{jk}\right) \otimes y_k \right) \right| \leq$$

$$\leq \sup_{(\alpha_k)\in B_{\ell_h^{r_0}}} \left\|\psi\right\| \pi_{r_0}\Big((\alpha_k)_{k=1}^{h}\Big) \left(\prod_{j=1}^{n} \varepsilon_{r'_j}\Big((x_{jk})_{k=1}^{h}\Big) \right) \varepsilon_{r'_{n+1}}\Big((y_k)_{k=1}^{h}\Big) \leq$$

$$\leq \left\|\psi\right\| \left(\prod_{j=1}^{n} \varepsilon_{r'_j}\Big((x_{jk})_{k=1}^{h}\Big) \right) \varepsilon_{r'_{n+1}}\Big((y_k)_{k=1}^{h}\Big).$$

By $\sigma(F'', F')$-density of F in F'' the latter inequality also holds when $y_k \in F'', 1 \leq k \leq h$. Hence $\mathbf{P_r}(T_\psi) \leq \left\|\psi\right\|$ and clearly $\varphi_{T_\psi} = \psi$, giving by 1) $\mathbf{P_r}(T_\psi) = \left\|\psi\right\|$. ■

The name of **r**-dominated multilinear maps is suggested by the following characterization.

Theorem 3 *Given Banach spaces $E_j, 1 \leq j \leq n$ and F and $T \in \mathcal{L}^n(\prod_{j=1}^{n} E_j, F)$, the following assertions are equivalent:*

1) $T \in \mathfrak{P}_{\mathbf{r}}(\prod_{j=1}^{n} E_j, F)$.

2) (Pietsch-Grothendieck's domination theorem) There are Radon probability measures $\mu_j, 1 \leq j \leq n$ (resp. ν) in the unit balls $B_{E'_j}$, (resp. in $B_{F''}$) and $C \geq 0$ such that, $\mathcal{B}_j$ (resp. $\mathcal{B}_{n+1}$) being the σ-algebra of Borel sets in $B_{E'_j}$ (resp. $B_{F''}$), for every $(x_j)_{j=1}^{n} \in \prod_{j=1}^{n} E_j$ and every $y' \in F'$ one has

$$\left|\left\langle T\Big((x_j)_{j=1}^{n}\Big), y'\right\rangle\right| \leq C \left\|f_{y'}\right\|_{L^{r'_{n+1}}(B_{F''},\, \mathcal{B}_{n+1},\, \nu)} \prod_{j=1}^{n} \left\|f_{x_j}\right\|_{L^{r'_j}(B_{E'_j},\, \mathcal{B}_j,\, \mu_j)} \tag{13}$$

Moreover, $\mathbf{P_r}(T) = \inf C$ taking the infimum over all $C \geq 0$ and μ_j, $1 \leq j \leq n$ and ν verifying (13).

3) (Generalized Kwapien's factorization theorem). There exist Banach spaces M_j and linear maps $A_j \in \mathfrak{P}_{r'_j}(E_j, M_j), 1 \leq j \leq n$ and an n-linear map $S : \prod_{j=1}^{n} M_j \longrightarrow F$ such that $T = S \circ ((A_1, A_2, ..., A_n))$ and the adjoint map $S' \in \mathfrak{P}_{r'_{n+1}}\big(F', \mathcal{L}^n\big(\prod_{j=1}^{n} M_j, \mathbb{R}\big)\big)$.

Proof. 1) $\Longrightarrow$ 2). Clearly, the restriction to $\mathcal{C}\big((B_{E'}, \sigma(E', E))\big)$ of each $\Psi \in \big(L^\infty(B_{E'})\big)'$ is a Radon measure. Then condition 2) follows from 1) directly by definition of **r**-dominated maps and the very general result of Defant $\big[$ [3], theorem 1 $\big]$. Moreover, the proof of that result allow us to obtain

$$\inf\left\{ C \geq 0 \;\middle|\; (13)\ \text{ holds} \right\} \leq \mathbf{P_r}(T). \tag{14}$$

2) $\Longrightarrow$ 3). Let $\mu_j, 1 \le j \le n$ and ν be probability Radon measures in the unit balls $B_{E'_j}$ and $B_{F''}$ respectively (with corresponding σ-algebras $\mathcal{B}_j$ and $\mathcal{B}_{n+1}$ of measurable sets) such that (13) holds.

Put $\Omega := \prod_{j=1}^n B_{E'_j}$ provided with the product measure $\mu := \otimes_{j=1}^n \mu_j$ and its corresponding σ-algebra $\mathcal{B}$ of measurable sets. For every $x_j \in E_j,\ 1 \le j \le n$, we define the map $G_{x_j} : \Omega \longrightarrow \mathbb{R}$ given by $G_{x_j}(\mathbf{x}') = \langle x_j, x'_j \rangle$ for every $\mathbf{x}' = (x'_1, x'_2, ..., x'_n) \in \Omega$. Clearly, as a consequence of Fubini's theorem, we have $G_{x_j} \in L^{r'_j}\big(\Omega, \mathcal{B}, \mu\big)$ and moreover, for each $y' \in F'$ the inequality

$$\Big|\Big\langle T\Big((x_j)_{j=1}^n\Big), y'\Big\rangle\Big| \le C \left\|f_{y'}\right\|_{L^{r'_{n+1}}(B_{F''},\, \mathcal{B}_{n+1},\, \nu)} \prod_{j=1}^n \left\|G_{x_j}\right\|_{L^{r'_j}(\Omega,\, \mathcal{B},\, \mu)} \tag{15}$$

holds still.

Define $A_j \in \mathcal{L}\big(E_j, L^{r'_j}(\Omega, \mathcal{B}, \mu)\big)$, as $A_j(x_j) = G_{x_j}$ for every $x_j \in E_j$ and $M_j := \overline{A_j(E_j)}$, taking the closure in $L^{r'_j}(\Omega, \mathcal{B}, \mu)$ and providing it with the induced topology. It is easy to check (classical Pietsch-Grothendieck's domination theorem) that

$$\forall\, 1 \le j \le n \quad A_j \in \mathfrak{P}_{r'_j}(E_j, M_j) \quad \text{ and } \quad \mathbf{P}_{r'_j}(A_j) \le 1. \tag{16}$$

Now we define the *multilinear* map $S : \prod_{j=1}^n A_j(E_j) \longrightarrow F$ as

$$\forall\, (x_j)_{j=1}^n \in \prod_{j=1}^n E_j \qquad S\big((G_{x_j})_{j=1}^n\big) = T\big((x_j)_{j=1}^n\big).$$

S is well defined because $\big(G_{x_j}\big)_{j=1}^n = \big(G_{\overline{x}_j}\big)_{j=1}^n$ implies $G_{x_j} = G_{\overline{x}_j} \in L^{r'_j}(\Omega, \mathcal{B}, \mu), 1 \le j \le n$ and

$$T\big((x_j)_{j=1}^n\big) - T\big((\overline{x}_j)_{j=1}^n\big) = \sum_{j=1}^n T\big(\overline{x}_1, ..., \overline{x}_{j-1}, x_j - \overline{x}_j, x_{j+1}, ..., x_n\big)$$

and by (15) we obtain $\big\|T\big((x_j)_{j=1}^n\big) - T\big((\overline{x}_j)_{j=1}^n\big)\big\| = 0$. (15) gives too the continuity of S and hence it can be *continuously* extended to a map (still denoted by S) in $\mathcal{L}^n\Big(\prod_{j=1}^n M_j, F\Big)$. To finish the proof we only need to see that $S' \in \mathfrak{P}_{r'_{n+1}}\big(F', \mathcal{L}^n\big(\prod_{j=1}^n M_j, \mathbb{R}\big)\big)$.

Given $\{y'_k\}_{k=1}^h \subset F', h \in \mathbb{N}$, fix a finite sequence $\{\alpha_k\}_{k=1}^h$ verifying $\Big\|(\alpha_k)_{k=1}^h\Big\|_{\ell_h^{r'_{n+1}}} = 1$. For every $\varepsilon > 0$, there are $G_{x_{jk}} \in B_{M_j},\ 1 \le k \le h,\ 1 \le j \le n$ such that

$$\forall\, 1 \le k \le h \quad \Big\|S'(y'_k)\Big\|_{\mathcal{L}^n\big(\prod_{j=1}^n M_j, \mathbb{R}\big)} \le \Big|\Big\langle S'(y'_k), (G_{x_{jk}})_{j=1}^n\Big\rangle\Big| + \varepsilon\, |\alpha_k|.$$

Hence, from Hölder's inequality and (13) we obtain

$$\pi_{r'_{n+1}}\Big(\big(S'(y'_k)\big)_{k=1}^h\Big) = \sup_{(\beta_k)\in B_{\ell_h^{r_{n+1}}}} \left|\sum_{k=1}^h \beta_k \left\|S'(y'_k)\right\|_{\mathcal{L}^n\left(\prod_{j=1}^n M_j,\mathbb{R}\right)}\right| \leq$$

$$\leq \sup_{(\beta_k)\in B_{\ell_h^{r_{n+1}}}} \left|\sum_{k=1}^h \beta_k \left(\left|\left\langle S'(y'_k), (G_{x_{jk}})_{k=1}^n\right\rangle\right| + \varepsilon|\alpha_k|\right)\right| \leq$$

$$\leq \sup_{(\beta_k)\in B_{\ell_h^{r_{n+1}}}} \left\|(\beta_k)\right\|_{\ell_h^{r_{n+1}}} \left(\sum_{k=1}^h \left|\left\langle y'_k, T(x_{jk})_{j=1}^n\right\rangle\right|^{r'_{n+1}}\right)^{\frac{1}{r'_{n+1}}} +$$

$$+\ \varepsilon \sup_{(\beta_k)\in B_{\ell_h^{r_{n+1}}}} \left\|(\beta_k)_{k=1}^h\right\|_{\ell_h^{r_{n+1}}} \left\|(\alpha_k)_{k=1}^h\right\|_{\ell_h^{r'_{n+1}}} \leq$$

$$\leq C \left(\sum_{k=1}^h \left(\left\|f_{y'_k}\right\|_{L^{r'_{n+1}}(B_{F''},\, \mathcal{B}_{n+1},\, \nu)}^{r'_{n+1}} \prod_{j=1}^n \left\|G_{x_{jk}}\right\|_{L^{r'_j}(\Omega,\, \mathcal{B},\, \mu)}^{r'_{n+1}}\right)\right)^{\frac{1}{r'_{n+1}}} + \varepsilon \leq$$

$$\leq C \left(\sum_{k=1}^h \left(\int_{B_{F''}} \left|\left\langle y'_k, y''\right\rangle\right|^{r'_{n+1}} d\nu(y'')\right)\right)^{\frac{1}{r'_{n+1}}} + \varepsilon =$$

$$= C \left(\int_{B_{F''}} \sum_{k=1}^h \left|\left\langle y'_k, y''\right\rangle\right|^{r'_{n+1}} d\nu(y'')\right)^{\frac{1}{r'_{n+1}}} + \varepsilon =$$

$$= C\, \varepsilon_{r'_{n+1}}\Big(\big(y'_k\big)_{k=1}^h\Big)\, \nu\Big(B_{F''}\Big)^{\frac{1}{r'_{n+1}}} + \varepsilon = C\, \varepsilon_{r'_{n+1}}\Big(\big(y'_k\big)_{k=1}^h\Big) + \varepsilon$$

and $\varepsilon > 0$ being arbitrary, the result follows. Moreover, by (16) and the definition of $\mathbf{P}_{r'_{n+1}}(S')$ we obtain

$$\mathbf{P}_{r'_{n+1}}(S') \prod_{j=1}^n \mathbf{P}_{r'_j}(A_j) \leq C. \tag{17}$$

3) $\Longrightarrow$ 1). Assume there there are Banach spaces M_j and maps $A_j \in \mathfrak{P}_{r'_j}(E_j, M_j)$, $1 \leq j \leq n$ and $S \in \mathcal{L}^n\big(\prod_{j=1}^n M_j, F\big)$ such that $S' \in \mathfrak{P}_{r'_{n+1}}\big(F', \mathcal{L}^n\big(\prod_{j=1}^n M_j,\ \mathbb{R}\big)\big)$ and $T = S \circ \big((A_j)_{j=1}^n\big)$. Given finite sequences $\{x_{jk}\}_{k=1}^h \subset E_j$ and $\{y'_k\}_{k=1}^h \subset F'$, $h \in \mathbb{N}$, using (2) and Hölder's inequality we have

$$\pi_{r'_0}\Big(\Big(\Big\langle T\Big((x_{jk})_{j=1}^n\Big), y'_k\Big\rangle\Big)_{k=1}^n\Big) = \sup_{(\alpha_k)\in B_{\ell_h^{r_0}}} \left|\sum_{k=1}^h \alpha_k \Big\langle \big(A_j(x_{jk})\big)_{j=1}^n\big), S'(y'_k)\Big\rangle\right| \leq$$

$$\leq \sup_{(\alpha_k)\in B_{\ell_h^{r_0}}} \sum_{k=1}^{h} |\alpha_k| \left\| S'(y'_k) \right\|_{\mathcal{L}^n\left(\prod_{j=1}^n M_j, \mathbb{R}\right)} \prod_{j=1}^{n} \left\| A_j(x_{jk}) \right\| \leq$$

$$\leq \sup_{(\alpha_k)\in B_{\ell_h^{r_0}}} \left\| (\alpha_k)_{k=1}^h \right\|_{\ell_h^{r_0}} \left(\prod_{j=1}^{n} \pi_{r'_j}\Big(\big(A_j(x_{jk})\big)_{k=1}^h \Big) \right) \pi_{r'_{n+1}}\Big(\big(S'(y'_k)\big)_{k=1}^h \Big) \leq$$

$$\leq \mathbf{P}_{r'_{n+1}}(S') \left(\prod_{j=1}^{n} \mathbf{P}_{r'_j}(A_j) \right) \varepsilon_{r'_{n+1}}\Big(\big(y'_k\big)_{k=1}^h \Big) \left(\prod_{j=1}^{n} \varepsilon_{r'_j}\Big(\big(x_{jk}\big)_{k=1}^h \Big) \right)$$

and hence $T \in \mathfrak{P}_{\mathbf{r}}\Big(\prod_{j=1}^n E_j, F\Big)$ and

$$\mathbf{P}_{\mathbf{r}}(T) \leq \mathbf{P}_{r'_{n+1}}(S') \prod_{j=1}^{n} \mathbf{P}_{r'_j}(A_j). \tag{18}$$

The assertions about $\mathbf{P}_{\mathbf{r}}(T)$ follow from (14), (17) and (18). ■

Theorem 3 can be used to find some equivalences between some tensor norms $\alpha_{\mathbf{r}}$ and $\alpha_{\mathbf{s}}$ derived from different admissible $(n+2)$-plas $\mathbf{r}$ and $\mathbf{s}$ on certain classes of Banach spaces. We present some results of this type which will be of fundamental importance in the final section of the paper.

