(n + 1)-tensor norms of Lapresté’s type

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Abstract

We study an (n + 1)-tensor norm $\alpha_r$ extending to (n + 1)-fold tensor
products the classical one of Lapresté in the case $n = 1$. We characterize the
maps of the minimal and the maximal multilinear operator ideals related to
$\alpha_r$ in the sense of Defant and Floret. As an application we give a complete
description of the reflexivity of the $\alpha_r$-tensor product ($\otimes_{j=1}^{n+1} E_j$, $\alpha_r$).

1 Introduction

In [14] Pietsch proposed building a systematic theory of ideals of multilinear
mappings between Banach spaces, similar to the already well-developed one regarding linear maps, as a first step to study ideals of more general non linear operators. Since then several classes of multilinear operators more or less related to classical absolutely $p$-summing operators has been studied although without to deal with aspects derived from a general organized theory.

Having in mind the close connection existing in linear case between problems of this kind and tensor products (see [2] for a systematic survey of the actual state of the art), in the present setting it is expected an analogous connection with multiple tensor products. However a systematic study of this approach has not been initiated until the works [4] and [5] of Floret, mainly motivated by the potential applications of the new theory to infinite holomorphy. In this way, classical notions of maximal operator ideals and its associated $\alpha$-tensor norm, dual tensor norm $\alpha'$ and the related $\alpha$-nuclear and $\alpha$-integral operators can be extended to the framework of multilinear operator ideals and multiple tensor products.

However, there are few concrete examples of multi-tensor norms to whose the general concepts of the theory have been applied and checked. The purpose of this paper is to study an (n + 1)-tensor norm $\alpha_r$ on tensor products $\bigotimes_{j=1}^{n+1} E_j$, 1 ≤ n,
of \( n + 1 \) Banach spaces \( E_j \), extending the classical one of Lapresté for \( n = 1 \), as well its associated \( \alpha_r \)-nuclear and \( \alpha_r \)-integral multilinear operators. Knowledge of such operators allows us to characterize the reflexivity of the corresponding tensor product \( \bigotimes_{j=1}^{n+1} \ell^u_j, \alpha_r \) of spaces \( \ell^u_j \).

The paper is organized as follows. First we introduce the notation and some general facts to be used. In section 2 we define the \((n+1)\)-fold tensor product \( \bigotimes_{n} \alpha_r \left( E_1, E_2, \ldots, E_n, F \right) \), \( n \in \mathbb{N} \) of type \( \alpha_r \) of Banach spaces \( E_j \), \( 1 \leq j \leq n \) and \( F \). We find its topological dual introducing the so called \( r \)-dominated maps and we obtain multilinear extensions of the classical theorems of Grothendieck-Pietsch and Kwapien (theorem 3). The latter one is the key to approximate \( r \)-dominated maps by multilinear maps of finite rank in many usual cases (theorem 7) and to compare different tensor norms \( \alpha_r \), a tool which will be very useful in our applications in the final section of the paper.

The elements of a completed \( \alpha_r \)-tensor product canonically lead to multilinear \( r \)-nuclear operators from \( \prod_{j=1}^{n} E_j \) into \( F \), which are considered in section 3 and characterized by means of suitable factorizations in theorem 9. According the pattern of the general theory of multi-tensor norms, the next step must be the study of the so called \( r \)-integral multilinear maps, i.e. the maps in the ideal associated to the \( \alpha_r \)-tensor norm in the sense of Defant-Floret [2]. To do this we need a technical result about the structure of some ultraproducts which follows easily from the work of Raynaud [15]. It will be presented in section 4 just before its use.

In section 4 we characterize the \( r \)-integral operators, obtaining as main result the "continuous" version of the previous factorizations of \( r \)-nuclear operators. Finally in section 5 we apply the characterizations of sections 3 and 4 to study the reflexivity of \( \alpha_r \)-tensor products and, more particularly, to characterize the reflexivity of \( \alpha_r \)-tensor products of \( \ell^u \) spaces, a result that, as far as we know, is new indeed for classical Lapresté’s tensor norms.

We shall deal always with vector spaces defined over the field \( \mathbb{R} \) of real numbers. Notation of the paper is standard in general. Some not so usual notations are settled now.

Given a normed space \( E \), we shall denote by \( B_E \) its closed unit ball and \( J_E : E \to E'' \) will be the canonical isometric inclusion of \( E \) into the bidual space \( E'' \). \( B_{E'} \) will be considered as a compact topological space \( (B_{E'}, \sigma(E', E)) \) when provided with the topology induced by the weak*-topology \( \sigma(E', E) \). For every \( x \in E \), we shall denote by \( f_x \) the continuous function defined on \( (B_{E'}, \sigma(E', E)) \) as \( f_x(x') = \langle x, x' \rangle \) for every \( x' \in B_{E'} \). The symbol \( E \approx F \) will mean that \( E \) and \( F \) are isomorphic normed spaces. The closed linear span in a Banach space \( E \) of a sequence \( \{x_m\}_{m=1}^{\infty} \subset E \) (respectively of a single vector \( x \)) will be represented by \( \left[ x_n \right]_{n=1}^{\infty} \) (resp. \( [x] \)).

As usual, \( e_k \) denotes the \( k \)-th standard unit vector in every \( \ell^p \), \( 1 \leq p \leq \infty \).
\(\ell^p_h\), \(h \in \mathbb{N}\) will be the \(\ell^p\)-space defined over the set \(\{1, 2, \ldots, h\}\) with the standard measure.

Given a normed space \(E\), a sequence \(\{x_m\}_{m=1}^k \subset E\), \(k \in \mathbb{N} \cup \{\infty\}\), and \(1 \leq p \leq \infty\), we define in the case \(p < \infty\)

\[
\pi_p\left((x_m)_{m=1}^k\right) := \left(\sum_{m=1}^k \|x_m\|^p\right)^{\frac{1}{p}}, \quad \varepsilon_p\left((x_m)_{m=1}^k\right) := \sup_{x' \in B_{E'}} \left(\sum_{m=1}^k \langle x_m, x'\rangle^p\right)^{\frac{1}{p}}
\]

and when \(p = \infty\)

\[
\pi_\infty\left((x_m)_{m=1}^k\right) := \varepsilon_\infty\left((x_m)_{m=1}^k\right) = \sup_{1 \leq m \leq k} \|x_m\|.
\]

A sequence \(\{x_m\}_{m=1}^\infty \subset E\) is called weakly \(p\)-absolutely summable, notation \((x_m)_{m=1}^\infty \in \ell^p(E)\), (resp. \(p\)-absolutely summable), if \(\varepsilon_p((x_m)_{m=1}^\infty) < \infty\) (resp. \(\pi_p((x_m)_{m=1}^\infty) < \infty\)).

Given Banach spaces \(E\) and \(F\), an operator or linear map \(T \in \mathcal{L}(E, F)\) is said to be \(p\)-absolutely summing if there exists \(C \geq 0\) such that

\[
(x_m)_{m=1}^\infty \in \ell^p(E) \implies \pi_p\left((T(x_m))_{m=1}^\infty\right) \leq C \varepsilon_p((x_m)_{m=1}^\infty).
\]

The linear space \(\Psi_p(E, F)\) of all \(p\)-absolutely summing operators from \(E\) into \(F\) becomes a Banach space under the norm \(\mathbf{P}_p(T) := \inf\{C \geq 0 \mid (1)\ \text{holds}\}\) for every \(T \in \Psi_p(E, F)\).

We consider always a finite cartesian product \(\prod_{m=1}^h E_m\) of normed spaces \(E_m, 1 \leq m \leq h \in \mathbb{N}\) as a normed space provided with the \(\ell^\infty\)-norm \(\|(x_m)_{m=1}^h\| = \sup_{m=1}^h \|x_m\|\).

If \(F\) is a Banach space we shall denote by \(\mathcal{L}^h(\prod_{m=1}^h E_m, F)\) the Banach space of all \(h\)-linear continuous maps from \(\prod_{m=1}^h E_m\) into \(F\). Given \(T \in \mathcal{L}^h(\prod_{m=1}^h E_m, F)\) we can define in a natural way the transposed linear map \(T' : F' \to \mathcal{L}^h(\prod_{m=1}^h E_m, \mathbb{R})\)

\[
\forall y' \in F' \quad \forall (x_m)_{m=1}^h \in \prod_{m=1}^h E_m \quad \left\langle T'(y'), (x_m)_{m=1}^h \right\rangle = \left\langle T((x_m)_{m=1}^h), y' \right\rangle.
\]

Given maps \(A_j \in \mathcal{L}(E_j, F_j)\) between normed spaces \(E_j\) and \(F_j, 1 \leq j \leq n\) we write

\[
(A_j)_{j=1}^n := (A_1, A_2, \ldots, A_n) : \prod_{j=1}^n E_j \to \prod_{j=1}^n F_j
\]

to denote the continuous linear map defined by

\[
\forall (x_j)_{j=1}^n \in \prod_{j=1}^n E_j \quad (A_j)_{j=1}^n((x_1, x_2, \ldots, x_n)) = (A_1(x_1), A_2(x_2), \ldots, A_n(x_n)).
\]
Some times we will write \((A_j)\) instead of \((A_j)_{j=1}^n\). Concerning \((n+1)\)-tensor norms, \(n \geq 1\) (or multi-tensor norms) we refer the reader to the pioneer works \([4]\) and \([5]\). If it is needed to emphasize, \(\alpha\left( z; \bigotimes_{j=1}^{n+1} M_j \right)\) or similar notations will denote the value of the multi-tensor norm \(\alpha\) of \(z \in \bigotimes_{j=1}^{n+1} M_j\).