Corollary 4 *Let* $\mathbf{r} = (r_j)_{j=0}^{n+1}$ *be such that* $r'_{n+1} \leq 2$ *and let* $\mathbf{s} = (s_j)_{j=0}^{n+1}$ *be an admissible* $(n+2)$*-pla such that* $s'_{n+1} \leq 2$, *and* $s'_j = r'_j,\ 1 \leq j \leq n$. *If* $E_j, 1 \leq j \leq n+1$ *are Banach spaces and* E''_{n+1} *has cotype* 2, *one has* $\big(\widehat{\bigotimes}_{j=1}^{n+1} E_j, \alpha_{\mathbf{r}}\big) \approx \big(\widehat{\bigotimes}_{j=1}^{n+1} E_j, \alpha_{\mathbf{s}}\big)$.

Proof. By theorem 2 and the open mapping theorem it is enough to see that $\mathfrak{P}_{\mathbf{s}}\big(\prod_{j=1}^n E_j, E'_{n+1}\big) = \mathfrak{P}_{\mathbf{r}}\big(\prod_{j=1}^n E_j, E'_{n+1}\big)$. Given $T \in \mathfrak{P}_{\mathbf{s}}\big(\prod_{j=1}^n E_j, E'_{n+1}\big)$ and using Kwapien's generalized theorem, we choose a factorization $T = C \circ (A_j)_{j=1}^n$ throughout some product $\prod_{j=1}^n M_j$ of Banach spaces in such a way that $A_j \in \mathfrak{P}_{s'_j}\big(E_j, M_j\big), 1 \leq j \leq n$ and $C' \in \mathfrak{P}_{s'_{n+1}}\big(E''_{n+1}, \mathcal{L}^n(\prod_{j=1}^n M'_j, \mathbb{R})\big)$. Being E''_{n+1} of cotype 2 and $r'_{n+1} \leq 2$, Maurey's theorem $\big[$ [2], corollary 3, §31.6 $\big]$ and Pietsch's inclusion theorem for absolutely p-summing maps give $C' \in \mathfrak{P}_1\big(E''_{n+1}, \mathcal{L}^n(\prod_{j=1}^n M'_j, \mathbb{R})\big) \subset \mathfrak{P}_{r'_{n+1}}\big(E''_{n+1}, \mathcal{L}^n(\prod_{j=1}^n M'_j, \mathbb{R})\big)$. As $r'_j = s'_j, 1 \leq j \leq n$, by the sufficient part of Kwapien's generalized theorem we obtain $T \in \mathfrak{P}_{\mathbf{r}}\big(\prod_{j=1}^n E_j, E'_{n+1}\big)$. In the same way we show $\mathfrak{P}_{\mathbf{r}}\big(\prod_{j=1}^n E_j, E'_{n+1}\big) \subset \mathfrak{P}_{\mathbf{s}}\big(\prod_{j=1}^n E_j, E'_{n+1}\big)$ and the proof is complete. ■

Corollary 5 *Let* $E_j, 1 \leq j \leq n+1$ *be Banach spaces and let* $\mathbf{r} = (r_j)_{j=0}^{n+1}$ *be an admissible* $(n+2)$*-pla such that* $r'_j \geq 2$ *for every* $1 \leq j \leq n+1$. *Let* $\mathbf{s} = (s_j)_{j=0}^{n+1}$ *be another admissible* $(n+2)$*-pla such that* $2 \leq s'_j$ *for every* $1 \leq j \leq n$ *and* $s_{n+1} = r_{n+1}$. *Then* $\big(\widehat{\bigotimes}_{j=1}^{n+1} E_j, \alpha_{\mathbf{r}}\big) \approx \big(\widehat{\bigotimes}_{j=1}^{n+1} E_j, \alpha_{\mathbf{s}}\big)$.

Proof. Arguing as above, we only need to show that $\mathfrak{P}_{\mathbf{s}}\big(\prod_{j=1}^{n} E_j, E'_{n+1}\big) = \mathfrak{P}_{\mathbf{r}}\big(\prod_{j=1}^{n} E_j,\ E'_{n+1}\big)$. The crucial step is the proof of the inclusion $\mathfrak{P}_{\mathbf{r}}\big(\prod_{j=1}^{n} E_j, E'_{n+1}\big) \subset \mathfrak{P}_{\mathbf{s}}\big(\prod_{j=1}^{n} E_j, E'_{n+1}\big)$ since the proof of the converse inclusion can be made exactly in the same way.

Let $T \in \mathfrak{P}_{\mathbf{r}}\big(\prod_{j=1}^{n} E_j, E'_{n+1}\big)$. By the proof of 2) $\Longrightarrow$ 3) in theorem 3 there are a probability space $(\Omega, \mathcal{B}, \mu)$, maps $A_j \in \mathfrak{P}_{r'_j}\big(E_j, L^{r'_j}(\Omega;\mu)\big), 1 \leq j \leq n$ and a map $S \in \mathcal{L}^n\big(\prod_{j=1}^{n} \overline{A_j(E_j)}, E'_{n+1}\big)$ such that $S' \in \mathfrak{P}_{r'_{n+1}}\big(E''_{n+1}, \mathcal{L}^n\big(\prod_{j=1} \overline{A_j(E_j)}, \mathbb{R}\big)\big)$ and $T = S \circ \big((A_j)_{j=1}^n\big)$. Consider the tensor products $\mathfrak{T}_\pi := \big(\widehat{\bigotimes}_{j=1}^{n} L^{r'_j}(\Omega,\mu)\big), \pi\big)$ and $\mathfrak{H}_\pi := L^{r_0}(\Omega,\mu)\widehat{\bigotimes}_\pi \mathfrak{T}_\pi$. The canonical linear map $\widetilde{\mathfrak{M}}_\mu$ from $\mathfrak{H}_\pi$ onto $L^{r_{n+1}}(\Omega,\mu)$, (recall the notation of introductory section) induces an isomorphism $\widehat{\mathfrak{M}}_\mu$ from the quotient space $K_1 := \mathfrak{H}_\pi / Ker(\widetilde{\mathfrak{M}}_\mu)$ onto $L^{r_{n+1}}(\Omega,\mu)$. As $r_{n+1} \leq 2$, K_1 has cotype 2.

Let $\Psi_1 : \mathfrak{H}_\pi \longrightarrow K_1$ be the canonical quotient map. For every $1 \leq j \leq n$ we consider the map $\psi_j \in \mathcal{L}\big(L^{r'_j}(\Omega), \mathfrak{H}_\pi\big)$ defined by

$$\psi_j : z \in L^{r'_j}(\Omega) \longrightarrow \big[\chi_\Omega\big] \otimes \big[\chi_\Omega\big] \otimes ... \otimes \big[\chi_\Omega\big] \otimes z \otimes \big[\chi_\Omega\big] \otimes ... \otimes \big[\chi_\Omega\big]$$

(z in the position $j+1$) and define $\mathfrak{T}_j := \psi_j\big(L^{r'_j}(\Omega)\big)$. $\big[\chi_\Omega\big]$ being of dimension 1 is complemented in each $L^p(\Omega,\mu), p \geq 1$. It follows that $\mathfrak{T}_j$ is a complemented (and hence closed) subspace of $\mathfrak{H}_\pi$. Define $F_j := \overline{A_j(E_j)}$. Clearly $H_j := \psi_j(F_j)$ is a closed subspace of $\mathfrak{T}_j$.

Claim. For every $1 \leq j \leq n$, $\Psi_1(\mathfrak{T}_j)$ is closed in K_1.

Proof of the claim. Fix $1 \leq j \leq n$. Let $P_j \in \mathcal{L}(\mathfrak{H}_\pi, \mathfrak{T}_j)$ be a projection and let $W_j := Ker(P_j) \oplus (Ker(\widetilde{\mathfrak{M}}_\mu) \cap \mathfrak{T}_j)$. The quotient space $K_{2j} := \mathfrak{H}_\pi / W_j$ is well defined. Let $\Psi_{2j} \in \mathcal{L}\big(\mathfrak{H}_\pi, K_{2j}\big)$ be the canonical quotient map. The map

$$\forall\, z \in \mathfrak{H}_\pi \quad L_j : \Psi_{2j}(z) \in K_{2j} \longrightarrow \Psi_1 \circ P_j(z) \in \Psi_1(\mathfrak{T}_j) \subset K_1$$

is well defined and continuous. In fact, given $z_1 = P_j(z_1) + (I_\pi - P_j)(z_1) \in \mathfrak{H}_\pi$ and $z_2 = P_j(z_2) + (I_\pi - P_j)(z_2) \in \mathfrak{H}_\pi$ (I_π denotes the identity map on $\mathfrak{H}_\pi$) such that $\Psi_{2j}(z_1) = \Psi_{2j}(z_2)$, as $(I_\pi - P_j)(z_1) \in Ker(P_j) \subset W$ and $(I_\pi - P_j)(z_2) \in Ker(P_j) \subset W$, we obtain $\Psi_{2j} \circ P_j(z_1) = \Psi_{2j} \circ P_j(z_2)$, i. e.

$$P_j(z_1) - P_j(z_2) \in W \quad \Longrightarrow \quad P_j(z_1) - P_j(z_2) \in Ker(\widetilde{\mathfrak{M}}_\mu) \cap \mathfrak{T}_j \subset Ker(\widetilde{\mathfrak{M}}_\mu)$$

and hence $L_j(z_1) = \Psi_1 \circ P_j(z_1) = \Psi_1 \circ P_j(z_2) = L_j(z_2)$ and L_j is well defined. On the other hand, given $\Psi_{2j}(z) \in K_{2j}$ there is $w \in \mathfrak{T}_\pi$ such that $\Psi_{2j}(w) = \Psi_{2j}(z)$ and $\|w\|_{\mathfrak{T}_\pi} \leq 2 \,\|\Psi_{2j}(z)\|_{K_{2j}}$. Then

$$\big\|L_j \circ \Psi_{2j}(z)\big\|_{K_1} = \big\|L_j \circ \Psi_{2j}(w)\big\|_{K_1} = \big\|\Psi_1 \circ P_j(w)\big\|_{K_1} \leq$$

$$\leq \|\Psi_1\| \, \|P_j\| \, \|w\|_{\mathfrak{H}_\pi} \leq 2 \, \|P_j\| \, \|\Psi_{2j}(z)\|_{K_{2j}}$$

and L_j turns out to be continuous. But, clearly, L_j is surjective. Then the canonical induced map $\widetilde{L}_j \in \mathcal{L}(K_{3j}, K_1)$ from the quotient space $K_{3j} := K_{2j}/Ker(L_j)$ onto K_1 is an isomorphism. Let $\Psi_{3j} \in \mathcal{L}(K_{2j}, K_{3j})$ be the canonical quotient map. Note that we have

$$\Psi_1 \circ P_j = L_j \circ \Psi_{2j} = \widetilde{L}_j \circ \Psi_{3j} \circ \Psi_{2j}. \tag{19}$$

Next take $z \in \overline{\Psi_1(\mathfrak{T}_j)}$. There is a sequence $\{z_m\}_{m=1}^{\infty} \subset \mathfrak{T}_j$ such that $z = \lim_{m\to\infty} \Psi_1(z_m)$ in K_1. Then $\{\widetilde{L}_j^{-1}(z_m)\}_{m=1}^{\infty}$ is a Cauchy sequence in K_{3j}. By a standard procedure (see $\big[$ [8], §14,4. (3)$\big]$ for instance) and switching to a suitable subsequence if necessary, we can assume that there is a sequence $\{w_m\}_{m=1}^{\infty} \subset \mathfrak{H}_\pi$ such that

$$\forall\, m \in \mathbb{N} \quad \Psi_{3j} \circ \Psi_{2j}(w_m) = \widetilde{L}_j^{-1}(z_m) = \Psi_{3j} \circ \Psi_{2j}(z_m) \tag{20}$$

and

$$\forall\, m, k \in \mathbb{N} \;\; \|w_m - w_k\|_{\mathfrak{H}_\pi} \leq 2 \, \|\Psi_{2j}(w_m) - \Psi_{2j}(w_k)\|_{K_{2j}} \leq 4 \|\widetilde{L}_j^{-1}(z_m) - \widetilde{L}_j^{-1}(z_k)\|_{K_{3j}}.$$

Then $\{w_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $\mathfrak{T}_\pi$ and there exists $w = \lim_{m\to\infty} w_m \in \mathfrak{H}_\pi$. By (20) we obtain

$$\Psi_{3j}\circ\Psi_{2j}(z_m) = \Psi_{3j}\circ\Psi_{2j}(w_m) = \Psi_{3j}\circ\Psi_{2j}\big(P_j(w_m) - (I_\pi - P_j)(w_m)\big) = \Psi_{3j}\circ\Psi_{2j}\circ P_j(w_m)$$

and since P_j is a projection and $P_j(z_m) = z_m$, by the definitions of Ψ_{3j} and L_j

$$\Psi_1(z_m) = \Psi_1 \circ P_j(z_m) = L_j \circ \Psi_{2j}(z_m) = L_j \circ \Psi_{2j} \circ P_j(w_m) = \Psi_1 \circ P_j(w_m)$$

and $\Psi_1 \circ P_j(w) = \lim_{m\to\infty} \Psi_1 \circ P_j(w_m) = \lim_{m\to\infty} \Psi_1(z_m) = z$. As $P_j(w) \in \mathfrak{T}_j$ we obtain $z \in \Psi_1(\mathfrak{T}_j)$ and $\Psi_1(\mathfrak{T}_j)$ is closed. ■

End of the proof of corollary 5. Let Φ_j be the restriction to $\mathfrak{T}_j$ of Ψ_1. Let Ψ_{4j} be the canonical quotient map from $\mathfrak{T}_j$ onto the quotient space $K_{4j} := \mathfrak{T}_j/\big(\mathfrak{T}_j \cap Ker(\mathfrak{M}_\mu)\big)$. The map $\widetilde{\Phi}_j : \Psi_{4j} \circ \psi_j(z_j) \in K_{4j} \longrightarrow \Phi_j \circ \psi_j(z_j) \in \Phi_j(\mathfrak{T}_j)$, $z_j \in F_j$ is well defined. In fact, if $\overline{z}_j \in F_j$ and $\Psi_{4j} \circ \psi_j(z_j - \overline{z}_j) = 0$, we will have $\psi_j(z_j - \overline{z}_j) \in Ker(\widetilde{\mathfrak{M}}_\mu)$ and hence, by definition of $\widetilde{\mathfrak{M}}_\mu$ and ψ_j, one has $z_j = \overline{z}_j$ and $\Phi_j \circ \psi_j(z_j) = \Phi_j \circ \psi_j(\overline{z}_j)$, turning $\widetilde{\Phi}_j$ well defined. The same argument shows that $\widetilde{\Phi}_j$ is injective. By the claim $\Phi_j(\mathfrak{T}_j)$ is closed in K_1. As $\widetilde{\Phi}_j$ is clearly surjective by the open map theorem it turns out that $\widetilde{\Phi}_j$ is an isomorphism from K_{4j} onto $\Phi_j(\mathfrak{T}_j)$.