As customary, for \(p \in [1, \infty]\), \(p'\) will be the conjugate extended real number such that \(1/p + 1/p' = 1\). Given \(n \geq 1\), in all the paper we denote by \(\mathbf{r}\) an \((n+2)\)-pla of extended real numbers \(\mathbf{r} = (r_0, r_1, r_2, \ldots, r_n, r_{n+1})\) such that \(1 < r_0 \leq \infty\), \(1 < r_j < \infty\), \(1 \leq j \leq n+1\), and

\[
1 = \frac{1}{r_0} + \frac{1}{r_1'} + \frac{1}{r_2'} + \ldots + \frac{1}{r_{n+1}'}. \tag{2}
\]

Such \(\mathbf{r}\) will be called an admissible \((n+2)\)-pla. Moreover, we define \(w\) such that

\[
\frac{1}{w} := \frac{1}{r_1'} + \frac{1}{r_2'} + \ldots + \frac{1}{r_n'}. \tag{3}
\]

which gives the equality

\[
n = \frac{1}{w} + \sum_{j=1}^n \frac{1}{r_j}. \tag{4}
\]

For later use we note that (2) implies

\[
1 = \frac{r_0}{r_1'} + \frac{r_0'}{r_2'} + \ldots + \frac{r_0'}{r_{n+1}'} \quad \text{and} \quad \frac{1}{r_{n+1}} = \frac{1}{r_0} + \frac{1}{r_1'} + \frac{1}{r_2'} + \ldots + \frac{1}{r_n'} \tag{5}
\]

as well

\[
\frac{1}{w} = \frac{1}{r_0} - \frac{1}{r_{n+1}} = \frac{1}{r_{n+1}} - \frac{1}{r_0} \quad \implies \quad 1 = \frac{1}{w} + \frac{1}{r_0}, \quad \frac{1}{r_{n+1}}. \tag{6}
\]

and moreover,

\[
\forall 1 \leq j \leq n \quad r_{n+1} < w < r_j', \tag{7}
\]

and

\[
\forall 1 \leq j \leq n+1 \quad r_j < r_0. \tag{8}
\]

To finish this introduction we consider the following construction which will be of fundamental importance in all the paper. Given any measure space \((\Omega, \mathcal{A}, \mu)\) and an admissible \((n+2)\)-pla \(\mathbf{r}\), as a direct consequence of generalized Hölder’s inequality and (2), we have a canonical \((n+1)\)-linear map \(\mathfrak{M}_\mu : L^{r_0}(\Omega, \mathcal{A}, \mu) \times \prod_{j=1}^n L^{r_j}(\Omega, \mathcal{A}, \mu) \rightarrow L^{r_{n+1}}(\Omega, \mathcal{A}, \mu)\) defined by the rule

\[
\forall (f_j)_{j=0}^n \in L^{r_0}(\Omega, \mu) \times \prod_{j=1}^n L^{r_j}(\Omega, \mu) \quad \mathfrak{M}_\mu((f_j)) = \prod_{j=0}^n f_j
\]
verifying \( \| M_n(f_j) \| \leq \| g \|_{L^0(\Omega)} \prod_{j=1}^n \| f_j \|_{L^j(\Omega)} \). If \((\Omega, \mathcal{A}, \mu)\) is \(\mathbb{N}\) with the counting measure we will write simply \( M \) instead of \( M_n \). Moreover, given \( g \in L^0(\Omega,\mu) \) we shall write \( D_g \) to denote the \( n \)-linear map from \( \prod_{j=1}^n L^j(\Omega,\mu) \) into \( L^{n+1}(\Omega,\mu) \) such that

\[
\forall \ (f_j)_{j=1}^n \in \prod_{j=1}^n L^j(\Omega,\mu) \quad D_g((f_j)_{j=1}^n) = M_n((g,f_1,\ldots,f_n)). \tag{9}
\]

It will be important for later applications to remark that \( M_n \) induces a linearization map \( \widetilde{M}_n : (L^0(\Omega,\mu) \hat{\otimes} (\bigotimes_{j=1}^n L^j(\Omega,\mu)), \pi) \rightarrow L^{n+1}(\Omega,\mu) \) and a canonical map \( \widetilde{M}_n : (L^0(\Omega,\mu) \hat{\otimes} (\bigotimes_{j=1}^n L^j(\Omega,\mu)), \pi)/\text{Ker}(M_n) \rightarrow L^{n+1}(\Omega,\mu) \)

such that \( \| \widetilde{M}_n \| \leq 1 \). Moreover, by (5) we obtain \( f = f^0 \prod_{j=1}^n f^j \) for every \( f \geq 0 \) in \( L^{n+1}(\Omega,\mu) \). As \( f = f^+ - f^- \) for every \( f \in L^{n+1}(\Omega,\mu) \) it turns out that \( M_n \) is a surjective map and \( \widetilde{M}_n \) becomes an isomorphism such that \( \| \widetilde{M}_n^{-1} \| \leq 2 \).

2 \( \alpha_r \)-tensor products and \( r \)-dominated multilinear maps

Let \( E_j, 1 \leq j \leq n+1 \) be normed spaces. Using classical methods we can show that

\[
\alpha_r \left( z ; \bigotimes_{j=1}^{n+1} E_j \right) := \inf \pi_{r_0} \left( (\lambda_m)_{m=1}^{n+1} \prod_{j=1}^{n+1} \varepsilon_{r_j}^h \right), \tag{10}
\]

taking the infimum over all representations of \( z \) of type

\[
z = \sum_{m=1}^h \lambda_m \left( \bigotimes_{j=1}^{n+1} x_{jm} \right) , \quad x_{jm} \in E_j \quad 1 \leq j \leq n+1, \quad 1 \leq m \leq h, \quad h \in \mathbb{N},
\]

is a norm on \( \bigotimes_{j=1}^{n+1} E_j \) which defines an \((n+1)\)-tensor norm in the class of normed spaces. It is interesting to note that if \( n = 1 \) we obtain the classical tensor norm \( \alpha_{r_2r_1} \) of Lapresté (see [2] for details).

The just defined normed tensor product space will be denoted by \( \bigotimes_{j=1}^{n+1} E_j, \alpha_r \) or \( \bigotimes_{\alpha_r}(E_1, E_2, \ldots, E_{n+1}) \) and its completion by \( \widehat{\bigotimes}_{\alpha_r}(E_1, E_2, \ldots, E_{n+1}) \). It is clear that for every permutation \( \sigma \) on the set \( \{1,2,\ldots,n+1\} \) the map

\[
I_\sigma : \sum_{i=1}^m \lambda_m \otimes_{j=1}^{n+1} x_{jm} \in \left( \bigotimes_{j=1}^{n+1} E_j, \alpha_r \right) \rightarrow \sum_{i=1}^m \lambda_m \otimes_{j=1}^{n+1} x_{\sigma(j)m} \in \left( \bigotimes_{j=1}^{n+1} E_{\sigma(j)}, \alpha_\sigma \right),
\]

5
where \( s \) is the admissible \((n+2)\)-pla \( s_0 := r_0 \) and \( s_j = r_{\sigma(j)} \), \( 1 \leq j \leq n + 1 \), is an isometry from \((\bigotimes_{j=1}^{n+1} E_j, \alpha_r)\) onto \((\bigotimes_{j=1}^{n+1} E_j, \alpha_s)\). We shall use this type of isomorphism in section 5 in the particular case of transpositions \( \sigma \) simply indicating the transposed indexes \( \sigma(j_0) = j_1, \sigma(j_1) = j_0 \) in the way \( j_0 \to j_1, j_1 \to j_0 \).

To compute the topological dual of an \( \alpha_r \)-tensor product we set a new definition:

**Definition 1** Let \( F \) and \( E_j, 1 \leq j \leq n \) be normed spaces. A map \( T \in \mathcal{L}^n(\prod_{j=1}^n E_j, F) \) is said to be \( r \)-dominated if there is \( C \geq 0 \) such that for every \( h \in \mathbb{N} \) and every set of finite sequences \( \{x_{jk}\}_{k=1}^h \subset E_j, 1 \leq j \leq n \) and \( \{y_k\}_{k=1}^h \subset F' \) the inequality

\[
\pi_{r_0} \left( \left( \left\langle \left( T(x_{1k}, x_{2k}, \ldots, x_{nk}), y_k \right) \right\rangle \right)_{k=1}^m \right) \leq C \left( \prod_{j=1}^n \varepsilon_{r_j} \left( \left( x_{jk} \right)_{k=1}^m \right) \right) \varepsilon_{r'_{n+1}} \left( \left( y_k \right)_{k=1}^h \right)
\]

(11)

holds.

It is easy to see that the linear space \( \mathfrak{P}_r(\prod_{j=1}^n E_j, F) \) of \( r \)-dominated \( n \)-linear maps from \( \prod_{j=1}^n E_j \) into \( F \) is normed setting \( \mathcal{P}_r(T) := \inf \left\{ C \geq 0 \mid (11) \text{ holds} \right\} \) for every \( T \in \mathfrak{P}_r(\prod_{j=1}^n E_j, F) \), becoming a Banach space when \( F \) does. The interest on \( r \)-dominated multilinear maps follows from the next result:

**Theorem 2** \( \left( \bigotimes_{\alpha_r} (E_1, E_2, \ldots, E_n, F) \right)' = \mathfrak{P}_r(\prod_{j=1}^n E_j, F') \) for all normed spaces \( F \) and \( E_j, 1 \leq j \leq n \).

**Proof.**

1. Given \( T \in \mathfrak{P}_r(\prod_{j=1}^n E_j, F') \) and \( z = \sum_{k=1}^h \lambda_k (\bigotimes_{j=1}^n x_{jk}) \otimes y_k \) in \( \left( \bigotimes_{j=1}^n E_j \right) \otimes F \) we define \( \varphi_r(z) = \sum_{k=1}^h \lambda_k \left( T\left( (x_{1k}, x_{2k}, \ldots, x_{nk}), y_k \right) \right) \). It follows directly from Hölder’s inequality, definition 1 and (10)

\[
|\varphi_r(z)| \leq \mathcal{P}_r(T) \alpha_r(z) \implies \|\varphi_r\| \leq \mathcal{P}_r(T).
\]

(12)

2) Conversely, let \( \psi \in \left( \bigotimes_{\alpha_r} (E_1, E_2, \ldots, E_n, F) \right)' \). We define \( T_\psi \in \mathcal{L}^n(\prod_{j=1}^n E_j, F') \) as

\[
\forall (x_j)_{j=1}^n \in \prod_{j=1}^n E_j, \forall y \in F, \left\langle T_\psi \left( (x_j)_{j=1}^n \right), y \right\rangle = \psi \left( x_1 \otimes x_2 \otimes \ldots x_n \otimes y \right).
\]

Given \( \{x_{jk}\}_{k=1}^h \subset E_j, 1 \leq j \leq n \) and \( \{y_k\}_{k=1}^h \subset F, h \in \mathbb{N} \) we have

\[
\pi_{r_0} \left( \left( \left\langle \left( T_\psi \left( (x_{jk})_{j=1}^n \right), y_k \right) \right\rangle \right)_{k=1}^h \right) = \sup_{(\alpha_k) \in B_{r_0}^h} \left| \sum_{k=1}^h \alpha_k \psi \left( \left( \bigotimes_{j=1}^n x_{jk} \right) \otimes y_k \right) \right| = \frac{1}{C} \left| \sum_{k=1}^h \alpha_k \psi \left( \left( \bigotimes_{j=1}^n x_{jk} \right) \otimes y_k \right) \right|
\]
Given Banach spaces \( \mathbb{B} \)

\[
\text{Problem 3:}
\]

Moreover, the proof of that result allow us to obtain \( \inf \{ C \geq 0 \mid (13) \text{ holds} \} \leq \mathcal{P}_r(T) \).
Let \( \mu_j, 1 \leq j \leq n \) and \( \nu \) be probability Radon measures in the unit balls \( B_{E'_j} \) and \( B_{F'} \), respectively (with corresponding \( \sigma \)-algebras \( B_j \) and \( B_{n+1} \) of measurable sets) such that (13) holds.

Put \( \Omega := \prod_{j=1}^{n} B_{E'_j} \) provided with the product measure \( \mu := \otimes_{j=1}^{n} \mu_j \) and its corresponding \( \sigma \)-algebra \( \mathcal{B} \) of measurable sets. For every \( x_j \in E_j, \ 1 \leq j \leq n \), we define the map \( G_{x_j} : \Omega \rightarrow \mathbb{R} \) given by \( G_{x_j}(\mathbf{x}') = \langle x_j, x'_j \rangle \) for every \( \mathbf{x}' = (x'_1, x'_2, \ldots, x'_n) \in \Omega \). Clearly, as a consequence of Fubini’s theorem, we have \( G_{x_j} \in L^{\ell'_j}(\Omega, \mathcal{B}, \mu) \) and moreover, for each \( y' \in F' \) the inequality

\[
\left| \left\langle T\left( (x_j)_{j=1}^{n}, y' \right) \right\rangle \right| \leq C \left\| f_{y'} \right\|_{L^{\ell'_{n+1}}(B_{F'}, \mathcal{B}_{n+1}, \nu)} \prod_{j=1}^{n} \left\| G_{x_j} \right\|_{L^{\ell'_j}(\Omega, \mathcal{B}, \mu)}
\]

holds still.

Define \( A_j \in \mathcal{L}(E_j, L^{\ell'_j}(\Omega, \mathcal{B}, \mu)) \), as \( A_j(x_j) = G_{x_j} \) for every \( x_j \in E_j \) and \( M_j := A_j(E_j) \), taking the closure in \( L^{\ell'_j}(\Omega, \mathcal{B}, \mu) \) and providing it with the induced topology. It is easy to check (classical Pietsch-Grothendieck’s domination theorem) that

\[
\forall 1 \leq j \leq n \quad A_j \in \mathfrak{P}_{\ell'_j}(E_j, M_j) \quad \text{and} \quad P_{\ell'_j}(A_j) \leq 1.
\]

Now we define the multilinear map \( S : \prod_{j=1}^{n} A_j(E_j) \rightarrow F \) as

\[
\forall (x_j)_{j=1}^{n} \in \prod_{j=1}^{n} E_j \quad S((G_{x_j})_{j=1}^{n}) = T\left( (x_j)_{j=1}^{n} \right).
\]

\( S \) is well defined because \( (G_{x_j})_{j=1}^{n} = (G_{\mathbf{\tau}})_{j=1}^{n} \) implies \( G_{x_j} = G_{\mathbf{\tau}} \in L^{\ell'_j}(\Omega, \mathcal{B}, \mu), 1 \leq j \leq n \) and

\[
T\left( (x_j)_{j=1}^{n} \right) - T\left( (\mathbf{\tau})_{j=1}^{n} \right) = \sum_{j=1}^{n} T(\mathbf{\tau}_1, \ldots, \mathbf{\tau}_{j-1}, x_j - \mathbf{\tau}_j, x_{j+1}, \ldots, x_n)
\]

and by (15) we obtain \( \|T\left( (x_j)_{j=1}^{n} \right) - T\left( (\mathbf{\tau})_{j=1}^{n} \right)\| = 0 \). (15) gives too the continuity of \( S \) and hence it can be continuously extended to a map (still denoted by \( S \)) in \( L^n\left(\prod_{j=1}^{n} M_j, F \right) \). To finish the proof we only need to see that \( S' \in \mathfrak{P}_{\ell'_n+1}(F', L^n\left(\prod_{j=1}^{n} M_j, \mathbb{R} \right)) \).

Given \( \{y'_k\}_{k=1}^{h} \subset F', h \in \mathbb{N} \), fix a finite sequence \( \{\alpha_k\}_{k=1}^{h} \) verifying \( \|\alpha_k\|_{\ell'_n+1} = 1 \). For every \( \varepsilon > 0 \), there are \( G_{x_{jk}} \in B_{M_j}, 1 \leq k \leq h, 1 \leq j \leq n \) such that

\[
\forall 1 \leq k \leq h \quad \left\| S'(y'_k) \right\|_{L^n(\prod_{j=1}^{n} M_j, \mathbb{R})} \leq \left| \left\langle S'(y'_k), (G_{x_{jk}})_{j=1}^{n} \right\rangle \right| + \varepsilon |\alpha_k|.
\]
Hence, from Hölder’s inequality and (13) we obtain

\[ \pi_{r_{n+1}}\left(\left(S'(y_k)\right)_{k=1}^h\right) = \sup_{(\beta_k) \in B_{r_h}^{r_{n+1}}} \left| \sum_{k=1}^h \beta_k \left\| S'(y_k) \right\|_{L^\infty(\prod_{j=1}^n M_j, \mathbb{R})} \right| \leq \]

\[ \leq \sup_{(\beta_k) \in B_{r_h}^{r_{n+1}}} \left| \left\langle S'(y_k), (G_{x,j})_{k=1}^n \right\rangle \right| + \epsilon |\alpha_k| \leq \]

\[ \leq \sup_{(\beta_k) \in B_{r_h}^{r_{n+1}}} \left\| (\beta_k)_{k=1}^h \right\|_{L^{r_{n+1}}(B_{r', B_{n+1}, \nu})} \left\| (\alpha_k)_{k=1}^h \right\|_{L^{r_{n+1}}(\Omega, \mathcal{B}, \mu)} \]

\[ \leq C \left( \sum_{k=1}^h \left\| f_{y_k} \right\|_{L^{r_{n+1}}(B_{r', B_{n+1}, \nu})} \prod_{j=1}^n \left\| G_{x,j} \right\|_{L^{r_{n+1}}(\Omega, \mathcal{B}, \mu)} \right)^{\frac{1}{r_{n+1}}} + \epsilon \leq \]

\[ \leq C \left( \sum_{k=1}^h \left( \int_{B_{r'}} \left\| y'_{k} \right\|_{L^{r_{n+1}}(B_{r', B_{n+1}, \nu})} \right)^{r_{n+1}} + \epsilon \right) \]

\[ = C \left( \frac{\varepsilon}{r_{n+1}} \left( y_{k}^{h} \right) \right) + \epsilon = C \varepsilon_{r_{n+1}} \left( y_{k}^{h} \right) + \epsilon = \]

and \( \epsilon > 0 \) being arbitrary, the result follows. Moreover, by (16) and the definition of \( P_{r_{n+1}}(S') \) we obtain

\[ P_{r_{n+1}}(S') \prod_{j=1}^n P_{r_j}(A_j) \leq C. \] (17)

3) \( \implies 1 \). Assume there there are Banach spaces \( M_j \) and maps \( A_j \in \mathfrak{P}_{r_j}(E_j, M_j) \), \( 1 \leq j \leq n \) and \( S \in \mathcal{L}^n(\prod_{j=1}^n M_j, F) \) such that \( S' \in \mathfrak{P}_{r_{n+1}}(F', \mathcal{L}^n(\prod_{j=1}^n M_j, \mathbb{R})) \) and

\[ T = S \circ (A_j)_{j=1}^n. \] Given finite sequences \( \{x_{jk}\}_{k=1}^h \subset E_j \) and \( \{y_{jk}\}_{k=1}^h \subset F', h \in \mathbb{N} \), using (2) and Hölder’s inequality we have

\[ \pi_{r_0}\left(\left\langle T(x_j k_{j=1}) k_{j=1}, y_k \right\rangle \right)_{k=1}^n = \sup_{(\alpha_k) \in B_{r_0}^{r_0}} \left| \sum_{k=1}^h \alpha_k \left\langle A_j(x_j k_{j=1}) k_{j=1}, S'(y_k) \right\rangle \right| \leq \]

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Let \( s \) be Banach spaces and let \( \pi \) be an \( n \)-summing map such that \( \sum_{k=1}^n \pi_k(x_{kj}) \leq 1 \) for every \( 1 \leq j \leq n+1 \). Let \( T \) be a \( (n+1) \)-summing map such that \( \sum_{k=1}^n \pi_k(x_{kj}) \leq 1 \) for every \( 1 \leq j \leq n+1 \). Let \( \alpha \) be an \( (n+1) \)-summing map such that \( \sum_{k=1}^n \alpha_k \leq 1 \) for every \( 1 \leq j \leq n+1 \). Let \( s \) be a Banach space and let \( (s_j)_{j=0}^n \) be an admissible \( (n+2) \)-pla such that \( s_j = s_j', 1 \leq j \leq n \). If \( E_j, 1 \leq j \leq n+1 \) are Banach spaces and \( E''_{n+1} \) has cotype 2, one has \( \otimes_{j=1}^{n+1} E_j, \alpha \approx \otimes_{j=1}^{n+1} E_j, \alpha \). 

The assertions about \( \mathcal{P}_r(T) \) follow from (14), (17) and (18). ■

**Theorem 3** Let \( r = (r_j)_{j=0}^{n+1} \) be such that \( r_j' \leq 2 \) and let \( s = (s_j)_{j=0}^{n+1} \) be an admissible \( (n+2) \)-pla such that \( s_j' \leq 2 \) and \( s_j' = r_j', 1 \leq j \leq n \). If \( E_j, 1 \leq j \leq n+1 \) are Banach spaces and \( E''_{n+1} \) has cotype 2, one has \( \otimes_{j=1}^{n+1} E_j, \alpha \approx \otimes_{j=1}^{n+1} E_j, \alpha \).