Next, remark that given $z_j \in L^{r'_j}(\Omega, \mu)$ and $\varepsilon > 0$, there is $\overline{z}_j \in L^{r'_j}(\Omega, \mu)$ such that $\Psi_{4j} \circ \psi_j(z_j) = \Psi_{4j} \circ \psi_j(\overline{z}_j)$ and

$$\|\psi_j(\overline{z}_j)\|_{\mathfrak{T}_j} \leq \|\Psi_{4j} \circ \psi_j(z_j)\|_{K_{4j}} + \varepsilon \leq \|\widetilde{\Phi}_j^{-1}\| \, \|\widetilde{\Phi}_j \circ \Psi_{4j} \circ \psi_j(z_j)\|_{K_1} + \varepsilon =$$

$$= \left\|\widetilde{\Phi}_j^{-1}\right\| \left\|\Phi_j \circ \psi_j(z_j)\right\|_{K_1} + \varepsilon \leq \left\|\widetilde{\Phi}_j^{-1}\right\| \left\|\psi_j(z_j)\right\|_{\mathfrak{T}_j} + \varepsilon.$$

But, as we have shown previously, $\Psi_{4j} \circ \psi_j(z_j) = \Psi_{4j} \circ \psi_j(\overline{z}_j)$ implies $z_j = \overline{z}_j$ and so $\psi_j(z_j) = \psi_j(\overline{z}_j)$. Then $\varepsilon > 0$ being arbitrary we obtain

$$\left\|\psi_j(z_j)\right\|_{\mathfrak{T}_j} \leq \left\|\widetilde{\Phi}_j^{-1}\right\| \left\|\Phi_j \circ \psi_j(z_j)\right\|_{K_1} \leq \left\|\widetilde{\Phi}_j^{-1}\right\| \left\|\psi_j(z_j)\right\|_{\mathfrak{T}_j}$$

which means that Φ_j is an isomorphism from $\mathfrak{T}_j$ onto $\Phi_j(\mathfrak{T}_j)$.

As a consequence the isomorphisms $F_j \approx H_j \approx \Phi_j(H_j)$ hold and F_j has cotype 2 because $\Phi_j(H_j)$ is a closed subspace of K_1 which has cotype 2. As $A_j \in \mathfrak{P}_{r'_j}(E_j, F_j)$, by Maurey's theorem $\big[$ [2], corollary 3, §31.6 $\big]$ and Pietsch's inclusion theorem for p-absolutely summing maps, we obtain $A_j \in \mathfrak{P}_2(E_j, F_j) \subset \mathfrak{P}_{s'_j}(E_j, F_j)$. It follow from the properties of S and from Kwapien's generalized theorem that $T \in \mathfrak{P}_{\mathbf{s}}\big(\prod_{j=1}^n E_j, E'_{n+1}\big)$ as desired. ■

Corollary 6 *Let $E_j, 1 \leq j \leq n+1$ be Banach spaces and let $\mathbf{r} = (r_j)_{j=0}^{n+1}$ be an admissible $(n+2)$-pla such that $r_{j\,0} \leq 2$ for some $1 \leq j_0 \leq n+1$ and $r'_{j_1} \geq 2$ for some $1 \leq j_1 \neq j_0 \leq n+1$. Choose $s_{j_0} < r_{j_0}$ and define $\frac{1}{s_0} := \frac{1}{r_0} + \frac{1}{r'_{j_0}} - \frac{1}{s'_{j_0}}$ and $s_j := r_j, 1 \leq j \neq j_0 \leq n+1$. Then $\mathbf{s} = (s_j)_{j=0}^{n+1}$ is an admissible $(n+2)$-pla such that $s_0 < \infty$ and $\big(\bigotimes_{j=1}^{n+1} E_j, \alpha_{\mathbf{r}}\big) \approx \big(\bigotimes_{j=1}^{n+1} E_j, \alpha_{\mathbf{s}}\big)$.*

Proof. After the eventual transposition $j_1 \to n+1, n+1 \to j_1$ we can assume that $j_1 = n+1$. Then the proof is essentially the same of corollary 5 because we have $r_{n+1} \leq 2$ and Maurey's theorem will be applicable still in the "axis" j_0. ■

Another application of theorem 3 concerns to the approximation of $\mathbf{r}$-dominated maps by finite rank maps.

Theorem 7 *Let $E_j, 1 \leq j \leq n+1$, be Banach spaces with duals E'_j having the metric approximation property and such that each $E'_j, 1 \leq j \leq n$ has the Radon-Nikodym property. Then $\mathfrak{P}_{\mathbf{r}}\Big(\prod_{j=1}^n E_j, E'_{n+1}\Big) = \big(\widehat{\bigotimes}_{j=1}^{n+1} E'_j, \alpha'_{\mathbf{r}}\big)$.*

Proof. Let $T \in \mathfrak{P}_{\mathbf{r}}\Big(\prod_{j=1}^n E_j, E'_{n+1}\Big)$. By Kwapien's theorem (theorem 3) there are Banach spaces M_j and operators $A_j \in \mathfrak{P}_{r'_j}(E_j, M_j), 1 \leq j \leq n$ and $S \in \mathfrak{L}^n\Big(\prod_{j=1}^n M_j, E'_{n+1}\Big)$ such that $T = S \circ (A_1, A_2, ..., A_n)$. Since every E'_j has the Radon-Nikodym property, by the result $\big[$ [11], page 228 $\big]$ of Makarov and Samarskii, each A_j is a quasi r'_j-nuclear operator. By $\big[$ [13], theorems 26 and 43 $\big]$ there is a sequence

$$\Big\{B_{jh} = \sum_{s_j=1}^{t_{jh}} x'_{jhs_j} \otimes m_{jhs_j}\Big\}_{h=1}^{\infty} \subset E'_j \otimes M_j,$$

of finite rank operators such that

$$\forall\, 1 \leq j \leq n \qquad \lim_{h\to\infty} \mathbf{P}_{r'_j}(A_j - B_{jh}) = 0. \tag{21}$$

In particular, every sequence $\{B_{jh}\}_{h=1}^{\infty}$ is a Cauchy sequence (and so bounded) in $\mathfrak{P}_{r'_j}(E_j, M_j)$, $1 \leq j \leq n$.

Since for every $(x_j)_{j=1}^n \in \prod_{j=1}^n E_j$ and $h \in \mathbb{N}$ we have

$$\Big(S \circ \big((B_{jh})_{j=1}^n\big)\big((x_j)_{j=1}^n\big) = S\Big(\Big(\sum_{s_j=1}^{t_{jh}} \langle x'_{jhs_j}, x_j\rangle m_{jhs_j}\Big)_{j=1}^n\Big) =$$

$$= \sum_{s_1=1}^{t_{1h}} \cdots \sum_{s_n=1}^{t_{nh}} \Big(\prod_{j=1}^n \langle x'_{jhs_j}, x_j\rangle\Big)\, S\big((m_{jhs_j})_{j=1}^n\big),$$

it turns out that $S \circ \big((B_{jh})_{j=1}^n\big) \in \mathcal{L}^n\big(\prod_{j=1}^n E_j, E'_{n+1}\big)$ has finite dimensional range and

$$S \circ \Big((B_{jh})_{j=1}^n\Big) = \sum_{s_1=1}^{t_{1h}} \cdots \sum_{s_n=1}^{t_{nh}} \Big(\otimes_{j=1}^n x'_{jhs_j}\Big) \otimes S\big((m_{jhs_j})_{j=1}^n\big) \in \bigotimes_{j=1}^{n+1} E'_j.$$

With a similar proof to the one given in [2] it can be seen that $\big(\widehat{\bigotimes}_{j=1}^{n+1} E'_j, \alpha'_{\mathbf{r}}\big)$ is a topological subspace of $\mathfrak{P}_{\mathbf{r}}\Big(\prod_{j=1}^n E_j, E'_{n+1}\Big)$. Hence by theorem 3, (18) and (21)

$$\alpha'_{\mathbf{r}}\Big(S \circ \big(B_{1h}, B_{2h}, ..., B_{nh}\big) - S \circ \big(B_{1k}, B_{2k}, ..., B_{nk}\big)\Big) =$$

$$= \mathbf{P}_{\mathbf{r}}\left(\sum_{j=1}^n \Big(S \circ B_{1k}, ..., B_{j-1,k}, B_{jh} - B_{jk}, B_{j+1,h}, ..., B_{nh}\Big)\right) \leq$$

$$\leq\ \mathbf{P}_{r'_{n+1}}(S')\ \sum_{j=1}^n \mathbf{P}_{r'_j}(B_{jh} - B_{jk})\Big(\prod_{1\leq s<j} \mathbf{P}_{r'_s}(B_{sk})\Big)\Big(\prod_{j<s\leq n} \mathbf{P}_{r'_s}(B_{sh})\Big)$$

is arbitrarily small when h and k lets to infinity and so there exists $z := \lim_{h\to\infty} S \circ \big(B_{1h}, B_{2h}, ..., B_{nh}\big) \in \big(\widehat{\bigotimes}_{j=1}^{n+1} E'_j, \alpha'_{\mathbf{r}}\big)$. On the other hand, it can be shown in an analogous way that

$$\lim_{h\to\infty} \mathbf{P}_{\mathbf{r}}\Big(T - S \circ \big((B_{jh})_{j=1}^n\big)\Big) = \lim_{h\to\infty} \mathbf{P}_{\mathbf{r}}\Big(S \circ \big((A_j)_{j=1}^n\big) - S \circ \big((B_{jh})_{j=1}^n\big)\Big) = 0$$

and hence $T = z$. ■

3 r-nuclear multilinear maps

With the same methods used in the classical case of Laprest\'e's tensor topologies, it can be shown that every element $z \in \widehat{\bigotimes}_{\alpha_{\mathbf{r}}}\big(E_1, E_2, ..., E_n, F\big)$ can be represented as a convergent series

$$z = \sum_{m=1}^{\infty} \lambda_m \Big(\otimes_{j=1}^{n} x_{jm} \Big) \ \otimes \ z_m \tag{22}$$

where $(\lambda_m) \in \ell^{r_0}$, $(x_{jm})_{m=1}^{\infty} \in \ell^{r'_j}(E_j)$, $j = 1, 2, ..., n$ and $(z_m)_{m=1}^{\infty} \in \ell^{r'_{n+1}}(F)$. Moreover, the norm of such elements z can be computed as in (10) but using representations (22) and $h = \infty$.

If F is a Banach space every $z \in \widehat{\bigotimes}_{\alpha_{\mathbf{r}}}\big(E_1, E_2, ..., E_n, F\big)$ defines canonically a multilinear map $T_z \in \mathcal{L}^n\Big(\prod_{j=1}^n E'_j, F\Big)$ by the rule

$$\forall \ \big(x'_j\big)_{j=1}^n \in \prod_{j=1}^n E'_j \qquad T_z\big((x'_j)_{j=1}^n\big) = \sum_{m=1}^{\infty} \lambda_m \left(\prod_{j=1}^n \Big\langle x_m^j, x'_m \Big\rangle \right) \ z_m. \tag{23}$$

Remark that T_z is independent on the representing series (22) for z as a consequence of theorem 2 and the easy fact that $\big(\bigotimes_{j=1}^n E'_j\big) \bigotimes F' \subset \mathfrak{P}_{\mathbf{r}}\Big(\prod_{j=1}^n E_j, \ F'\Big)$ canonically. In this way we have defined a canonical *linear map*

$$\Phi : z \in \widehat{\bigotimes}_{\alpha_{\mathbf{r}}}\big(E_1, E_2, ..., E_n, F\big) \longrightarrow T_z \in \mathcal{L}^n\Big(\prod_{j=1}^n E'_j, F\Big) \tag{24}$$

which suggest the next definition:

Definition 8 *A multilinear map $A \in \mathcal{L}^n\Big(\prod_{j=1}^n E_j, F\Big)$ is said to be $\mathbf{r}$-nuclear if it is the restriction $R(T_z)$ to $\prod_{j=1}^n E_j$ of a map T_z for some $z \in \widehat{\bigotimes}_{\alpha_{\mathbf{r}}}\big(E'_1, E'_2, ..., E'_n, F\big)$.*

It can be shown that the set $\mathfrak{N}_{\mathbf{r}}\Big(\prod_{j=1}^n E_j, F\Big)$ of all n-linear $\mathbf{r}$-nuclear maps from $\prod_{j=1}^n E_j$ into F becomes a Banach space under the $\mathbf{r}$-nuclear norm

$$\mathbf{N}_{\mathbf{r}}(A) = \inf\Big\{ \alpha_{\mathbf{r}}(z) \ \Big| \ A = R(T_z), \ z \in \widehat{\otimes}_{\alpha_{\mathbf{r}}}\big(E'_1, E'_2, ..., E'_n, F\big) \Big\}$$

if all $E_j, 1 \le j \le n$ and F are Banach spaces. $\mathbf{r}$-nuclear maps can be characterized by means of suitable factorizations as follows.

Theorem 9 *Let F and $E_j, 1 \le j \le n$ be Banach spaces and $T \in \mathcal{L}^n\big(\prod_{i=1}^n E_j, F\big)$. T is $\mathbf{r}$-nuclear if and only if there are maps $A_j \in \mathcal{L}(E_j, \ell^{r'_j}), 1 \le j \le n$, $C \in \mathcal{L}(\ell^{r_{n+1}}, F)$ and $\lambda := (\lambda_m) \in \ell^{r_0}$ such that T factorizes in the way*

$$
\begin{array}{ccc}
\prod_{j=1}^{n} E_j & \xrightarrow{\quad T \quad} & F \\
\Big\downarrow {\scriptstyle (A_j)_{j=1}^{n}} & & \Big\uparrow {\scriptstyle C} \\
\prod_{j=1}^{n} \ell^{r'_j} & \xrightarrow[\quad D_\lambda \quad]{} & \ell^{r_{n+1}}
\end{array}
.
$$

Moreover $\mathbf{N_r}(T) = \inf \left(\prod_{j=1}^{n} \|A_j\|\right) \|D_\lambda\| \, \|C\|$ *taking the infimum over all factorizations as above.*

Proof. The proof being quite standard (compare with [10]) is omitted.

Remark. By theorem 9, (2) and the compactness result $\big(\big[$ [1], theorem 4.2 $\big]\big)$ of Alencar and Floret, if $r_0 < \infty$, every $\mathbf{r}$-nuclear mapping is compact.

As an application of theorem 7 we can obtain a sufficient condition in order that the map Φ be injective. Although the formulation of this condition is far to be optimal, it will be enough for our applications in the sequel.

Corollary 10 *Let $E_j, 1 \le j \le n$ be reflexive Banach spaces having the approximation property. Then, for every Banach space E_{n+1} such that E'_{n+1} has the metric approximation property, the map Φ in (24) is injective and so $\big(\widehat{\bigotimes}_{j=1}^{n+1} E_j, \alpha_{\mathbf{r}}\big) = \mathfrak{N}_{\mathbf{r}}\big(\prod_{j=1}^{n} E'_j, E_{n+1}\big)$.*

Proof. Since we have actually $\Phi \in \mathcal{L}\Big(\big(\widehat{\bigotimes}_{j=1}^{n+1} E_j, \alpha_{\mathbf{r}}\big), \mathfrak{N}_{\mathbf{r}}\big(\prod_{j=1}^{n} E'_j, E_{n+1}\big)\Big)$, it is enough to show that this map is injective. Is easy to see that $\bigotimes_{j=1}^{n+1} E'_j \subset \Big(\mathfrak{N}_{\mathbf{r}}\big(\prod_{j=1}^{n} E'_j, E_{n+1}\big)\Big)'$. Now theorem 7 implies that the transposed map

$$\Phi' : \Big(\mathfrak{N}_{\mathbf{r}}\big(\prod_{j=1}^{n} E'_j, E_{n+1}\big)\Big)' \longrightarrow \mathfrak{P}_{\mathbf{r}}\Big(\prod_{j=1}^{n} E_j, E'_{n+1}\Big)$$

has dense range, getting the injectivity of Φ. ∎

4 r-integral multilinear maps

Definition 11 *Let $E_j, 1 \le j \le n$, and F be Banach spaces. A continuous n-linear map T from $\prod_{j=1}^{n} E_j$ into F is called* **r**-*integral if $J_F T \in \big(\widehat{\bigotimes}_{\alpha'_{\mathbf{r}}}\big(E_1, E_2, ..., E_n, F'\big)\big)'$.*

The norm of $J_F T$ in that dual space is taken as definition of the $\mathbf{r}$-integral norm $\mathbf{I_r}(T)$ of a map $T \in \mathfrak{I}_{\mathbf{r}}\big(\prod_{j=1}^{n} E_j, F\big)$, the set of $\mathbf{r}$-integral multilinear maps from $\prod_{j=1}^{n} E_j$

into F. $(\mathfrak{I}_{\mathbf{r}}, \mathbf{I}_{\mathbf{r}})$ turns out to be the maximal ideal of multilinear maps associated to the $(n+1)$-tensor norm $\alpha_{\mathbf{r}}$ in the sense of Defant and Floret (see [2] and theorem 4.5 in [5]). The next theorem gives the prototype of $\mathbf{r}$-integral maps.

Theorem 12 *Given a measure space* $(\Omega, \mathcal{A}, \mu)$ *and* $g \in L^{r_0}(\Omega, \mathcal{A}, \mu)$, *the canonical multilinear map* $D_g : \prod_{j=1}^{n} L^{r_j'}(\Omega, \mathcal{A}, \mu) \longrightarrow L^{r_{n+1}}(\Omega, \mathcal{A}, \mu)$ *is* $\mathbf{r}$*-integral.*

Proof. Let $\mathcal{S}_j, 1 \leq j \leq n$ be the subspace of $L^{r_j'}(\Omega, \mu)$ of simple functions with support of finite measure. Every $\mathcal{S}_j$ being dense in $L^{r_j'}(\Omega, \mu)$, it is enough so see that $D_g \in \big(\bigotimes_{\alpha_{\mathbf{r}}'}\big(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n, L^{r_{n+1}'}(\Omega, \mu)\big)\big)'$ (density lemma for $(n+1)$-tensor norms).

Fix $z \in \bigotimes_{\alpha_{\mathbf{r}}'}\big(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n, L^{r_{n+1}'}(\Omega, \mu)\big)$. There exist finite dimensional subspaces $M_j \subset \mathcal{S}_j, 1 \leq j \leq n$ generated by the characteristic functions $\{\chi_{B_k}\}_{k=1}^{h}$ of a finite family of pairwise disjoints sets of finite measure $\{B_k\}_{k=1}^{h} \subset \mathcal{A}$ and there exists a finite dimensional subspace $N \subset L^{r_{n+1}'}(\Omega, \mu)$ such that $z \in \otimes\big(M_1, M_2, ..., M_n, N\big)$. Then for every $f_j \in M_j,\ 1 \leq j \leq n$ and $f_{n+1} \in N$, using (4)

$$\big\langle \otimes_{j=1}^{n+1} f_j, D_g \big\rangle = \left\langle \Big(\otimes_{j=1}^{n} \sum_{k=1}^{h} \alpha_{jk} \chi_{B_k}\Big) \otimes f_{n+1}, D_g \right\rangle = \sum_{k=1}^{h} \left(\prod_{j=1}^{n} \alpha_{jk}\right) \ \Big\langle \chi_{B_k}\ g, f_{n+1} \Big\rangle =$$

$$= \sum_{k=1}^{h} \frac{1}{\mu(B_k)^n} \left(\prod_{j=1}^{n} \left(\int_{B_k} f_j \ d\mu\right)\right) \ \Big\langle \chi_{B_k}\ g, f_{n+1} \Big\rangle =$$

$$= \sum_{k=1}^{h} \left(\int_{B_k} |g|^{r_0} \ d\mu\right)^{\frac{1}{r_0}} \left(\prod_{j=1}^{n} \left(\frac{1}{\mu(B_k)^{\frac{1}{r_j}}} \int_{B_k} f_j \ d\mu\right)\right) \ \left\langle \frac{\left(\int_{B_k} |g|^{r_0} \ d\mu\right)^{-\frac{1}{r_0}}}{\mu(B_k)^{\frac{1}{w}}} \chi_{B_k} g, f_{n+1} \right\rangle .$$

As a consequence

$$\forall\ z \in \bigotimes\big(M_1, M_2, ..., M_n, N\big) \quad \big\langle z, D_g \big\rangle = \big\langle z, V \big\rangle \tag{25}$$

where we have defined

$$V := \sum_{k=1}^{h} \left(\int_{B_k} |g|^{r_0} \ d\mu\right)^{\frac{1}{r_0}} \big(\otimes_{j=1}^{n} \varphi_{jk}\big) \otimes \frac{\left(\int_{B_k} |g|^{r_0} \ d\mu\right)^{-\frac{1}{r_0}}}{\mu(B_k)^{\frac{1}{w}}} \ \chi_{B_k} g$$

and where φ_{jk} is the class in $L^{r_j}(\Omega, \mu)/M_j^{\perp} = M_j'$ of the function $\mu(B_k)^{-\frac{1}{r_j}}\ \chi_{B_k}$ for every $\forall\ 1 \leq j \leq n, \quad 1 \leq k \leq h$. Moreover, (the class of) $\chi_{B_k} g \in N'$ for every $1 \leq k \leq h$ since $\chi_{B_k} g \in L^{r_0}(\Omega, \mu)$ and by (7) we obtain $\chi_{B_k} g \in L^{r_{n+1}}(\Omega, \mu)$, B_k being of finite measure.

Note that, by finite dimensionality

$$V \in \bigotimes_{\alpha_{\mathbf{r}}}\big(M'_1, M'_2, ..., M'_n, N'\big) = \big(\bigotimes_{\alpha'_{\mathbf{r}}}\big(M_1, M_2, ..., M_m, N\big)\big)'. \tag{26}$$

Now we perform some computations. The first one is

$$\pi_{r_0}\left(\left(\left(\int_{B_k}|g|^{r_0}\,d\mu\right)^{\frac{1}{r_0}}\right)_{k=1}^{h}\right) = \left(\sum_{k=1}^{h}\int_{B_k}|g|^{r_0}\,d\mu\right)^{\frac{1}{r_0}} = \big\|g\big\|_{L^{r_0}(\Omega)} \tag{27}$$

In second time, for every $1 \le j \le n$, using (4) and Hölder's inequality, we obtain

$$\varepsilon_{r'_j}\left(\left(\varphi_{j,k}\right)_{k=1}^{h}\right) = \sup_{\|f\|_{L^{r'_j}(\Omega)}\le 1}\left(\sum_{k=1}^{h}\frac{1}{\mu(B_k)^{\frac{r'_j}{r_j}}}\left(\int_{B_k}f\,d\mu\right)^{r'_j}\right)^{\frac{1}{r'_j}} \le$$

$$\le \sup_{\|f\|_{L^{r'_j}(\Omega)}\le 1}\left(\sum_{k=1}^{h}\frac{1}{\mu(B_k)^{\frac{r'_j}{r_j}}}\left(\int_{B_k}|f|^{r'_j}\,d\mu\right)\,\mu(B_k)^{\frac{r'_j}{r_j}}\right)^{\frac{1}{r'_j}} \le$$

$$\le \sup_{\|f\|_{L^{r'_j}(\Omega)}\le 1}\left(\sum_{k=1}^{h}\int_{B_k}|f|^{r'_j}\,d\mu\right)^{\frac{1}{r'_j}} = \sup_{\|f\|_{L^{r'_j}(\Omega)}\le 1}\big\|f\big\|_{L^{r'_j}(\Omega)} = 1. \tag{28}$$

Finally, by Hölder's inequality and (6) we have

$$\varepsilon_{r'_{n+1}}\left(\left(\mu(B_k)^{-\frac{1}{w}}\left(\int_{B_k}|g|^{r_0}\,d\mu\right)^{-\frac{1}{r_0}}\chi_{B_k}\,g\right)_{k=1}^{h}\right) =$$

$$= \sup_{\|f\|_{L^{r'_{n+1}}(\Omega)}\le 1}\left(\sum_{k=1}^{h}\mu(B_k)^{-\frac{r'_{n+1}}{w}}\left(\int_{B_k}|g|^{r_0}\,d\mu\right)^{-\frac{r'_{n+1}}{r_0}}\left(\int_{B_k}g\,f\,d\mu\right)^{r'_{n+1}}\right)^{\frac{1}{r'_{n+1}}} \le$$

$$\le \sup_{\|f\|_{L^{r'_{n+1}}(\Omega)}\le 1}\left(\sum_{k=1}^{h}\int_{B_k}|f|^{r'_{n+1}}\,d\mu\right)^{\frac{1}{r'_{n+1}}} = \sup_{\|f\|_{L^{r'_{n+1}}(\Omega)}\le 1}\left(\int_{\Omega}|f|^{r'_{n+1}}\,d\mu\right)^{\frac{1}{r'_{n+1}}} = 1. \tag{29}$$

Then, by (25), (26), (27), (28) and (29)

$$\big|\big\langle z, D_g\big\rangle\big| \le \alpha'_{\mathbf{r}}\big(z; \bigotimes\big(M_1, M_2, ..., M_n, N\big)\big)\ \alpha_{\mathbf{r}}\big(V; \bigotimes\big(M'_1, M'_2, ..., M'_n, N'\big)\big) \le$$

$$\leq \alpha'_{\mathbf{r}}\big(z; \bigotimes\big(M_1, M_2, ..., M_n, N\big)\big) \; \big\|g\big\|_{L^{r_0}(\Omega)}$$

and, $\alpha'_{\mathbf{r}}$ being a finite generated $(n+1)$-tensor norm,

$$\big|\langle z, D_g\rangle\big| \leq \alpha'_{\mathbf{r}}\big(z; \bigotimes\big(\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n, L^{r_{n+1}}(\Omega,\mu)\big)\big) \; \big\|g\big\|_{L^{r_0}(\Omega)},$$

which means $\mathbf{I}_{\mathbf{r}}(D_g) \leq \big\|g\big\|_{L^{r_0}(\Omega)}$. ■

To find a characterization of $\mathbf{r}$-integral maps we need to use ultraproducts $(E_\gamma)_{\mathcal{U}}$ of a given family $\{E_\gamma, \gamma \in \mathfrak{G}\}$ of Banach spaces over an ultrafilter $\mathcal{U}$ on the index set $\mathfrak{G}$. For this topic our main reference is [17]. We use the natural notation $(x_\gamma)_{\mathcal{U}}$ for every element in $(E_\gamma)_{\mathcal{U}}$.