**Proof.** By theorem 2 and the open mapping theorem it is enough to see that \( \mathcal{P}_s(\prod_{j=1}^n E_j, E''_{n+1}) = \mathcal{P}_r(\prod_{j=1}^n E_j, E''_{n+1}) \). Given \( T \in \mathcal{P}_s(\prod_{j=1}^n E_j, E''_{n+1}) \) and using Kwapien’s generalized theorem, we choose a factorization \( T = C \circ (A_j)_{j=1}^n \) throughout some product \( \prod_{j=1}^n M_j \) of Banach spaces in such a way that \( A_j \in \mathcal{P}_s(E_j, M_j), 1 \leq j \leq n \) and \( C' \in \mathcal{P}_{s'}(E''_{n+1}, L^n(\prod_{j=1}^n M'_j, \mathbb{R})) \) being \( E''_{n+1} \) of cotype 2 and \( r_j' \leq 2 \), Maurey’s theorem [2, corollary 3, §31.6] and Pietsch’s inclusion theorem for absolutely \( p \)-summing maps give \( C' \in \mathcal{P}_s(E''_{n+1}, L^n(\prod_{j=1}^n M_j, \mathbb{R})) \subset \mathcal{P}_{s'}(E''_{n+1}, L^n(\prod_{j=1}^n M'_j, \mathbb{R})) \). As \( r_j' = s_j', 1 \leq j \leq n \), by the sufficient part of Kwapien’s generalized theorem we obtain \( T \in \mathcal{P}_r(\prod_{j=1}^n E_j, E''_{n+1}) \). In the same way we show \( \mathcal{P}_r(\prod_{j=1}^n E_j, E''_{n+1}) \subset \mathcal{P}_s(\prod_{j=1}^n E_j, E''_{n+1}) \) and the proof is complete. ■

**Corollary 4** Let \( E_j, 1 \leq j \leq n+1 \) be Banach spaces and let \( r = (r_j)_{j=0}^{n+1} \) be an admissible \( (n+2) \)-pla such that \( r_j' \geq 2 \) for every \( 1 \leq j \leq n+1 \). Let \( s = (s_j)_{j=0}^{n+1} \) be another admissible \( (n+2) \)-pla such that \( 2 \leq s_j' \) for every \( 1 \leq j \leq n \) and \( s_{n+1} = r_{n+1} \). Then \( \otimes_{j=1}^{n+1} E_j, \alpha_r \approx \otimes_{j=1}^{n+1} E_j, \alpha_s \).
Proof. Arguing as above, we only need to show that $\Psi_s(\prod_{j=1}^n E_j, E'_{n+1}) = \Psi_r(\prod_{j=1}^n E_j, E'_{n+1})$. The crucial step is the proof of the inclusion $\Psi_s(\prod_{j=1}^n E_j, E'_{n+1}) \subset \Psi_s(\prod_{j=1}^n E_j, E'_{n+1})$ since the proof of the converse inclusion can be made exactly in the same way.

Let $T \in \Psi_r(\prod_{j=1}^n E_j, E'_{n+1})$. By the proof of 2) $\implies$ 3) in theorem 3 there are a probability space $(\Omega, B, \mu)$, maps $A_j \in \Psi_{r_j}(E_j, L^{r_j}(\Omega, \mu)), 1 \leq j \leq n$ and a map $S \in L^n(\prod_{j=1}^n A_j(E_j), E'_{n+1})$ such that $S' \in \Psi_{r_{n+1}}(E''_{n+1}, L^n(\prod_{j=1}^n A_j(E_j), \mathbb{R}))$ and $T = S \circ ((A_j)'_{n=1})$. Consider the tensor products $\otimes = (\otimes_{j=1}^n L^{r_j}(\Omega, \mu), \pi)$ and $\otimes = L^{n_2}(\Omega, \mu) \otimes \pi$. The canonical linear map $\bar{\theta}_\pi$ from $\otimes$ onto $L^{n_2}(\Omega, \mu)$, (recall the notation of introductory section) induces an isomorphism $\bar{\theta}_\pi$ as $(\bar{\theta}_\pi)^{-1}$ and define $\otimes = (\otimes_{j=1}^n L^{r_j}(\Omega, \mu), \pi)$ and $\otimes = L^{n_2}(\Omega, \mu) \otimes \pi$. The canonical linear map $\bar{\theta}_\pi$ from $\otimes$ onto $L^{n_2}(\Omega, \mu)$, (recall the notation of introductory section) induces an isomorphism $\bar{\theta}_\pi$.

Let $\Psi_1 : \otimes \longrightarrow K_1$ be the canonical quotient map. For every $1 \leq j \leq n$ we consider the map $\psi_j \in L(L^{r_j}(\Omega), \otimes)$ defined by

$$
\psi_j : z \in L^{r_j}(\Omega) \longrightarrow [\chi_\alpha] \otimes [\chi_\alpha] \otimes \ldots \otimes [\chi_\alpha] \otimes z \otimes [\chi_\alpha] \otimes \ldots \otimes [\chi_\alpha]
$$

(z in the position $j + 1$) and define $\otimes = \psi_j(L^{r_j}(\Omega))$. $[\chi_\alpha]$ being of dimension 1 is complemented in each $L^{r_j}(\Omega, \mu), p \geq 1$. It follows that $\otimes$ is a complemented (and hence closed) subspace of $\otimes$. Define $F_j := A_j(E_j)$.

Claim. For every $1 \leq j \leq n, \Psi_1(\otimes)$ is closed in $K_1$.

Proof of the claim. Fix $1 \leq j \leq n$. Let $P_j \in L(\otimes, \otimes)$ be a projection and let $W_j := Ker(P_j) \oplus (Ker(\bar{\theta}_\pi) \cap \otimes)$. The quotient space $\Psi_2 \in L(\otimes, \otimes)$ is well defined. Let $\Psi_2 \in L(\otimes)$ be the canonical quotient map. The map $\forall z \in \otimes, L_j : \Psi_2(z) \in K_2 \longrightarrow \Psi_1 \circ P_j(z) \in \Psi_1(\otimes) \subset K_1$

is well defined and continuous. In fact, given $z_1 = P_j(z_1) + (I_\pi - P_j)(z_1) \in \otimes$ and $z_2 = P_j(z_2) + (I_\pi - P_j)(z_2) \in \otimes$, $I_\pi$ denotes the identity map on $\otimes$ such that $\Psi_2(z_1) = \Psi_2(z_2)$, as $(I_\pi - P_j)(z_1) \in Ker(P_j) \subset W$ and $(I_\pi - P_j)(z_2) \in Ker(P_j) \subset W$, we obtain $\Psi_2 \circ P_j(z_1) = \Psi_2 \circ P_j(z_2)$, i.e.

$$
P_j(z_1) - P_j(z_2) \in W \implies P_j(z_1) - P_j(z_2) \in Ker(\bar{\theta}_\pi) \cap \otimes \subset Ker(\bar{\theta}_\pi)
$$

and hence $L_j(z_1) = \Psi_1 \circ P_j(z_1) = \Psi_1 \circ P_j(z_2) = L_j(z_2)$ and $L_j$ is well defined. On the other hand, given $\Psi_2(z) \in K_2$ there is $w \in \otimes$ such that $\Psi_2(z) = \Psi_2(z)$ and $\|w\|_{\otimes} \leq 2 \|\Psi_2(z)\|_{K_2}$. Then

$$
\|L_j \circ \Psi_2(z)\|_{K_1} = \|L_j \circ \Psi_2(z)\|_{K_1} = \|\Psi_1 \circ P_j(z)\|_{K_1} \leq
$$

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that we have
\[ e \] and since
\[ \Phi \]
Let \( \Psi \) and since
\[ \psi \]
Next take \( z \in \Psi_1(\Sigma_j) \). There is a sequence \( \{z_m\}_{m=1}^{\infty} \subset \Sigma_j \) such that \( z = \lim_{m \to \infty} \Psi_1(z_m) \) in \( K_1 \). Then \( \{\tilde{L}^{-1}(z_m)\}_{m=1}^{\infty} \) is a Cauchy sequence in \( K_{3j} \). By a standard procedure (see [8], §14.4. (3)) for instance) and switching to a suitable subsequence if necessary, we can assume that there is a sequence \( \{w_m\}_{m=1}^{\infty} \subset \Theta_\pi \) such that
\[ \forall \ m \in \mathbb{N} \quad \Psi_{3j} \circ \Psi_{2j}(w_m) = \tilde{L}^{-1}_j(z_m) = \Psi_{3j} \circ \psi_j(z_m) \] (20)
and
\[ \forall \ m, k \in \mathbb{N} \quad \|w_m - w_k\|_{\beta_n} \leq 2 \||\Psi_{2j}(w_m) - \Psi_{2j}(w_k)\|_{K_{3j}} \leq 4 \|\tilde{L}^{-1}_j(z_m) - \tilde{L}^{-1}_j(z_k)\|_{K_{3j}}. \]
Then \( \{w_m\}_{m=1}^{\infty} \) is a Cauchy sequence in \( \Theta_\pi \) and there exists \( w = \lim_{m \to \infty} w_m \in \Theta_\pi \).
By (20) we obtain
\[ \Psi_{3j} \circ \Psi_{2j}(z_m) = \Psi_{3j} \circ \Psi_{2j}(w_m) = \Psi_{3j} \circ \Psi_{2j}(P_j(w_m) - (I_{\pi} - P_j)(w_m)) = \Psi_{3j} \circ \Psi_{2j}(P_j(w_m)) \]
and since \( P_j \) is a projection and \( P_j(z_m) = z_m \), by the definitions of \( \Psi_{3j} \) and \( L_j \)
\[ \Psi_1(z_m) = \Psi_1 \circ P_j(z_m) = L_j \circ \Psi_{2j}(z_m) = L_j \circ \Psi_{2j}(w_m) = \Psi_1 \circ P_j(w_m) \]
and \( \Psi_1 \circ P_j(w) = \lim_{m \to \infty} \Psi_1 \circ P_j(w_m) = \lim_{m \to \infty} \Psi_1(z_m) = z \). As \( P_j(w) \in \Sigma_j \) we obtain \( z \in \Psi_1(\Sigma_j) \) and \( \Psi_1(\Sigma_j) \) is closed.

**End of the proof of corollary 5.** Let \( \Phi_j \) be the restriction to \( \Sigma_j \) of \( \Psi_1 \).
Let \( \Psi_{4j} \) be the canonical quotient map from \( \Sigma_j \) onto the quotient space \( K_{4j} := \Sigma_j / (\Sigma_j \cap Ker(\mathcal{M}_\mu)) \). The map \( \tilde{\Phi}_j : \Psi_{4j} \circ \psi_j(z_j) \in K_{4j} \mapsto \Phi_j \circ \psi_j(z_j) \in \Phi_j(\Sigma_j), z_j \in F_j \) is well defined. In fact, if \( z_j \in F_j \) and \( \Psi_{4j} \circ \psi_j(z_j - z_j) = 0 \), we will have \( \psi_j(z_j - z_j) \in Ker(\mathcal{M}_\mu) \) and hence, by definition of \( \mathcal{M}_\mu \) and \( \psi_j \), one has \( z_j = z_j \) and \( \Phi_j \circ \psi_j(z_j) = \Phi_j \circ \psi_j(z_j) \), turning \( \tilde{\Phi}_j \) well defined. The same argument shows that \( \tilde{\Phi}_j \) is injective. By the claim \( \Phi_j(\Sigma_j) \) is closed in \( K_1 \). As \( \tilde{\Phi}_j \) is clearly surjective by the open map theorem it turns out that \( \tilde{\Phi}_j \) is an isomorphism from \( K_{4j} \) onto \( \Phi_j(\Sigma_j) \).

Next, remark that given \( z_j \in L^{s_j}(\Omega, \mu) \) and \( \varepsilon > 0 \), there is \( z_j \in L^{s_j}(\Omega, \mu) \) such that \( \Psi_{4j} \circ \psi_j(z_j) = \Psi_{4j} \circ \psi_j(z_j) \) and
\[ \|\psi_j(z_j)\|_{\mathcal{M}_\mu} \leq \|\Psi_{4j} \circ \psi_j(z_j)\|_{K_{4j}} + \varepsilon \leq \|\tilde{\Phi}_j^{-1}\| \|\tilde{\Phi}_j \circ \Psi_{4j} \circ \psi_j(z_j)\|_{K_1} + \varepsilon = \]
Let $\Phi_j \in \text{sequence } S$ type 2 because $\Phi_j$ are Banach spaces. Theorem 7 that $j \leq 1$ some $s_j \leq s_0 \leq 1$ and $r_j \geq 2$ for some $1 \leq j \neq j_0 \leq n + 1$. Choose $s_{j0} < r_{j0}$ and define $\frac{1}{s_0} := \frac{1}{r_{j0}} + \frac{1}{s_{j0}}$ and $s := r_j, 1 \leq j \neq j_0 \leq n + 1$. Then $s = (s_j)_{j=0}^{n+1}$ is an admissible $(n + 2)$-pla such that $s_0 < \infty$ and $(\otimes_{j=1}^{n+1} E_j, \alpha_s) \approx (\otimes_{j=1}^{n+1} E_j, \alpha_r)$.

**Proof.** After the eventual transposition $j_1 \to n + 1, n + 1 \to j_1$ we can assume that $j_1 = n + 1$. Then the proof is essentially the same of corollary 5 because we have $r_{n+1} \leq 2$ and Maurey’s theorem will be applicable still in the ”axis” $j_0$.

Another application of theorem 3 concerns to the approximation of $r$-dominated maps by finite rank maps.

**Theorem 7** Let $E_j, 1 \leq j \leq n+1$, be Banach spaces with duals $E_j'$ having the metric approximation property and such that each $E_j', 1 \leq j \leq n$ has the Radon-Nikodym property. Then $\mathfrak{P}_r(\prod_{j=1}^{n} E_j, E'_{n+1}) = (\otimes_{j=1}^{n+1} E_j, \alpha_{r})$.

**Proof.** Let $T \in \mathfrak{P}_r(\prod_{j=1}^{n} E_j, E'_{n+1})$. By Kwapien’s theorem (theorem 3) there are Banach spaces $M_j$ and operators $A_j \in \mathfrak{P}_{r_j}(E_j, M_j), 1 \leq j \leq n$ and $S \in \mathcal{L}^{n}(\prod_{j=1}^{n} M_j, E'_{n+1})$ such that $T = S \circ (A_1, A_2, ..., A_n)$. Since every $E_j$ has the Radon-Nikodym property, by the result [11, page 228] of Makarov and Samarskii, each $A_j$ is a quasi $r_j'$-nuclear operator. By [13, theorems 26 and 43] there is a sequence

$$\left\{B_{jh} = \sum_{s_j=1}^{t_{jh}} x'_{jhs_j} \otimes m_{jhs_j}\right\}_{h=1}^{\infty} \subset E_j' \otimes M_j,$$
of finite rank operators such that

$$\forall \ 1 \leq j \leq n \quad \lim_{h \to \infty} P_{r_j}(A_j - B_jh) = 0. \quad (21)$$

In particular, every sequence $\{B_jh\}_{h=1}^\infty$ is a Cauchy sequence (and so bounded) in $\mathfrak{P}_{r_j}(E_j, M_j)$, $1 \leq j \leq n$.

Since for every $(x_j)_{j=1}^n \in \prod_{j=1}^n E_j$ and $h \in \mathbb{N}$ we have

$$S \circ ((B_jh)^n_{j=1})((x_j)^n_{j=1}) = S\left(\left(\sum_{s_j=1}^{t_jh} \langle x'_{jhs_j}, x_j \rangle m_{jhs_j}\right)_{j=1}^n\right) =$$

$$= \sum_{s_1=1}^{t_{1h}} \cdots \sum_{s_n=1}^{t_{nh}} \left(\prod_{j=1}^n \langle x'_{jhs_j}, x_j \rangle\right) S\left((m_{jhs_j})_{j=1}^n\right),$$

it turns out that $S \circ ((B_jh)^n_{j=1}) \in \mathcal{L}^n(\prod_{j=1}^n E_j, E_{n+1}')$ has finite dimensional range and

$$S \circ ((B_jh)^n_{j=1}) = \sum_{s_1=1}^{t_{1h}} \cdots \sum_{s_n=1}^{t_{nh}} \left(\bigotimes_{j=1}^n \langle x'_{jhs_j}, x_j \rangle\right) \otimes S\left((m_{jhs_j})_{j=1}^n\right) \in \bigotimes_{j=1}^{n+1} E_j'.$$

With a similar proof to the one given in [2] it can be seen that $(\bigotimes_{j=1}^{n+1} E_j', \alpha'_r)$ is a topological subspace of $\mathfrak{P}_r \left(\prod_{j=1}^n E_j, E_{n+1}'\right)$. Hence by theorem 3, (18) and (21)

$$\alpha'_r \left(S \circ (B_{1h}, B_{2h}, ..., B_{nh}) - S \circ (B_{1k}, B_{2k}, ..., B_{nk})\right) =$$

$$= \mathfrak{P}_r \left(\sum_{j=1}^n \left(S \circ B_{1k}, ..., B_{j-1,k}, B_{j,h} - B_{j,k}, B_{j+1,h}, ..., B_{nh}\right)\right) \leq$$

$$\leq \mathfrak{P}_{r_{n+1}'}(S) \sum_{j=1}^n \mathfrak{P}_{r_j}(B_{jh} - B_{jk}) \left(\prod_{1 \leq s < j} \mathfrak{P}_{r_{s+1}'}(B_{sk})\right) \left(\prod_{j < s \leq n} \mathfrak{P}_{r_{s+1}'}(B_{sh})\right)$$

is arbitrarily small when $h$ and $k$ lets to infinity and so there exists $z := \lim_{h \to \infty} S \circ (B_{1h}, B_{2h}, ..., B_{nh}) \in (\bigotimes_{j=1}^{n+1} E_j', \alpha'_r)$. On the other hand, it can be shown in an analogous way that

$$\lim_{h \to \infty} \mathfrak{P}_r \left(T - S \circ ((B_jh)_{j=1}^n)\right) = \lim_{h \to \infty} \mathfrak{P}_r \left((A_j)_{j=1}^n - S \circ ((B_jh)_{j=1}^n)\right) = 0$$

and hence $T = z$.  ■
3 r-nuclear multilinear maps

With the same methods used in the classical case of Lapresté’s tensor topologies, it can be shown that every element $z \in \bigotimes_{\alpha_{r}}(E_1, E_2, ..., E_n, F)$ can be represented as a convergent series

$$z = \sum_{m=1}^{\infty} \lambda_{m} \bigotimes_{j=1}^{n} x_{jm} \otimes z_{m} \quad (22)$$

where $(\lambda_{m}) \in \ell^{r_{0}}$, $(x_{jm})_{m=1}^{\infty} \in \ell^{r_{j}}(E_j)$, $j = 1, 2, ..., n$ and $(z_{m})_{m=1}^{\infty} \in \ell^{r_{n+1}}(F)$. Moreover, the norm of such elements $z$ can be computed as in (10) but using representations (22) and $h = \infty$.

If $F$ is a Banach space every $z \in \bigotimes_{\alpha_{r}}(E_1, E_2, ..., E_n, F)$ defines canonically a multilinear map $T_{z} \in L^{n}(\prod_{j=1}^{n} E_{j}', F)$ by the rule

$$\forall (x_{j}')_{j=1}^{n} \in \prod_{j=1}^{n} E_{j}' \quad T_{z}((x_{j}')_{j=1}^{n}) = \sum_{m=1}^{\infty} \lambda_{m} \left( \prod_{j=1}^{n} \langle x_{m}', x_{j}' \rangle \right) z_{m}. \quad (23)$$

Remark that $T_{z}$ is independent on the representing series (22) for $z$ as a consequence of theorem 2 and the easy fact that $(\bigotimes_{j=1}^{n} E_{j}') \otimes F' \subset \mathfrak{R}_{r}(\prod_{j=1}^{n} E_{j}, F')$ canonically. In this way we have defined a canonical linear map

$$\Phi: z \in \bigotimes_{\alpha_{r}}(E_1, E_2, ..., E_n, F) \longrightarrow T_{z} \in L^{n}(\prod_{j=1}^{n} E_{j}', F) \quad (24)$$

which suggest the next definition:

Definition 8 A multilinear map $A \in L^{n}(\prod_{j=1}^{n} E_{j}, F)$ is said to be r-nuclear if it is the restriction $R(T_{z})$ to $\prod_{j=1}^{n} E_{j}$ of a map $T_{z}$ for some $z \in \bigotimes_{\alpha_{r}}(E_1', E_2', ..., E_n', F')$.