Given a family $\big\{T_\gamma \in \mathcal{L}^n\big(\prod_{j=1}^n E_\gamma^j, F_\gamma\big), \mid \gamma \in \mathfrak{G}\big\}$ of maps between the cartesian product $\prod_{j=1}^n E_\gamma^j$ of Banach spaces E_γ^j and $F_\gamma, 1 \leq j \leq n, \; \gamma \in \mathfrak{G}$, such that $\sup_{\gamma\in\mathfrak{G}}\big\|T_\gamma\big\| < \infty$, there is a canonical n-linear continuous ultraproduct map $(T_\gamma)_{\mathcal{U}}$ from the ultraproduct $\big(\prod_{j=1}^n E_\gamma^j\big)_{\mathcal{U}}$ into the ultraproduct $(F_\gamma)_{\mathcal{U}}$ such that for every $\mathbf{x} := \big(\big(x_\gamma^j\big)_{j=1}^n\big)_{\mathcal{U}} \in \big(\prod_{j=1}^n E_\gamma^j\big)_{\mathcal{U}}$ we have $\big(T_\gamma\big)_{\mathcal{U}}(\mathbf{x}) = \big(T_\gamma\big(\big(x_\gamma^j\big)_{j=1}^n\big)\big)_{\mathcal{U}}$. The main result we shall need is the following factorization theorem:

Lemma 13 *Consider a family of canonical maps $D_{g_\gamma} : \prod_{j=1}^n \ell^{r'_j} \longrightarrow \ell^{r_{n+1}}, \; \gamma \in \mathfrak{G} \neq \emptyset$ defined by a family of elements $\{g_\gamma \big| \; \gamma \in \mathfrak{G}\} \subset \ell^{r_0}$ such that $0 < \sup_{\gamma\in\mathfrak{G}} \big\|D_{g_\gamma}\big\| < \infty$. There exist a decomposable measure space $(\Omega, \mathcal{M}, \mu)$, a function $g \in L^{r_0}(\Omega, \mathcal{M}, \mu)$ and order onto isometries $\mathfrak{X}_j : \big(\ell^{r'_j}\big)_{\mathcal{U}} \longrightarrow L^{r'_j}(\Omega, \mathcal{M}, \mu), \; 1 \leq j \leq n, \; \mathfrak{X}_0 : \big(\ell^{r_0}\big)_{\mathcal{U}} \longrightarrow L^{r_0}(\Omega, \mathcal{M}, \mu)$ and $\mathfrak{X}_{n+1} : \big(\ell^{r_{n+1}}\big)_{\mathcal{U}} \longrightarrow L^{r_{n+1}}(\Omega, \mathcal{M}, \mu)$ such that the diagram*

$$\begin{array}{ccc}
\Big(\prod_{j=1}^n \ell^{r'_j}\Big)_{\mathcal{U}} & \xrightarrow{\quad (D_{g_\gamma})_{\mathcal{U}} \quad} & (\ell^{r_{n+1}})_{\mathcal{U}} \\
\big(\mathfrak{X}_j\big)_{j=1}^n \Big\downarrow & & \Big\uparrow \mathfrak{X}_{n+1}^{-1} \\
\prod_{j=1}^n L^{r'_j}(\Omega) & \xrightarrow{\quad D_g \quad} & L^{r_{n+1}}(\Omega).
\end{array}$$

is commutative. Moreover, $\big\|D_g\big\| = \big\|(D_{g_\gamma})_{\mathcal{U}}\big\|$.

Proof. By (5) and a factorization result of Raynaud, $\big[$ [15], theorem 5.1 $\big]$ there are a decomposable measure space $(\Omega, \mathcal{M}, \mu)$ and isometric order isomorphisms

$$\mathfrak{X}_0 : \big(\ell^{r_0}\big)_{\mathcal{U}} \longrightarrow L^{r_0}(\Omega, \mathcal{M}, \mu), \quad \mathfrak{X}_j : \big(\ell^{r'_j}\big)_{\mathcal{U}} \longrightarrow L^{r'_j}(\Omega, \mathcal{M}, \mu), \; 1 \leq j \leq n,$$

and $\mathfrak{X}_{n+1} : \big(\ell^{r_{n+1}}\big)_{\mathcal{U}} \longrightarrow L^{r_{n+1}}(\Omega, \mathcal{M}, \mu)$ such that, $\mathfrak{M}_\gamma$ being the map corresponding to $\gamma \in \mathfrak{G}$ (recall the notations introduced in section 1), we have $(\mathfrak{M}_\gamma)_{\mathcal{U}} = \mathfrak{X}_{n+1}^{-1} \circ \mathfrak{M}_\mu \circ \big((\mathfrak{X}_j)_{j=1}^n\big)$. The lemma follows taking $g = \mathfrak{X}_0\big((g_\gamma)_{\mathcal{U}}\big)$. ■

Now we can obtain the following characterization:

Theorem 14 *Let $E_j, 1 \leq j \leq n$ and F be Banach spaces and $T \in \mathcal{L}^n(\prod_{j=1}^n E_j, F)$. The following are equivalent:*

1) T is $\mathbf{r}-$integral.

2) $J_F T$ can be factorized as

$$\begin{array}{ccccc} \prod_{j=1}^n E_j & \xrightarrow{\quad T \quad} & F & \xrightarrow{\quad J_F \quad} & F'' \\ \Big\downarrow (A_j)_{j=1}^n & & & & \Big\uparrow C \\ \prod_{j=1}^n L^{r'_j}(\Omega, \mathcal{M}, \mu) & & \xrightarrow[\quad D_g \quad]{} & & L^{r_{n+1}}(\Omega, \mathcal{M}, \mu) \end{array} \tag{30}$$

where $A_j \in \mathcal{L}\big(E_j, L^{r'_j}(\Omega, \mathcal{M}, \mu)\big)$, $1 \leq j \leq n$, $C \in \mathcal{L}\big(L^{r_{n+1}}(\Omega, \mathcal{M}, \mu), F''\big)$ and D_g is the multilinear diagonal operator corresponding to some $g \in L^{r_0}(\Omega, \mathcal{M}, \mu)$. Moreover

$$\mathbf{I_r}(T) = \inf \big\|D_g\big\| \, \big\|C\big\| \, \prod_{j=1}^n \big\|A_j\big\| \tag{31}$$

taking the infimum over all factorizations as in the previous diagram.

3) $J_F T$ can be factorized as above but $(\Omega, \mathcal{M}, \mu)$ being a finite measure space and $g = \chi_\Omega$. Formula (31) holds too taking the infimum over the factorizations of that type.

Proof. 1) $\Longrightarrow$ 2). This can be done using standard methods with help of theorem 9 and lemma 13 (see for instance [10] for a detailed development of the method, used in a similar framework).

2) $\Longrightarrow$ 3). Given $\varepsilon > 0$, select a factorization of type (30) with $g \in L^{r_0}(\Omega, \mathcal{M}, \mu)$ and such that

$$\big\|g\big\|_{L^{r_0}(\Omega,\mu)} \big\|C\big\| \prod_{j=1}^n \big\|A_j\big\| \leq \mathbf{I_r}(T) + \varepsilon. \tag{32}$$

After projection onto the sectional subspaces $L^{r'_j}(Supp(g)), 1 \leq j \leq n$ if necessary, we can assume that $\Omega = Supp(g)$. Consider the new *finite measure* ν on $(\Omega, \mathcal{M})$ defined by

$$\forall\, M \in \mathcal{M} \quad \nu(M) = \int_M |g|^{r_0} \, d\mu$$

and the mappings

$$\forall\, 1 \leq j \leq n \quad H_j : f_j \in L^{r'_j}(\Omega, \mu) \longrightarrow H_j(f_j) = f_j \, |g|^{-\frac{r_0}{r'_j}} \in L^{r'_j}(\Omega, \nu)$$

and

$$H_{n+1} : f \in L^{r_{n+1}}(\Omega,\mu) \longrightarrow H_{n+1}(f) = f\ |g|^{-\frac{r_0}{r_{n+1}}} \in L^{r_{n+1}}(\Omega,\nu).$$

By Radon-Nikodym's theorem

$$\Big\|H_{n+1}(f)\Big\|_{L^{r_{n+1}}(\Omega,\nu)} = \Big\|f\Big\|_{L^{r_{n+1}}(\Omega,\mu)}, \quad \Big\|H_j(f_j)\Big\|_{L^{r'_j}(\Omega,\nu)} = \Big\|f_j\Big\|_{L^{r'_j}(\Omega,\mu)},\ 1 \le j \le n \tag{33}$$

and for every $(f_j)_{j=1}^n \in \prod_{j=1}^n L^{r'_j}(\Omega,\mu)$, using (2)

$$\Big(H_{n+1}^{-1}\circ D_{\chi_\Omega}\circ(H_j)_{j=1}^n\Big)\big((f_j)_{j=1}^n\big) = |g|^{\frac{r_0}{r_{n+1}}}\ \prod_{j=1}^n f_j\ |g|^{-\frac{r_0}{r'_j}} = |g|^{r_0\left(\frac{1}{r_{n+1}}-\sum_{j=1}^n\frac{1}{r'_j}\right)}\prod_{j=1}^n f_j =$$

$$= |g|^{r_0\left(\frac{1}{r_{n+1}}-1+\frac{1}{r_0}+\frac{1}{r'_{n+1}}\right)}\ \prod_{j=1}^n f_j = g\ \prod_{j=1}^n f_j = D_g\Big(\big(f_j\big)_{j=1}^n\Big). \tag{34}$$

As $\chi_\Omega \in L^{r_0}(\Omega,\nu)$, joining the factorization (34) with the initial one we get our goal and moreover, by (33) and (32)

$$\mathbf{I_r}(T) \le \big\|C\circ H_{n+1}^{-1}\big\|\ \big\|D_{\chi_\Omega}\big\|\ \prod_{j=1}^n \big\|H_j\circ A_j\big\| \le$$

$$\le \big\|C\big\|\ \big\|H_{n+1}\circ D_g\circ H_j^{-1}\big\|\ \prod_{j=1}^n\big\|A_j\big\| \le \mathbf{I_r}(T)+\varepsilon. \tag{35}$$

3) $\Longrightarrow$ 1). It is immediate by theorem 12 and the ideal properties of multilinear **r**-integral operators. ■

5 Applications to reflexivity

Previous results allows us to obtain some information about the reflexivity of completed tensor products of type $\alpha_{\mathbf{r}}$.

Theorem 15 *Let $E_j, 1 \le j \le n \in \mathbb{N}$ and F be reflexive Banach spaces such that $E'_j,\ 1 \le j \le n$ and F' have the metric approximation property. Given an admissible $(n+2)$-pla $\mathbf{r}$, the space $\widehat{\bigotimes}_{\alpha_{\mathbf{r}}}\big(E_1, E_2, ..., E_n, F\big)$ is reflexive if and only if*

$$\mathfrak{N}_{\mathbf{r}}\Big(\prod_{j=1}^n E'_j,\ F\Big) = \mathfrak{I}_{\mathbf{r}}\Big(\prod_{j=1}^n E'_j,\ F\Big). \tag{36}$$

Proof. If (36) holds, by theorem 7 and corollary 10 we obtain

$$\Big(\widehat{\bigotimes}_{\alpha_{\mathbf{r}}}\big(E_1, E_2, ..., E_n, F\big)\Big)'' = \Big(\mathfrak{P}_{\mathbf{r}}\Big(\prod_{j=1}^{n} E_j,\ F'\Big)\Big)' = \Big(\widehat{\bigotimes}_{\alpha'_{\mathbf{r}}}\big(E'_1, E'_2, ..., E'_n, F'\big)\Big)' =$$

$$= \mathfrak{I}_{\mathbf{r}}\Big(\prod_{j=1}^{n} E'_j,\ F\Big) = \mathfrak{N}_{\mathbf{r}}\Big(\prod_{j=1}^{n} E'_j,\ F\Big) = \widehat{\otimes}_{\alpha_{\mathbf{r}}}\big(E_1, E_2, ..., E_n, F\big).$$

Conversely, if $\widehat{\otimes}_{\alpha_{\mathbf{r}}}\big(E_1, E_2, ..., E_n, F\big)$ is reflexive, by definition of **r**-integral maps, theorem 7 and corollary 10 we obtain

$$\mathfrak{I}_{\mathbf{r}}\Big(\prod_{j=1}^{n} E'_j,\ F\Big) = \Big(\mathfrak{P}_{\mathbf{r}}\Big(\prod_{j=1}^{n} E_j,\ F'\Big)\Big)' = \widehat{\bigotimes}_{\alpha_{\mathbf{r}}}\big(E_1, E_2, ..., E_n, F\big) = \mathfrak{N}_{\mathbf{r}}\Big(\prod_{j=1}^{n} E'_j,\ F\Big). \quad \blacksquare$$

We apply theorem 15 to characterize the reflexivity of $\big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\big)$. First, we need a lemma.

Lemma 16 *Let* $\mathbf{r} = (r_j)_{j=0}^{n+1}$ *an admissible* $(n+2)$*-pla verifying* $r_0 = \infty$ *and let* $1 < u'_j \leq r'_j$ *for every* $1 \leq j \leq n+1$. *Then there exists a non compact map* $T \in \mathfrak{I}_{\mathbf{r}}\big(\prod_{j=1}^{n} \ell^{u'_j}, \ell^{u_{n+1}}\big)$.