It can be shown that the set $\mathfrak{R}_{r}(\prod_{j=1}^{n} E_{j}, F)$ of all $n$-linear r-nuclear maps from $\prod_{j=1}^{n} E_{j}$ into $F$ becomes a Banach space under the r-nuclear norm

$$\mathcal{N}_{r}(A) = \inf \left\{ \alpha_{r}(z) \mid A = R(T_{z}), \ z \in \bigotimes_{\alpha_{r}}(E_1', E_2', ..., E_n', F) \right\}$$

if all $E_{j}$, $1 \leq j \leq n$ and $F$ are Banach spaces. r-nuclear maps can be characterized by means of suitable factorizations as follows.

Theorem 9 Let $F$ and $E_{j}$, $1 \leq j \leq n$ be Banach spaces and $T \in L^{n}(\prod_{i=1}^{n} E_{j}, F)$. $T$ is r-nuclear if and only if there are maps $A_{j} \in L(E_{j}, \ell^{r_{j}})$, $1 \leq j \leq n$, $C \in L(\ell^{r_{n+1}}, F)$ and $\lambda := (\lambda_{m}) \in \ell^{r_{0}}$ such that $T$ factorizes in the way
Moreover $N_r(T) = \inf \left( \prod_{j=1}^{n} \|A_j\| \right) \left\| D_\lambda \right\| \left\| C \right\|$ taking the infimum over all factorizations as above.

**Proof.** The proof being quite standard (compare with [10]) is omitted.

**Remark.** By theorem 9, (2) and the compactness result ([1], theorem 4.2) of Alencar and Floret, if $r_0 < \infty$, every $r$-nuclear mapping is compact.

As an application of theorem 7 we can obtain a sufficient condition in order that the map $\Phi$ be injective. Although the formulation of this condition is far to be optimal, it will be enough for our applications in the sequel.

**Corollary 10** Let $E_j, 1 \leq j \leq n$ be reflexive Banach spaces having the approximation property. Then, for every Banach space $E_{n+1}$ such that $E'_{n+1}$ has the metric approximation property, the map $\Phi$ in (24) is injective and so $(\bigotimes_{j=1}^{n+1} E_j, \alpha_r) = \mathcal{N}_r(\prod_{j=1}^{n} E'_j, E_{n+1})$.

**Proof.** Since we have actually $\Phi \in \mathcal{L} \left( (\bigotimes_{j=1}^{n+1} E_j, \alpha_r), \mathcal{N}_r(\prod_{j=1}^{n} E'_j, E_{n+1}) \right)$, it is enough to show that this map is injective. Is easy to see that $\bigotimes_{j=1}^{n+1} E'_j \subset \left( \mathcal{N}_r(\prod_{j=1}^{n} E'_j, E_{n+1}) \right)'$. Now theorem 7 implies that the transposed map

$$\Phi' : \left( \mathcal{N}_r(\prod_{j=1}^{n} E'_j, E_{n+1}) \right)' \rightarrow \mathcal{P}_r(\prod_{j=1}^{n} E_j, E'_{n+1})$$

has dense range, getting the injectivity of $\Phi$.  

4  **r-integral multilinear maps**

**Definition 11** Let $E_j, 1 \leq j \leq n$, and $F$ be Banach spaces. A continuous $n$-linear map $T$ from $\prod_{j=1}^{n} E_j$ into $F$ is called $r$-integral if $J_FT \in (\bigotimes_{j=1}^{n} (E_1, E_2, \ldots, E_n, F'))'$.

The norm of $J_FT$ in that dual space is taken as definition of the $r$-integral norm $I_r(T)$ of a map $T \in \mathcal{I}_r(\prod_{j=1}^{n} E_j, F)$, the set of $r$-integral multilinear maps from $\prod_{j=1}^{n} E_j$.
Given a measure space \((\Omega, \mathcal{A}, \mu)\) and \(g \in L^r(\Omega, \mathcal{A}, \mu)\), the canonical multilinear map \(D_g : \prod_{j=1}^n L^{s_j}(\Omega, \mathcal{A}, \mu) \to L^{r+1}(\Omega, \mathcal{A}, \mu)\) is \(r\)-integral.

**Proof.** Let \(S_j, 1 \leq j \leq n\) be the subspace of \(L^{s_j}(\Omega, \mu)\) of simple functions with support of finite measure. Every \(S_j\) being dense in \(L^{s_j}(\Omega, \mu)\), it is enough so see that \(D_g \in \bigotimes_{\alpha_j^r} (S_1, S_2, \ldots, S_n, L^{s_{n+1}}(\Omega, \mu))'\) (density lemma for \((n + 1)\)-tensor norms).

Fix \(z \in \bigotimes_{\alpha_j^r} (S_1, S_2, \ldots, S_n, L^{s_{n+1}}(\Omega, \mu))\). Then exist finite dimensional subspaces \(M_j \subset S_j, 1 \leq j \leq n\) generated by the characteristic functions \(\{\chi_{B_k}\}_{k=1}^h\) of of a finite family of pairwise disjoints sets of finite measure \(\{B_k\}_{k=1}^h \subset \mathcal{A}\) and there exists a finite dimensional subspace \(N \subset L^{s_{n+1}}(\Omega, \mu)\) such that \(z \in \otimes (M_1, M_2, \ldots, M_n, N)\). Then for every \(f_j \in M_j, 1 \leq j \leq n\) and \(f_{n+1} \in N\), using (4)

\[
\langle \otimes_{j=1}^{n+1} f_j, D_g \rangle = \left( \otimes_{j=1}^n \frac{1}{\mu(B_k)^{s_j}} \left( \prod_{j=1}^n \left( \int_{B_k} f_j \, d\mu \right) \right) \chi_{B_k} g, f_{n+1} \right) = \sum_{k=1}^h \frac{1}{\mu(B_k)^{r+1}} \left( \prod_{j=1}^n \left( \int_{B_k} f_j \, d\mu \right) \right) \chi_{B_k} g, f_{n+1} \rangle
\]

As a consequence

\[
\forall z \in \bigotimes (M_1, M_2, \ldots, M_n, N) \quad \langle z, D_g \rangle = \langle z, V \rangle
\]

where we have defined

\[
V := \sum_{k=1}^h \left( \int_{B_k} |g|^{r_0} \, d\mu \right) \chi_{B_k} g
\]

and where \(\varphi_{jk}\) is the class in \(L^{s_j}(\Omega, \mu)/M_j'\) of the function \(\mu(B_k)^{-\frac{1}{r_0}} \chi_{B_k}\) for every \(\forall 1 \leq j \leq n, 1 \leq k \leq h\). Moreover, (the class of ) \(\chi_{B_k} g \in N'\) for every \(1 \leq k \leq h\) since \(\chi_{B_k} g \in L^{r_0}(\Omega, \mu)\) and by (7) we obtain \(\chi_{B_k} g \in L^{r_{n+1}}(\Omega, \mu), B_k\) being of finite measure.
Note that, by finite dimensionality

\[ V \in \bigotimes_{\alpha_r}(M'_1, M'_2, ..., M'_n, N') = (\bigotimes_{\alpha'_r}(M_1, M_2, ..., M_m, N))'. \]  

(26)

Now we perform some computations. The first one is

\[
\pi_{r_0} \left( \left( \left( \int_{B_k} |g|^{r_0} \, d\mu \right)^{\frac{1}{r_0}} \right)_{k=1}^{h} \right) = \left( \sum_{k=1}^{h} \int_{B_k} |g|^{r_0} \, d\mu \right)^{\frac{1}{r_0}} = \|g\|_{L^{r_0}(\Omega)}
\]

(27)

In second time, for every \( 1 \leq j \leq n \), using (4) and Hölder’s inequality, we obtain

\[
\varepsilon_{r'_j} \left( \left( \varphi_{j,k} \right)_{k=1}^{h} \right) = \sup_{\|f\|_{L^{r'_j}(\Omega)} \leq 1} \left( \sum_{k=1}^{h} \frac{1}{\mu(B_k)^{\frac{r'_j}{r_j}}} \left( \int_{B_k} |f|^{r'_j} \, d\mu \right)^{\frac{1}{r'_j}} \right) \leq \left( \sum_{k=1}^{h} \frac{1}{\mu(B_k)^{\frac{r'_j}{r_j}}} \left( \int_{B_k} |f|^{r'_j} \, d\mu \right)^{\frac{1}{r'_j}} \right)^{\frac{1}{r_j}} \leq \sup_{\|f\|_{L^{r'_j}(\Omega)} \leq 1} \|f\|_{L^{r'_j}(\Omega)} = 1.
\]

(28)

Finally, by Hölder’s inequality and (6) we have

\[
\varepsilon_{r'_{n+1}} \left( \left( \mu(B_k)^{-\frac{1}{r_0}} \left( \int_{B_k} |g|^{r_0} \, d\mu \right)^{-\frac{1}{r_0}} \chi_{B_k} \, g \right)_{k=1}^{h} \right) = \sup_{\|f\|_{L^{r'_{n+1}}(\Omega)} \leq 1} \left( \sum_{k=1}^{h} \mu(B_k)^{-\frac{r'_{n+1}}{r_0}} \left( \int_{B_k} |f|^{r_0} \, d\mu \right)^{-\frac{r'_{n+1}}{r_0}} \left( \int_{B_k} g \, f \, d\mu \right)^{\frac{r'_{n+1}}{r_0}} \right)^{\frac{-1}{r'_{n+1}}} \leq \left( \sum_{k=1}^{h} \int_{B_k} |f|^{r'_{n+1}} \, d\mu \right)^{\frac{1}{r'_{n+1}}} = \sup_{\|f\|_{L^{r'_{n+1}}(\Omega)} \leq 1} \left( \int_{\Omega} |f|^{r'_{n+1}} \, d\mu \right)^{\frac{1}{r'_{n+1}}} = 1.
\]

(29)

Then, by (25), (26), (27), (28) and (29)

\[
|\langle z, Dg \rangle| \leq \alpha_{r}(z; \bigotimes(M_1, M_2, ..., M_n, N)) \alpha_{r}(V; \bigotimes(M'_1, M'_2, ..., M'_n, N')) \leq
\]

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Consider a family of canonical maps \( M \) to \( \gamma_X \) and are a decomposable measure space \( (\Omega,\mu) \). For every element in \( \bigotimes\limits_{n=1}^{\infty} F_\gamma \) being a finite generated \((n+1)\)-tensor norm,

\[
\left\langle z, D_g \right\rangle \leq \alpha'_r(z; \bigotimes\limits_{n=1}^{\infty} (S_1, S_2, \ldots, S_n, L^{r_{n+1}}(\Omega, \mu))) \|g\|_{L^r(\Omega)};
\]

which means \( I_r(D_g) \leq \|g\|_{L^r(\Omega)} \) \( \blacksquare \).