Proof. Let $I_1 := \big[0, \frac{1}{2}\big[$ and $I_m := \big[\sum_{i=1}^{m} \frac{1}{2^i}, \sum_{i=1}^{m+1} \frac{1}{2^i}\big[$ if $m > 1$. The map $A_j : (\beta_i) \in \ell^{u'_j} \longrightarrow \sum_{m=1}^{\infty} \beta_m\ \mu(I_m)^{-\frac{1}{r'_j}}\ \chi_{I_m} \in L^{r'_j}([0,1], \mu), 1 \leq j \leq n$ (μ is the Lebesgue measure on $[0,1]$), is well defined and continuous since

$$\big\|A_j\big((\beta_m)\big)\big\| = \left(\sum_{m=1}^{\infty} \frac{|\beta_m|^{r'_j}}{\mu(I_m)}\ \mu(I_m)\right)^{\frac{1}{r'_j}} \leq \big\|(\beta_m)\big\|_{\ell^{u'_j}}.$$

Take $g = \chi_{[0,1]} \in L^\infty([0,1], \mu)$. Consider now the closed linear subspace F generated by the set $\{\chi_{I_m}, m \in \mathbb{N}\}$ in $L^{r_{n+1}}([0,1])$. The map

$$Q : f \in L^{r_{n+1}}([0,1]) \longrightarrow \sum_{m=1}^{\infty} \frac{1}{\mu(I_m)}\ \left(\int_{I_m} f\ d\mu\right)\ \chi_{I_m} \in F$$

is continuous since, by Hölder's inequality

$$\big\|Q(f)\big\|_F = \left(\sum_{m=1}^{\infty} \left(\int_{I_m} f\ d\mu\right)^{r_{n+1}}\ \mu(I_m)^{1-r_{n+1}}\right)^{\frac{1}{r_{n+1}}} \leq$$

$$\leq \left(\sum_{m=1}^{\infty} \left(\int_{I_m} |f|^{r_{n+1}} \, d\mu \right) \mu(I_m)^{\frac{r_{n+1}}{r'_{n+1}} + 1 - r_{n+1}} \right)^{\frac{1}{r_{n+1}}} = \|f\|_{L^{r_{n+1}}([0,1])}.$$

It is immediate that Q is a projection from $L^{r_{n+1}}([0,1])$ onto F. Finally consider the map

$$C : f = \sum_{m=1}^{\infty} \beta_m \, \chi_{I_m} \in F \longrightarrow \left(\beta_m \, \mu(I_m)^{\frac{1}{r_{n+1}}} \right) \in \ell^{u_{n+1}}$$

is continuous since $r_{n+1} \leq u_{n+1}$ and

$$\|C(f)\|_{\ell^{u_{n+1}}} = \left(\sum_{m=1}^{\infty} |\beta_m|^{u_{n+1}} \mu(I_m)^{\frac{u_{n+1}}{r_{n+1}}} \right)^{\frac{1}{u_{n+1}}} \leq \left(\sum_{m=1}^{\infty} |\beta_m|^{r_{n+1}} \mu(I_m) \right)^{\frac{1}{r_{n+1}}} = \|f\|_F.$$

Hence $T := C \circ Q \circ D_g \circ \left((A_j)_{j=1}^n \right) \in \mathfrak{I}_{\mathbf{r}} \left(\prod_{j=1}^n \ell^{u'_j}, \ell^{u_{n+1}} \right)$ but T is not compact since, using (2)

$$\forall \, m \in \mathbb{N} \quad T\big((\mathbf{e}_m, \mathbf{e}_m, ..., \mathbf{e}_m) \big) = \frac{1}{\mu(I_m)^{\frac{1}{r_{n+1}}}} \, \mu(I_m)^{\frac{1}{r_{n+1}}} \, \mathbf{e}_m = \mathbf{e}_m. \quad \blacksquare$$

We can state now the main result of this section:

Theorem 17 *If* $1 < u_j < \infty$ *for every* $1 \leq j \leq n+1$, $\left(\widehat{\bigotimes}_{i=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}} \right)$ *is reflexive if and only if at least one of the following set of conditions holds:*

*S*1). *There is* $1 \leq j_0 \leq n+1$ *such that* $u'_j > 2$ *and* $u'_j > r'_j$ *for all* $1 \leq j \neq j_0 \leq n+1$.

*S*2). *There exists* $1 \leq j_0 \leq n+1$ *such that* $u'_j > 2$ *for every* $1 \leq j \neq j_0 \leq n+1$ *and*

$$\frac{1}{r_{j_0}} > \sum_{1 \leq j \neq j_0}^{n+1} \frac{1}{u'_j}. \tag{37}$$

and moreover, there exists $1 \leq j_1 \neq j_0 \leq n+1$ *such that* $r'_j \geq 2$ *for every* $1 \leq j \neq j_1 \leq n+1$.

*S*3). *We have* $u'_j > 2$ *for every* $1 \leq j \leq n+1$, *and there exists* $1 \leq j_0 \leq n+1$ *such that* $r'_{j_0} \leq 2$ *and*

$$\frac{1}{2} > \sum_{1 \leq j \neq j_0}^{n+1} \frac{1}{u'_j}. \tag{38}$$

*S*4). *There is* $1 \leq j_0 \leq n+1$ *such that* $u'_{j_0} = 2, r'_{j_0} \leq 2$, $u'_j > 2$ *for every* $1 \leq j \neq j_0 \leq n+1$ *and*

$$\frac{1}{2} > \sum_{1 \leq j \neq j_0}^{n+1} \frac{1}{u'_j}. \tag{39}$$

Proof. Sufficient conditions. Case S1). After the transposition $j_0 \longrightarrow n+1, n+1 \longrightarrow j_0$ if necessary, we can assume $j_0 = n+1$ and so $u'_j > 2$ and $u'_j > r'_j$ for every $1 \leq j \leq n$.

By theorem 14, given $T \in \mathfrak{I}_{\mathbf{r}}\Big(\prod_{j=1}^{n} \ell^{u'_j}, \ell^{u_{n+1}}\Big)$ there are a finite measure space $(\Omega, \mathcal{M}, \mu)$ and mappings $A_j \in \mathcal{L}(\ell^{u'_j}, L^{r'_j}(\Omega,\mu)), 1 \leq j \leq n$ and $C \in \mathcal{L}(L^{r_{n+1}}(\Omega,\mu), \ell^v)$ such that $T = C \circ D_{\chi_\Omega} \circ \big(A_j\big)_{j=1}^{n}$. By Rosenthal's result $\big[$ [16],theorem A.2 $\big]$ every A_j is compact, and by the metric approximation property of ℓ^{u_j}, there is a bounded sequence

$$\left\{ A_{jm} = \sum_{k=1}^{k_{jm}} \mathbf{x}_{jk} \otimes f_{jm}^{k} \right\}_{m=1}^{\infty} \subset \ell^{u_j} \otimes L^{r'_j}(\Omega,\mu) \tag{40}$$

such that

$$\forall\, 1 \leq j \leq n \qquad \lim_{m\to\infty} \Big\| A_j - A_{jm} \Big\|_{\mathcal{L}(\ell^{u'_j},\, L^{r'_j}(\Omega,\mu))} = 0. \tag{41}$$

Define $T_m := C \circ D_{\chi_\Omega} \circ \big((A_{jm})_{j=1}^{n}\big)$ for every $m \in \mathbb{N}$. Arguing as in theorem 7 and using theorem 14 we obtain for every $1 \leq j \leq n$ and $m \in \mathbb{N}$

$$\big\{C \circ D_{\chi_\Omega} \circ \big(A_{1m}, ..., A_{j-1,m}, A_j - A_{jm}, A_{j+1,m}, ..., A_{nm}\big)\big\}_{m=1}^{\infty} \subset \mathfrak{I}_{\mathbf{r}}\Big(\prod_{j=1}^{n} \ell^{u'_j}, \ell^{u_{n+1}}\Big)$$

and by (41)

$$\mathbf{I}_{\mathbf{r}}(T - T_m) \leq \sum_{j=1}^{n} \mathbf{I}_{\mathbf{r}}\big(C \circ D_{\chi_\Omega} \circ \big(A_{1m}, ..., A_{j-1,m}, A_j - A_{jm}, A_{j+1}, ..., A_n\big)\big) \leq$$

$$\leq \mu(\Omega)^{\frac{1}{r_0}} \; \|C\| \; \sum_{j=1}^{n} \|A_j - A_{jm}\| \left(\prod_{1\leq s<j} \|A_{sm}\| \right) \left(\prod_{j<s\leq n} \|A_s\| \right) \tag{42}$$

which approach to 0 if $m \longrightarrow \infty$. But actually we have

$$T_m = \sum_{k=1}^{k_{jm}} \big(\otimes_{j=1}^{n} \mathbf{x}_{jk}\big) \otimes \big(C \circ D_{\chi_\Omega} \circ \big((f_{jm}^{k})\big)\big) \in \mathfrak{N}_{\mathbf{r}}\Big(\prod_{j=1}^{n} \ell^{u'_j}, \ell^{u_{n+1}}\Big).$$

It follows from theorem 7 that $\mathbf{N}_{\mathbf{r}}(T_m - T_s) = \mathbf{I}_{\mathbf{r}}(T_m - T_s)$ for $m, s \in \mathbb{N}$ and using (42), it turns out that $\big\{T_m\big\}_{m=1}^{\infty}$ is a Cauchy sequence in $\mathfrak{N}_{\mathbf{r}}\big(\prod_{j=1}^{n} \ell^{u'_j}, \ell^{u_{n+1}}\big)$. Then $T \in \mathfrak{N}_{\mathbf{r}}\big(\prod_{j=1}^{n} \ell^{u'_j}, \ell^{u_{n+1}}\big)$ and by theorem 15 $\big(\widehat{\bigotimes}_{i=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\big)$ is reflexive.

Case S2). Let $1 \leq j_0 \neq j_1 \leq n+1$ such that $u'_j > 2, 1 \leq j \neq j_0 \leq n+1$, $r'_j \geq 2, 1 \leq j \neq j_1 \leq n+1$ and (37) holds. In a first step we are going to see that we can assume $r'_{j_1} \geq 2$ too.

Consider the case that $r'_{j_1} < 2$. In such a case we have $u'_{j_1} > 2$ because $j_0 \neq j_1$. If $j_1 = n+1$, defining $s'_{n+1} = 2, s'_j := r'_j, 1 \leq j \leq n$ and $\frac{1}{s_0} := \frac{1}{r_0} + \frac{1}{r'_{n+1}} - \frac{1}{2}$ we obtain an admissible $(n+2)$-pla $\mathbf{s} = (s_j)_{j=0}^{n+1}$ verifying (37) still and such that, $\ell^{u_{n+1}}$ having cotype 2, by corollary 4, we have $(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}) \approx (\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{s}})$. If $1 \leq j_1 \leq n$, a transposition $j_1 \to n+1, n+1 \to j_1$ would reduce the situation to the just considered case. So, in the formulation of $S1$) we can assume that $r'_j \geq 2, 1 \leq j \leq n+1$.

After the eventual transposition $j_0 \longrightarrow n+1, n+1 \longrightarrow j_0$ we can assume that $u'_j > 2$ for every $1 \leq j \leq n$, $r'_j \geq 2$ for every $1 \leq j \leq n+1$ and (37) holds for $j_0 = n+1$. Using (5) this last condition can be written in the way

$$\frac{1}{r_0} + \sum_{\{j \,|r'_j < u'_j\}}^{n} \left(\frac{1}{r'_j} - \frac{1}{u'_j} \right) > \sum_{\{j \,|r'_j \geq u'_j\}}^{n} \left(\frac{1}{u'_j} - \frac{1}{r'_j} \right). \tag{43}$$

For every $1 \leq j \leq n$ such that $r'_j \geq u'_j$, choose $2 \leq t'_j < u'_j$ close enough to u'_j in order that

$$\frac{1}{t_0} := \frac{1}{r_0} + \sum_{\{j \,|r'_j < u'_j\}}^{n} \left(\frac{1}{r'_j} - \frac{1}{u'_j} \right) - \sum_{\{j \,|r'_j \geq u'_j\}}^{n} \left(\frac{1}{t'_j} - \frac{1}{r'_j} \right) > 0. \tag{44}$$

Now define $t'_j := r'_j$ if $r'_j < u'_j, 1 \leq j \leq n$ and $t_{n+1} := r_{n+1}$. By (2) we have

$$\frac{1}{t_{n+1}} = \sum_{j=1}^{n} \frac{1}{t'_j} + \sum_{\{j \,|r'_j < u'_j\}}^{n} \left(\frac{1}{r'_j} - \frac{1}{t'_j} \right) + \sum_{\{j \,|r'_j \geq u'_j\}}^{n} \left(\frac{1}{r'_j} - \frac{1}{t'_j} \right) + \frac{1}{r_0}$$

and it turns out that $\mathbf{t} = (t_j)_{j=0}^{n+1}$ is an admissible $(n+2)$-pla such that $2 \leq t'_j < u'_j$ and $t'_j \leq r'_j$ for every $1 \leq j \leq n$ and moreover, by corollary 5 we have $\left(\bigotimes_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\right) \approx \left(\bigotimes_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{t}}\right)$. Hence by case $S1$), $\left(\bigotimes_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\right)$ is reflexive.

Case S3). Once again after the transposition $j_0 \longrightarrow n+1, n+1 \longrightarrow j_0$ we can assume that $r'_{n+1} \leq 2$, $u'_j > 2$ for every $1 \leq j \leq n+1$ and (38) holds for $j_0 = n+1$, or in an equivalent way (by (2)),

$$\frac{1}{r_0} + \frac{1}{r'_{n+1}} - \frac{1}{2} + \sum_{\{j \,|r'_j < u'_j\}}^{n} \left(\frac{1}{r'_j} - \frac{1}{u'_j} \right) > \sum_{\{j \,|r'_j \geq u'_j\}}^{n} \left(\frac{1}{u'_j} - \frac{1}{r'_j} \right).$$

Remark that, by (2) we have necessarily $r'_j \geq 2, 1 \leq j \leq n$. Since $\ell^{u_{n+1}}$ has cotype 2, by corollary 4 there exists an $(n+2)$-pla $\mathbf{s} = (s_j)_{j=0}^{n+1}$ such that $s'_{n+1} = 2$, $s'_j := r'_j, 1 \leq j \leq n$ and $\frac{1}{s_0} := \frac{1}{r_0} + \frac{1}{r'_{n+1}} - \frac{1}{2}$ and $\left(\bigotimes_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{s}}\right) \approx \left(\bigotimes_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\right)$. Then $\left(\bigotimes_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{s}}\right)$ is reflexive by the case S2) and so $\left(\bigotimes_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\right)$ does.

Case S4). Assume the existence of $1 \le j_0 \le n+1$ such that $u'_{j_0} = 2, r'_{j_0} \le 2$, $u'_j > 2$ for every $1 \le j \neq j_0 \le n+1$ and (39) holds. Consider the admissible $(n+2)$-pla $\mathbf{s} = (s_j)_{j=0}^{n+1}$ such that $s_{j_0} := 2, s_j := r_j$ for every $1 \le j \neq j_0 \le n+1$ and $\frac{1}{s_0} := \frac{1}{r_0} + \frac{1}{r'_{j_0}} - \frac{1}{2}$. We obtain from Kwapien's generalized theorem and Pietsch's inclusion theorem that $\mathfrak{P}_{\mathbf{r}}\big(\prod_{j=1}^n \ell^{u_j}, \ell^{u'_{n+1}}\big) \subset \mathfrak{P}_{\mathbf{s}}\big(\prod_{j=1}^n \ell^{u_j}, \ell^{u'_{n+1}}\big)$. The reverse inclusion is true by Kwapien's factorization theorem and Maurey's theorem $\big[$ [2], corollary 3, §31.6 $\big]$ because $\ell^{u_{j_0}} = \ell^2$ has cotype 2 and $r'_{j_0} < 2$ give $\mathfrak{P}_2(\ell^2, M) = \mathfrak{P}_{r'_{j_0}}(\ell^2, M)$ for every Banach space M. Then $\big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\big) \approx \big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{s}}\big)$ and $\big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{s}}\big)$ is reflexive by (39) and the case $S2$).