To find a characterization of \( r \)-integral maps we need to use ultraproducts \((E_\gamma)_U\) of a given family \( \{E_\gamma, \gamma \in G\} \) on the index set \( G \). For this topic our main reference is [17]. We use the natural notation \((x_\gamma)_U\) for every element in \((E_\gamma)_U\).

Given a family \( \{T_\gamma \in L^n(\prod_{j=1}^n E_j^j, F_\gamma), \quad \gamma \in G\} \) of maps between the cartesian product \( \prod_{j=1}^n E_j^j \) of Banach spaces \( E_j^j \) and \( F_\gamma, 1 \leq j \leq n, \quad \gamma \in G \), such that \( \sup_{\gamma \in G} \|T_\gamma\| < \infty \), there is a canonical \( n \)-linear continuous ultraproduct map \((T_\gamma)_U\) from the ultraproduct \((\prod_{j=1}^n E_j^j)_U\) into the ultraproduct \((F_\gamma)_U\) such that for every \( \mathbf{x} := (x_\gamma^n_{j=1})_U \subset (\prod_{j=1}^n E_j^j)_U \) we have \((T_\gamma)_U(\mathbf{x}) = (T_\gamma((x_\gamma^n_{j=1}))_U)\). The main result we shall need is the following factorization theorem:

**Lemma 13** Consider a family of canonical maps \( D_{g_\gamma} : \prod_{j=1}^n E_j^j \rightarrow E_{r_{n+1}}, \quad \gamma \in G \neq \emptyset \) defined by a family of elements \( \{g_\gamma | \gamma \in G\} \subset E_{r_0} \) such that \( 0 < \sup_{\gamma \in G} \|D_{g_\gamma}\| < \infty \). There exist a decomposable measure space \((\Omega,\mathcal{M},\mu)\), a function \( g \in L^{r_0}(\Omega,\mathcal{M},\mu) \) and order onto isometries \( \mathbf{x}_j : (E_j^j)_U \rightarrow L^{r_j}(\Omega,\mathcal{M},\mu) \), \( 1 \leq j \leq n \), \( \mathbf{x}_0 : (E_0^0)_U \rightarrow L^{r_0}(\Omega,\mathcal{M},\mu) \) and \( \mathbf{x}_{n+1} : (E_{r_{n+1}})_U \rightarrow L^{r_{n+1}}(\Omega,\mathcal{M},\mu) \) such that the diagram

\[
\begin{array}{ccc}
(\prod_{j=1}^n E_j^j)_U & \xrightarrow{(D_{g_\gamma})_U} & (E_{r_{n+1}})_U \\
(\mathbf{x}_j)_{j=1}^n & \downarrow & \mathbf{x}_{n+1}^{-1} \\
(\prod_{j=1}^n L^{r_j}(\Omega))_U & \xrightarrow{D_g} & L^{r_{n+1}}(\Omega).
\end{array}
\]

is commutative. Moreover, \( \|D_g\| = \|(D_{g_\gamma})_U\| \).

**Proof.** By (5) and a factorization result of Raynaud, [15], theorem 5.1] there are a decomposable measure space \((\Omega,\mathcal{M},\mu)\) and isometric order isomorphisms

\[
\mathbf{x}_0 : (E_0^0)_U \rightarrow L^{r_0}(\Omega,\mathcal{M},\mu), \quad \mathbf{x}_j : (E_j^j)_U \rightarrow L^{r_j}(\Omega,\mathcal{M},\mu), \quad 1 \leq j \leq n,
\]

and \( \mathbf{x}_{n+1} : (E_{r_{n+1}})_U \rightarrow L^{r_{n+1}}(\Omega,\mathcal{M},\mu) \) such that, \( M_\gamma \) being the map corresponding to \( \gamma \in G \) (recall the notations introduced in section 1), we have \((M_\gamma)_U = \mathbf{x}_{n+1}^{-1} \circ M_\mu \circ ((\mathbf{x}_j)_{j=1}^n)_U \). The lemma follows taking \( g = \mathbf{x}_0(g_\gamma)_U \) \( \blacksquare \).

Now we can obtain the following characterization:
Theorem 14 Let $E_j, 1 \leq j \leq n$ and $F$ be Banach spaces and $T \in \mathcal{L}^n(\prod_{j=1}^n E_j, F)$. The following are equivalent:

1) $T$ is $r-$integral.
2) $J_F T$ can be factorized as

\[
\begin{array}{cccc}
\prod_{j=1}^n E_j & T & F & J_F \\
(A_j)_{j=1}^n & & & F'' \\
\prod_{j=1}^n L^{r_j}(\Omega, \mathcal{M}, \mu) & D_g & & L^{r_n+1}(\Omega, \mathcal{M}, \mu)
\end{array}
\]

where $A_j \in \mathcal{L}(E_j, L^{r_j}(\Omega, \mathcal{M}, \mu))$, $1 \leq j \leq n$, $C \in \mathcal{L}(L^{r_n+1}(\Omega, \mathcal{M}, \mu), F'')$ and $D_g$ is the multilinear diagonal operator corresponding to some $g \in L^{r_0}(\Omega, \mathcal{M}, \mu)$. Moreover

\[
\text{I}_r(T) = \inf \|D_g\| \|C\| \prod_{j=1}^n \|A_j\| \tag{31}
\]

taking the infimum over all factorizations as in the previous diagram.

3) $J_F T$ can be factorized as above but $(\Omega, \mathcal{M}, \mu)$ being a finite measure space and $g = \chi_\Omega$. Formula (31) holds too taking the infimum over the factorizations of that type.

Proof. 1) $\implies$ 2). This can be done using standard methods with help of theorem 9 and lemma 13 (see for instance [10] for a detailed development of the method, used in a similar framework).

2) $\implies$ 3). Given $\varepsilon > 0$, select a factorization of type (30) with $g \in L^{r_0}(\Omega, \mathcal{M}, \mu)$ and such that

\[
\|g\|_{L^{r_0}(\Omega, \mu)} \|C\| \prod_{j=1}^n \|A_j\| \leq \text{I}_r(T) + \varepsilon. \tag{32}
\]

After projection onto the sectional subspaces $L^{r_j}(\text{Supp}(g))$, $1 \leq j \leq n$ if necessary, we can assume that $\Omega = \text{Supp}(g)$. Consider the new finite measure $\nu$ on $(\Omega, \mathcal{M})$ defined by

\[
\forall M \in \mathcal{M} \quad \nu(M) = \int_M |g|^{r_0} \, d\mu
\]

and the mappings

\[
\forall 1 \leq j \leq n \quad H_j : f_j \in L^{r_j}(\Omega, \mu) \rightarrow H_j(f_j) = f_j |g|^{r_0} \in L^{r_j}(\Omega, \nu)
\]
and
\[ H_{n+1} : f \in L^r_{n+1}(\Omega, \mu) \longrightarrow H_{n+1}(f) = f |g|^{-\frac{r_0}{r_{n+1}}} \in L^r_{n+1}(\Omega, \nu). \]

By Radon-Nikodym’s theorem
\[ \left\| H_{n+1}(f) \right\|_{L^r_{n+1}(\Omega, \nu)} = \left\| f \right\|_{L^r_{n+1}(\Omega, \mu)}, \]
and for every \( (f_j)^n_{j=1} \in \prod_{j=1}^n L^r_j(\Omega, \nu) \), using (2)
\[ \left( H_{n+1}^{-1} \circ D_{\chi_\Omega} \circ (H_j)^n_{j=1} \right) \left( (f_j)^n_{j=1} \right) = |g|^{-\frac{r_0}{r_{n+1}}} \prod_{j=1}^n f_j \left| g \right|^{-\frac{r_0}{r_j}} = |g|^{-\frac{1}{r_{n+1}} - \sum_{j=1}^n \frac{r_j}{r_j}} \prod_{j=1}^n f_j = \]
\[ = |g|^{-\frac{1}{r_{n+1}} - 1 + \frac{r_0}{r_0 + r_{n+1}}} \prod_{j=1}^n f_j = |g|^{-\frac{1}{r_{n+1}} - \sum_{j=1}^n \frac{1}{r_j}} \prod_{j=1}^n f_j = D_g \left( (f_j)^n_{j=1} \right). \]

As \( \chi_\Omega \in L^{r_0}(\Omega, \nu) \), joining the factorization (34) with the initial one we get our goal and moreover, by (33) and (32)
\[ I_r(T) \leq \left\| C \circ H_{n+1}^{-1} \right\| \left\| D_{\chi_\Omega} \right\| \prod_{j=1}^n \left\| H_j \circ A_j \right\| \leq \]
\[ \leq \left\| C \right\| \left\| H_{n+1} \circ D_g \circ H_j^{-1} \right\| \prod_{j=1}^n \left\| A_j \right\| \leq I_r(T) + \varepsilon. \]

3) \( \Rightarrow 1 \). It is immediate by theorem 12 and the ideal properties of multilinear \( r \)-integral operators. \( \blacksquare \)

5 Applications to reflexivity

Previous results allows us to obtain some information about the reflexivity of completed tensor products of type \( \alpha_r \).