Necessary conditions. We are going to see that $\big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\big)$ is not reflexive if none of the previous conditions holds. It is enough to consider the following cases.

Case N1). Assume there exist $1 \le j_0 \le n$ such that $u'_{j\,0} \le 2$ and $1 \le j_0 \neq j_1 \le n+1$ such that $u_{j_1} \ge 2$. After the transposition $j_1 \longrightarrow n+1, n+1 \longrightarrow j_1$ on$\{1, 2, ..., n+1\}$ if necessary, we can assume that $j_1 = n+1$, i.e. $u_{n+1} \ge 2$.

For every $1 < p < \infty$, let $\{R_{p,h}\}_{h=1}^{\infty}$ be the sequence of Rademacher functions in $L^p([0,1])$. It is well known that the sequence $\{R_{p,h}\}_{h=1}^{\infty}$ is equivalent to the standard unit basis of ℓ^2 and its closed linear span X_p is complemented in $L^p([0,1])$ (Khintchine's inequality and $\big[$ [12], proposition 5 $\big]$).

Let $P_{n+1} \in \mathcal{L}\big(L^{r_{n+1}}([0,1]), X_{r_{n+1}}\big)$ be a projection. Let $S_{j_0} : \ell^{u'_{j_0}} \longrightarrow X_{r'_{j_0}}$ be the continuous linear map such that $S_{j_0}(\mathbf{e}_h) = R_{r'_{j_0},h}$. On the other hand, for every $1 \le j \neq j_0 \le n$ fix a sequence $(\alpha_{jh})_{h=1}^{\infty} \in \ell^2$ such that $\alpha_{j1} = 1$ and denote by $S_j : \ell^{u'_j} \longrightarrow X_{r'_j}$ the continuous linear map such that $S_j(\mathbf{e}_h) = \alpha_{jh}\ R_{r'_j,h}$ (remark that

$$\big\|S_j\big((\beta_h)\big)\big\| \le C_j\ \big\|(\alpha_{jh}\beta_h)\big\|_{\ell^2} \le C_j\ \big\|(\alpha_{jh})\big\|_{\ell^2}\ \big\|(\beta_h)\big\|_{\ell^\infty} \le C_j\ \big\|(\alpha_{jh})\big\|_{\ell^2}\ \big\|(\beta_h)\big\|_{\ell^{u'_j}}$$

for some $C_j > 0$ by Khintchine's inequality).

Take $g := \prod_{j=1, j\neq j\,0}^{n} R_{r'_j,1} \in L^{r_0}([0,1])$, and consider the well defined map $T_{n+1} \in \mathcal{L}(X_{r_{n+1}}, \ell^{u_{n+1}})$ such that $T_{n+1}(R_{r_{n+1},h}) = \mathbf{e}_h$ for $h \in \mathbb{N}$. Then

$$T := T_{n+1} \circ P_{n+1} \circ D_g \circ \big(S_j\big)_{j=1}^{n}$$

is $\mathbf{r}$-integral by theorem 14. Let $\{z_{j_0,h}\}_{h=1}^{\infty} := \{(a_{1h}, a_{2h}, ..., a_{nh})\}_{h=1}^{\infty} \subset \prod_{j=1}^n \ell^{u'_j}$ such that $a_{jh} = \mathbf{e}_1$ if $j \neq j_0$ and $a_{j_0h} = \mathbf{e}_h$, for every $h \in \mathbb{N}$. We obtain $T(z_{j_0,h}) = \mathbf{e}_h$ for every $h \in \mathbb{N}$ and so T is not compact. If $r_0 \neq \infty$, by the remark after theorem 9 we have $T \notin \mathfrak{N}_{\mathbf{r}}\big(\prod_{j=1}^n \ell^{u'_j}, \ell^{u_{n+1}}\big)$ and by theorem 15, $\big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\big)$ is not reflexive.

In the case $r_0 = \infty$ we need to consider several possibilities. First assume that there are $1 \le j_2 \neq j_0 \le n+1$ and $1 \le j_3 \neq j_2 \le n+1$ such that $r'_{j_2} \ge 2$ and $r'_{j_3} \ge 2$. By corollary 6 there is an admissible $(n+2)$-pla $\mathbf{s} = (s_j)_{j=0}^{n+1}$ such that $s_0 \neq \infty$ and

$\big(\widehat{\bigotimes}_{j=1}^{n+1}\ell^{u_j},\alpha_{\mathbf{r}}\big) \approx \big(\widehat{\bigotimes}_{j=1}^{n+1}\ell^{u_j},\alpha_{\mathbf{s}}\big)$. Then by the previous case with $r_0 \neq \infty$, we see that $\big(\widehat{\bigotimes}_{j=1}^{n+1}\ell^{u_j},\alpha_{\mathbf{r}}\big)$ is not reflexive.

Finally, having (2) in mind, it remains to consider the case that $r'_{j_0} \leq 2$ and $n = 1$. We are dealing with $\ell^{u_1}\widehat{\bigotimes}_{\alpha_{\mathbf{r}}}\ell^{u_2}$ where $u'_1 \leq 2, r'_1 \leq 2$ and $u_2 \geq 2$. By theorems 2 and 7 we have $\big(\ell^{u_1}\widehat{\bigotimes}_{\alpha_{\mathbf{r}}}\ell^{u_2}\big)' = \ell^{u'_1}\widehat{\bigotimes}_{\alpha'_{\mathbf{r}}}\ell^{u'_2}$. The set $K := \big\{\mathbf{e}_i \otimes \mathbf{e}_i,\ i \in \mathbb{N}\big\} \subset \ell^{u'_1} \bigotimes_{\alpha'_{\mathbf{r}}} \ell^{u'_2}$ is bounded. If $\ell^{u_1}\widehat{\bigotimes}_{\alpha_{\mathbf{r}}}\ell^{u_2}$ were reflexive, $\ell^{u'_1}\widehat{\bigotimes}_{\alpha'_{\mathbf{r}}}\ell^{u'_2}$ would be reflexive too and by Smul'yan's theorem, switching to a suitable subsequence if necessary, we would assume that $\{\mathbf{e}_i \otimes \mathbf{e}_i\}_{i=1}^{\infty}$ is weakly convergent to some $z \in \ell^{u'_1}\widehat{\bigotimes}_{\alpha'_{\mathbf{r}}}\ell^{u'_2}$. It follows from boundedness of K and the density of $\big[\mathbf{e}_h\big]_{h=1}^{\infty} \bigotimes \big[\mathbf{e}_h\big]_{h=1}^{\infty}$ in $\ell^{u_1}\widehat{\bigotimes}_{\alpha_{\mathbf{r}}}\ell^{u_2}$ that given $T \in \ell^{u_1}\widehat{\bigotimes}_{\alpha_{\mathbf{r}}}\ell^{u_2}$ and $\rho > 0$, there exist $w \in \bigcup_{k=1}^{\infty}\big[\mathbf{e}_h\big]_{h=1}^{k} \bigotimes \big[\mathbf{e}_h\big]_{h=1}^{k}$ and $m_0 \in \mathbb{N}$ such that

$$\forall\, m \geq m_0 \quad \big|\big\langle T, z\big\rangle\big| \leq \big|\big\langle T, z - \mathbf{e}_m \otimes \mathbf{e}_m\big\rangle\big| + \big|\big\langle T - w, \mathbf{e}_m \otimes \mathbf{e}_m\big\rangle\big| + \big|\big\langle w, \mathbf{e}_m \otimes \mathbf{e}_m\big\rangle\big| \leq$$

$$\leq \big|\big\langle T, z - \mathbf{e}_m \otimes \mathbf{e}_m\big\rangle\big| + \sup_{k\in\mathbb{N}}\big|\big\langle T - w, \mathbf{e}_k \otimes \mathbf{e}_k\big\rangle\big| + \big|\big\langle w, \mathbf{e}_m \otimes \mathbf{e}_m\big\rangle\big| \leq \rho$$

because $\langle w, \mathbf{e}_m \otimes \mathbf{e}_m\rangle = 0$ if m is large enough. Then $z = 0$. But we are assuming that $\mathfrak{I}_{\mathbf{r}}(\ell^{u'_1}, \ell^{u_2}) = \big(\ell^{u'_1}\widehat{\bigotimes}_{\alpha'_{\mathbf{r}}}\ell^{u'_2}\big)' = \ell^{u_1}\widehat{\bigotimes}_{\alpha_{\mathbf{r}}}\ell^{u_2}$ and so, by the construction made in the case $r_0 \neq \infty$ there is $T \in \ell^{u_1}\widehat{\bigotimes}_{\alpha_{\mathbf{r}}}\ell^{u_2}$ such that $\big\langle T(\mathbf{e}_i), \mathbf{e}_i\big\rangle = \big\langle \mathbf{e}_i, \mathbf{e}_i\big\rangle = 1$ for every $i \in \mathbb{N}$, a contradiction. Then $\ell^{u_1}\widehat{\bigotimes}_{\alpha_{\mathbf{r}}}\ell^{u_2}$ is not reflexive.

Case N2). Assume that $u'_j \geq 2$ for every $1 \leq j \leq n$, $r'_j \geq 2$ for every $1 \leq j \leq n+1$, $u'_{n+1} \leq r'_{n+1}$ and $\frac{1}{r_{n+1}} \leq \sum_{j=1}^{n}\frac{1}{u'_j}$, or equivalently (by (5))

$$\frac{1}{r_0} + \sum_{\{j\ |r'_j < u'_j\}}^{n}\left(\frac{1}{r'_j} - \frac{1}{u'_j}\right) \leq \sum_{\{j\ |r'_j \geq u'_j\}}^{n}\left(\frac{1}{u'_j} - \frac{1}{r'_j}\right). \tag{45}$$

Given $1 \leq j \leq n$, if $r'_j < u'_j$ and $t'_j \in \big[u'_j, \infty\big[$ it turns out that we have

$$\frac{1}{r_0} + \sum_{\{j\ |r'_j < u'_j\}}^{n}\left(\frac{1}{r'_j} - \frac{1}{t'_j}\right) \in \left[\frac{1}{r_0} + \sum_{\{j\ |r'_j < u'_j\}}^{n}\left(\frac{1}{r'_j} - \frac{1}{u'_j}\right)\ ,\ \frac{1}{r_0} + \sum_{\{j\ |r'_j < u'_j\}}^{n}\frac{1}{r'_j}\right[.$$

On the other hand, if $r'_j \geq u'_j$ and $t'_j \in \big[u'_j, r'_j\big]$ we have

$$\sum_{\{j\ |r'_j \geq u'_j\}}^{n}\left(\frac{1}{t'_j} - \frac{1}{r'_j}\right) \in \left[0, \sum_{\{j\ |r'_j \geq u'_j\}}^{n}\left(\frac{1}{u'_j} - \frac{1}{r'_j}\right)\right].$$

Then it follows from (45) that we can choose $t'_j \geq u'_j$ for every $1 \leq j \leq n$ such that $r'_j < u'_j$ and $u'_j \leq t'_j \leq r'_j$ for every $1 \leq j \leq n$ which verifies $u'_j \leq r'_j$ in order that

$$\frac{1}{r_0} + \sum_{\{j \mid r'_j < u'_j\}}^{n} \left(\frac{1}{r'_j} - \frac{1}{t'_j} \right) = \sum_{\{j \mid r'_j \geq u'_j\}}^{n} \left(\frac{1}{t'_j} - \frac{1}{r'_j} \right).$$

By (2) we have

$$\frac{1}{r_{n+1}} = \sum_{j=1}^{n} \frac{1}{t'_j} + \sum_{\{j \mid r'_j < u'_j\}}^{n} \left(\frac{1}{r'_j} - \frac{1}{t'_j} \right) + \sum_{\{j \mid r'_j \geq u'_j\}}^{n} \left(\frac{1}{r'_j} - \frac{1}{t'_j} \right) + \frac{1}{r_0} = \sum_{j=1}^{n} \frac{1}{t'_j}.$$

Taking $t_0 = \infty$ and $t_{n+1} = r_{n+1}$ we obtain an admissible $(n+2)$-pla $\mathbf{t} = (t_j)_{j=0}^{n+2}$ such that $t'_j \geq u'_j \geq 2$ for every $1 \leq j \leq n$. By corollary 5 we have $\big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\big) \approx \big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{t}}\big)$ and so $\mathfrak{I}_{\mathbf{t}}\big(\prod_{j=1}^{n} \ell^{u'_j}, \ell^{u_{n+1}}\big) = \mathfrak{I}_{\mathbf{r}}\big(\prod_{j=1}^{n} \ell^{u'_j}, \ell^{u_{n+1}}\big)$. But by lemma 16 there is a non compact map $S \in \mathfrak{I}_{\mathbf{t}}\big(\prod_{j=1}^{n} \ell^{u'_j}, \ell^{u_{n+1}}\big)$. Now we take $s'_j = t'_j$ if $1 \leq j \leq n$, $s'_{n+1} > t'_{n+1}$ and define $s_0 < \infty$ such that $\frac{1}{s_0} := \frac{1}{t'_{n+1}} - \frac{1}{s'_{n+1}}$. Then $\mathbf{s} = (s_j)_{j=0}^{n+1}$ is another admissible $(n+2)$-pla verifying $\big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{s}}\big) \approx \big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{t}}\big)$ corollary 6 and $S \in \mathfrak{I}_{\mathbf{s}}\big(\prod_{j=1}^{n} \ell^{u'_j}, \ell^{u_{n+1}}\big)$. By remark after theorem 9 we have $S \notin \mathfrak{N}_{\mathbf{s}}\big(\prod_{j=1}^{n} \ell^{u'_j}, \ell^{u_{n+1}}\big)$ and by theorem 15 $\big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\big) \approx \big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{s}}\big)$ turns out to be not reflexive.

Case N3). Assume that $u'_j \geq 2$ for every $1 \leq j \leq n+1$, $r'_{n+1} \leq 2$ and $\frac{1}{2} \leq \sum_{j=1}^{n} \frac{1}{u'_j}$, or, in an equivalent form (by (2))

$$\frac{1}{r_0} + \frac{1}{r'_{n+1}} - \frac{1}{2} + \sum_{\{j \mid r'_j < u'_j\}}^{n} \left(\frac{1}{r'_j} - \frac{1}{u'_j} \right) \leq \sum_{\{j \mid r'_j \geq u'_j\}}^{n} \left(\frac{1}{u'_j} - \frac{1}{r'_j} \right).$$

By (2) we have $r'_j \geq 2, 1 \leq j \leq n$. Defining $\frac{1}{s_0} := \frac{1}{r_0} + \frac{1}{r'_{n+1}} - \frac{1}{2}$, $s'_j := r'_j, 1 \leq j \leq n$ and $s_{n+1} := 2$ we obtain an admissible $(n+2)$-pla $\mathbf{s} = (s_j)_{j=0}^{n+1}$ such that, $\ell^{u_{n+1}}$ having cotype 2, by corollary 4 one has $\big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{s}}\big) \approx \big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\big)$. Then $\big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{s}}\big)$ is not reflexive by the case $N2$), obtaining the desired conclusion by isomorphism.

Case N4). Assume there are $1 \leq j_0 \leq n$ and $1 \leq j_1 \neq j_0 \leq n+1$ such that $u'_{j_0} < 2, r'_{j_0} < 2$ and $r_{j_1} \leq u_{j_1}$.

a) First we consider the case that $n \geq 2$. By (2) necessarily exist $1 \leq j_2 \neq j_3 \leq n+1$ such that $r'_{j_2} \geq 2$ and $r'_{j_3} \geq 2$ and so, by corollary 6 and eventually switching to an isomorphic tensor product $\big(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{s}}\big)$, we can suppose moreover, that $r_0 < \infty$.

After the transposition $j_1 \longrightarrow n+1, n+1 \longrightarrow j_1$ if necessary we can assume that $j_1 = n+1$, i. e. $r_{n+1} \leq u_{n+1}$ indeed. If there exists $1 \leq j_4 \neq j_0 \leq n+1$ such that $u'_{j_4} \leq 2$, the result follows from case $N1$). Hence we can assume $u'_j > 2$ for every $1 \leq j \neq j_0 \leq n+1$.

Fix $t < 2$ such that $r'_{j_0} < t$, $u'_{j_0} < t$ and $u_{n+1} < t$. Let $\{\varphi_k\}_{k=1}^{\infty}$ be a sequence of standard independent identically distributed t-stable random variables in $[0,1]$. It is known that the norm $K_{t,p} := \left\|\varphi_k\right\|_{L^p([0,1])}$, $k \in \mathbb{N}$ is only dependent on t and p for every $1 \leq p < 2$ and that $\left\{\Phi_{k,p} := \frac{\varphi_k}{K_{t,p}}\right\}_{k=1}^{\infty}$ is isometrically equivalent in $L^p([0,1])$, $1 \leq p < t$ to the canonical basis of ℓ^t (see $\big[$ [6], proposition IV.4.10 $\big]$ for example). Then $\{\Phi_{k,r_{n+1}}\}_{k=1}^{\infty}$ is a normalized basis in the reflexive subspace $\left[\Phi_{k,r_{n+1}}\right]_{k=1}^{\infty} \approx \ell^t$ of $L^{r_{n+1}}([0,1])$ and thus it is weakly convergent to 0 in $L^{r_{n+1}}([0,1])$ (see $\big[$ [7], footnote page 169 $\big]$ for instance). Switching to a suitable subsequence if necessary, by $\big[$[18], chapter III, theorem 1.8 $\big]$, the sequence $\{\Phi_k, r_{n+1}\}_{k=1}^{\infty}$ can be enlarged to obtain a normalized basis $\mathcal{B} := \{\Phi_{k,r_{n+1}}\}_{k=1}^{\infty} \cup \{\Psi_m\}_{m=1}^{\infty}$ in $L^{r_{n+1}}([0,1])$. By reflexivity the sequence $\{\Phi^*_{k,r_{n+1}}\}_{k=1}^{\infty} \cup \{\Psi^*_m\}_{m=1}^{\infty}$ of associated coefficient functionals to $\mathcal{B}$ is a basis in $L^{r'_{n+1}}([0,1])$. From $\big[$ [18], chapter I, theorem 3.1 $\big]$ we find $1 \leq M \in \mathbb{R}$ such that $1 \leq \left\|\Phi^*_{k,r_{n+1}}\right\| \leq M$ and $1 \leq \left\|\Psi^*_k\right\| \leq M$ for every $k \in \mathbb{N}$. As above we obtain that $\{\Phi^*_{k,r_{n+1}}\}_{k=1}^{\infty}$ must be weakly convergent to 0. As $r'_{n+1} > 2$, by the result $\big[$ [7], corollary 5$\big]$ of Kadec and Pełcińsky, switching to a subsequence again, it can be assumed that $\{\Phi^*_{k,r_{n+1}}\}_{k=1}^{\infty}$ is equivalent to the standard unit basis in $\ell^{r'_{n+1}}$ or to the standard unit basis in ℓ^2. By $\big[$ [7], corollary 1 $\big]$, the latter possibility would imply that $\left[\Phi^*_{k,r_{n+1}}\right]_{k=1}^{\infty}$ would be complemented in $L^{r'_{n+1}}([0,1])$ and by reflexivity and duality, we would have the isomorphisms $\left(\left[\Phi^*_{k,r_{n+1}}\right]_{k=1}^{\infty}\right)' \approx \left[\Phi_{k,r_{n+1}}\right]_{k=1}^{\infty} \approx \ell^t \approx \ell^2$ which is not possible. Then $\left\{\Phi^*_{k,r_{n+1}}\right\}_{k=1}^{\infty}$ is equivalent to the standard basis of $\ell^{r'_{n+1}}$ and so, the map $V \in \mathcal{L}\left(\ell^{u'_{n+1}}, L^{r'_{n+1}}([0,1])\right)$ such that $V(\mathbf{e}_h)) = \Phi^*_{h,r_{n+1}}$, $h \in \mathbb{N}$ is well defined.

Let $S_j \in \mathcal{L}\left(\ell^{u'_j}, L^{r'_j}([0,1])\right)$, $1 \leq j \neq j_0 \leq n$ be defined as in previous case $N1$) and consider $S_{j_0} \in \mathcal{L}\left(\ell^{u'_{j_0}}, L^{r'_{j_0}}([0,1])\right)$ such that $S_{j_0}(\mathbf{e}_k) = \Phi_{k,r'_{j_0}}$ for every $k \in \mathbb{N}$. Taking g as in case $N1$), the map $T := V' \circ D_g \circ \left((S_j)\right)_{j=1}^{n}$ is $\mathbf{r}$-integral. However, for every $k \in \mathbb{N}$ and every $(\gamma_h) \in \ell^{u'_{n+1}}$ we have

$$\left\langle T(z_{j_0,k}), (\gamma_h)\right\rangle = \left\langle \frac{K_{t,r_{n+1}}}{K_{t,r'_{j_0}}}\ \Phi_{k,r_{n+1}}, \sum_{h=1}^{\infty} \gamma_h \Phi^*_{h,r_{n+1}}\right\rangle = \frac{K_{t,r_{n+1}}}{K_{t,r'_{j_0}}}\ \gamma_k$$

and so $T(z_{j_0,k}) = \frac{K_{t,r_{n+1}}}{K_{t,r'_{j_0}}}\ \mathbf{e}_k$ and T is not compact. By remark after theorem 9 we obtain $T \notin \mathfrak{N}_{\mathbf{r}}\left(\prod_{j=1}^{n} \ell^{u'_j}, \ell^{u_{n+1}}\right)$ and by theorem 15 $\left(\widehat{\bigotimes}_{j=1}^{n+1} \ell^{u_j}, \alpha_{\mathbf{r}}\right)$ is not reflexive.

b) Now we consider the case $n=1$. If $r_0 \neq \infty$ the previous argumentation can be used still and $\ell^{u_1}\widehat{\bigotimes}_{\alpha_{\mathbf{r}}}\ell^{u_2}$ is not reflexive. If $r_0=\infty$, after an eventual transposition, we will be dealing with the case $u_1' \leq 2, r_1' < 2$ and $r_2 \leq u_2$. If $u_2 \geq 2$ the result follows from $N1$). If $u_2 < 2$ and $u_1' = 2$ we repeat the proof given in this case for $n \geq 2$ and $\ell^{u_1}\widehat{\bigotimes}_{\alpha_{\mathbf{r}}}\ell^{u_2}$ turns out to be non reflexive. If $u_2 < 2$ and $u_1' < 2$ the same construction just used in the case $n \geq 2$ show the existence of a map $T \in \mathfrak{I}_{\mathbf{r}}(\ell^{u_1'}, \ell^{u_2})$ such that $T(\mathbf{e}_i) = \frac{K_{t,r_2}}{K_{t,r_1'}}\,\mathbf{e}_i$ for every $i \in \mathbb{N}$. Then we can repeat the argumentation used in the last part of N1) with the set $K := \{\mathbf{e}_i \otimes \mathbf{e}_i,\ i \in \mathbb{N}\} \subset \ell^{u_1'} \bigotimes \ell^{u_2'}$ to conclude that $\left(\widehat{\bigotimes}_{j=1}^{n+1}\ell^{u_j}, \alpha_{\mathbf{r}}\right)$ is not reflexive.

Finally we check that the proof of theorem 17 is complete. Assume that neither condition $S1$), $S2$), $S3$), $S4$) holds.

a) First case: assume there is $1 \leq j_0 \leq n+1$ such that $u_{j_0}' \leq 2$. After an eventual transposition with any $1 \leq k \neq j_0 \leq n+1$, we can take $j_0 \leq n$. If there is some $1 \leq j_1 \neq j_0 \leq n+1$ such that $u_{j_1}' \leq 2$, by $N1$), $\left(\widehat{\bigotimes}_{j=1}^{n+1}\ell^{u_j}, \alpha_{\mathbf{r}}\right)$ is not reflexive. Then we can assume $u_j' > 2,\ 1 \leq j \neq j_0 \leq n+1$. As $S1$) does not holds, there exists $j_1 \neq j_0$ such that $r_{j_1} \leq u_{j_1}$. If it would be $u_{j_0}' < 2$ and $r_{j_0}' < 2$, $\left(\widehat{\bigotimes}_{j=1}^{n+1}\ell^{u_j}, \alpha_{\mathbf{r}}\right)$ would be not reflexive by $N4$). If $u_{j_0}' = 2$ and $r_{j_0}' < 2$, as $S4$) does not holds, after the transposition $j_0 \to n+1, n+1 \to j_0$, by $N3$) $\left(\widehat{\bigotimes}_{j=1}^{n+1}\ell^{u_j}, \alpha_{\mathbf{r}}\right)$ is not reflexive.

In the case $r_{j_0}' \geq 2$, by (2) there is at most an unique $1 \leq j_2 \leq n+1$ such that $r_{j_2}' < 2$. Necessarily $j_2 \neq j_0$. As $S2$) does not holds, after an eventual transposition $j_0 \to n+1, n+1 \to j_0$, we see that $u_{n+1}' \leq 2 \leq r_{n+1}'$ and by $N2$) $\left(\widehat{\bigotimes}_{j=1}^{n+1}\ell^{u_j}, \alpha_{\mathbf{r}}\right)$ is not reflexive.

b) Second case: assume that $u_j' > 2, 1 \leq j \leq n+1$. As $S1$) does not holds, after an eventual transposition, it turns out that $u_{n+1}' \leq r_{n+1}'$. But $S3$) is not verified. Then for every $1 \leq j_0 \leq n+1$ we have $r_{j_0}' > 2$ or (38) does not holds. If it would be $r_j' > 2$ for every $1 \leq j \leq n+1$, as $S2$) is not verified, $\left(\widehat{\bigotimes}_{j=1}^{n+1}\ell^{u_j}, \alpha_{\mathbf{r}}\right)$ would be not reflexive by $N3$). If it would exists $1 \leq j_1 \leq n+1$ such that $r_{j_1}' \leq 2$, then (38) would fails for this index j_1. After an evident transposition, by $N3$) $\left(\widehat{\bigotimes}_{j=1}^{n+1}\ell^{u_j}, \alpha_{\mathbf{r}}\right)$ would be not reflexive. ∎

The application of theorem 17 to the case $n=1$ gives the following characterization of reflexivity of classical Lapresté's tensor products:

Corollary 18 *Let $n=1$ and let $\mathbf{r}=(r_0,r_1,r_2)$ be an admissible triple. If $1 < u_1, u_2 < \infty$, $\ell^{u_1}\widehat{\bigotimes}_{\alpha_{\mathbf{r}}}\ell^{u_2}$ is reflexive if and only if one of the following sets of conditions holds*

1) $u_1' > 2, u_1' > r_1'$.
2) $u_2' > 2, u_2' > r_2'$.

3) $u_1' > 2, r_2 \leq 2$.
4) $u_2' > 2, r_1 \leq 2$.
5) $u_1' \geq 2, u_2' > 2$.
6) $u_1' > 2, u_2' \geq 2$.

Proof. By theorem 17, $\ell^{u_1}\widehat{\bigotimes}_{\alpha_{\mathbf{r}}}\ell^{u_2}$ is reflexive if and only if one of the following sets of conditions holds
$a)$ $u_1' > 2, u_1' > r_1'$.
$b)$ $u_2' > 2, u_2' > r_2'$.
$c)$ $u_1' > 2, u_1' > r_2, r_1' \geq 2$.
$d)$ $u_2' > 2, u_2' > r_1, r_2' \geq 2$.
$e)$ $u_1' > 2, u_2' > 2, r_1' \leq 2$.
$f)$ $u_1' > 2, u_2' > 2, r_2' \leq 2$.
$g)$ $u_1' = 2, u_2' > 2, r_1' \leq 2$.
$h)$ $u_2' = 2, u_1' > 2, r_2' \leq 2$.
Clearly $c)$ and 3) (resp. $d)$ and 4)) are equivalent. On the other hand, if 5) holds and $r_1' \leq 2$ then $e)$ or $g)$ holds. If 5) and $r_1' > 2$ are true we have $r_1 < 2 < u_2'$ and $d)$ is verified. The remaining of the proof is similar or trivial. ■

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