**Theorem 15** Let \( E_j, 1 \leq j \leq n \in \mathbb{N} \) and \( F \) be reflexive Banach spaces such that \( E_j', 1 \leq j \leq n \) and \( F' \) have the metric approximation property. Given an admissible \( (n + 2) \)-pla \( r \), the space \( \mathcal{R}_r \left( \bigotimes_{\alpha_r} (E_1, E_2, ..., E_n, F) \right) \) is reflexive if and only if
\[ \mathcal{M}_r \left( \prod_{j=1}^n E_j', F \right) = \mathcal{J}_r \left( \prod_{j=1}^n E_j', F \right). \]
Proof. If (36) holds, by theorem 7 and corollary 10 we obtain

\[
\left(\bigotimes_{\alpha_r} (E_1, E_2, ..., E_n, F)\right)'' = \left(\mathcal{P}_r \left(\prod_{j=1}^{n} E_j, F'\right)\right)' = \left(\bigotimes_{\alpha_r'} (E'_1, E'_2, ..., E'_n, F')\right)'
\]

\[
= \mathcal{J}_r \left(\prod_{j=1}^{n} E'_j, F\right) = \mathcal{M}_r \left(\prod_{j=1}^{n} E'_j, F\right) = \bigotimes_{\alpha_r} (E_1, E_2, ..., E_n, F).
\]

Conversely, if \(\bigotimes_{\alpha_r} (E_1, E_2, ..., E_n, F)\) is reflexive, by definition of \(r\)-integral maps, theorem 7 and corollary 10 we obtain

\[
\mathcal{J}_r \left(\prod_{j=1}^{n} E'_j, F\right) = \mathcal{M}_r \left(\prod_{j=1}^{n} E'_j, F\right) = \bigotimes_{\alpha_r'} (E'_1, E'_2, ..., E'_n, F').
\]

We apply theorem 15 to characterize the reflexivity of \(\bigotimes_{j=1}^{n+1} \ell^{u_j}, \alpha_r\). First, we need a lemma.

Lemma 16 Let \(r = (r_j)_{j=0}^{n+1}\) an admissible \((n + 2)\)-plá verifying \(r_0 = \infty\) and let \(1 < u_j' \leq r_j'\) for every \(1 \leq j \leq n + 1\). Then there exists a non compact map \(T \in \mathcal{J}_r(\prod_{j=1}^{n} E'_j, \ell^{u_{n+1}})\).

Proof. Let \(I_1 := [0, \frac{1}{2}]\) and \(I_m := [\sum_{i=1}^{m} \frac{1}{2^i}, \sum_{i=1}^{m+1} \frac{1}{2^i}]\) if \(m > 1\). The map \(A_j : (\beta_i) \in \ell^{u_j} \to \sum_{m=1}^{\infty} \beta_m \mu(I_m)^{-\frac{1}{r_j'}} \chi_{I_m} \in L^{r_j'}([0, 1], \mu), 1 \leq j \leq n\) (\(\mu\) is the Lebesgue measure on \([0, 1]\)), is well defined and continuous since

\[
\|A_j((\beta_m))\| = \left(\sum_{m=1}^{\infty} \frac{|\beta_m|^{r_j'}}{\mu(I_m)} \right)^{\frac{1}{r_j'}} \leq \|\beta_m\|_{\ell^{u_j'}}.
\]

Take \(g = \chi_{[0,1]} \in L^\infty([0, 1], \mu)\). Consider now the closed linear subspace \(F\) generated by the set \(\{\chi_{I_m}, m \in \mathbb{N}\}\) in \(L^{r_{n+1}}([0, 1])\). The map

\[
Q : f \in L^{r_{n+1}}([0, 1]) \to \sum_{m=1}^{\infty} \frac{1}{\mu(I_m)} \left(\int_{I_m} f \, d\mu\right) \chi_{I_m} \in F
\]

is continuous since, by Hölder’s inequality

\[
\|Q(f)\|_F = \left(\sum_{m=1}^{\infty} \left(\int_{I_m} f \, d\mu\right)^{r_{n+1}} \mu(I_m)^{1-r_{n+1}}\right)^{\frac{1}{r_{n+1}}} \leq
\]

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\[ \left( \sum_{m=1}^{\infty} \left( \int_{I_m} |f|^{r_{n+1}} \, d\mu \right) \mu(I_m)^{r_{n+1}^{-1} + 1 - r_{n+1}} \right)^{1/r_{n+1}} = \|f\|_{L^{r_{n+1}}([0,1])}. \]

It is immediate that \( Q \) is a projection from \( L^{r_{n+1}}([0,1]) \) onto \( F \). Finally consider the map

\[ C : f = \sum_{m=1}^{\infty} \beta_m x_{I_m} \in F \longrightarrow \left( \beta_m \mu(I_m)^{1/r_{n+1}} \right) \in \ell^{u_{n+1}} \]

is continuous since \( r_{n+1} \leq u_{n+1} \) and

\[ \|C(f)\|_{\ell^{u_{n+1}}} = \left( \sum_{m=1}^{\infty} |\beta_m|^{u_{n+1}} \mu(I_m)^{u_{n+1}^{-1} + 1 - u_{n+1}} \right)^{1/u_{n+1}} \leq \left( \sum_{m=1}^{\infty} |\beta_m|^{r_{n+1}} \mu(I_m)^{r_{n+1}^{-1}} \right)^{1/r_{n+1}} = \|f\|_F. \]

Hence \( T := C \circ Q \circ D_g \circ ((A_j)_{j=1}^n) \in \mathcal{F}_r(\prod_{j=1}^n \ell^{u_j}, \ell^{u_{n+1}}) \) but \( T \) is not compact since, using (2)

\[ \forall m \in \mathbb{N} \quad T((e_m, e_m, \ldots, e_m)) = \frac{1}{\mu(I_m)^{1/r_{n+1}}} \mu(I_m)^{1/r_{n+1}} e_m = e_m. \]

We can state now the main result of this section:

**Theorem 17** If \( 1 < u_j < \infty \) for every \( 1 \leq j \leq n + 1 \), \( \bigotimes_{i=1}^{n+1} \ell^{u_j}, \alpha_r \) is reflexive if and only if at least one of the following set of conditions holds:

S1). There is \( 1 \leq j_0 \leq n + 1 \) such that \( u'_j > 2 \) and \( u'_j > r'_j \) for all \( 1 \leq j \neq j_0 \leq n + 1 \).

S2). There exists \( 1 \leq j_0 \leq n + 1 \) such that \( u'_j > 2 \) for every \( 1 \leq j \neq j_0 \leq n + 1 \)

\[ \frac{1}{r'_{j_0}} > \sum_{1 \leq j \neq j_0}^{n+1} \frac{1}{u'_j}. \]  \hfill (37)

and moreover, there exists \( 1 \leq j_1 \neq j_0 \leq n + 1 \) such that \( r'_j \geq 2 \) for every \( 1 \leq j \neq j_1 \leq n + 1 \).

S3). We have \( u'_j > 2 \) for every \( 1 \leq j \leq n + 1 \), and there exists \( 1 \leq j_0 \leq n + 1 \) such that \( r'_j \leq 2 \)

\[ \frac{1}{2} > \sum_{1 \leq j \neq j_0}^{n+1} \frac{1}{u'_j}. \]  \hfill (38)

S4). There is \( 1 \leq j_0 \leq n + 1 \) such that \( u'_{j_0} = 2, r'_{j_0} \leq 2, u'_j > 2 \) for every \( 1 \leq j \neq j_0 \leq n + 1 \)

\[ \frac{1}{2} > \sum_{1 \leq j \neq j_0}^{n+1} \frac{1}{u'_j}. \]  \hfill (39)
Proof. Sufficient conditions. Case S1). After the transposition $j_0 \rightarrow n+1$, $n+1 \rightarrow j_0$ if necessary, we can assume $j_0 = n+1$ and so $u_j' > 2$ and $r_j' > r_j'$ for every $1 \leq j \leq n$.

By theorem 14, given $T \in \mathcal{J}_r\left(\prod_{j=1}^{n} \ell^u_j, \ell^{u_n+1}\right)$ there are a finite measure space $(\Omega, \mathcal{M}, \mu)$ and mappings $A_j \in \mathcal{L}(\ell^u_j, L^r_j(\Omega, \mu)), 1 \leq j \leq n$ and $C \in \mathcal{L}(L^{r_{n+1}}(\Omega, \mu), \ell^v)$ such that $T = C \circ D_{\chi_{\Omega}} \circ (A_j)_{j=1}^{n}$. By Rosenthal’s result [16, theorem A.2] every $A_j$ is compact, and by the metric approximation property of $\ell^u_j$, there is a bounded sequence

$$\left\{ A_{jm} = \sum_{k=1}^{k_{im}} x_{jk} \otimes f_{jm}^k \right\}_{m=1}^{\infty} \subset \ell^u_j \otimes L^r_j(\Omega, \mu) \tag{40}$$

such that

$$\forall 1 \leq j \leq n \quad \lim_{m \to \infty} \left\| A_j - A_{jm} \right\|_{\mathcal{L}(\ell^u_j, L^r_j(\Omega, \mu))} = 0. \tag{41}$$

Define $T_m := C \circ D_{\chi_{\Omega}} \circ (A_{jm})_{j=1}^{n}$ for every $m \in \mathbb{N}$. Arguing as in theorem 7 and using theorem 14 we obtain for every $1 \leq j \leq n$ and $m \in \mathbb{N}$

$$\{ C \circ D_{\chi_{\Omega}} \circ (A_{1m}, \ldots, A_{j-1,m}, A_j - A_{jm}, A_{j+1,m}, \ldots, A_{nm}) \}_{m=1}^{\infty} \subset \mathcal{J}_r\left(\prod_{j=1}^{n} \ell^u_j, \ell^{u_n+1}\right)$$

and by (41)

$$\mathbf{I}_r(T - T_m) \leq \sum_{j=1}^{n} \mathbf{I}_r(C \circ D_{\chi_{\Omega}} \circ (A_{1m}, \ldots, A_{j-1,m}, A_j - A_{jm}, A_{j+1}, \ldots, A_n)) \leq$$

$$\leq \mu(\Omega)^{1/6} \left\| C \right\| \sum_{j=1}^{n} \left\| A_j - A_{jm} \right\| \left( \prod_{1 \leq s < j} \left\| A_{sm} \right\| \right) \left( \prod_{j < s \leq n} \left\| A_s \right\| \right) \tag{42}$$

which approach to 0 if $m \to \infty$. But actually we have

$$T_m = \sum_{k=1}^{k_{im}} \left( \bigotimes_{j=1}^{n} x_{jk} \right) \otimes (C \circ D_{\chi_{\Omega}} \circ (f_{jm}^k)) \in \mathcal{N}_r\left(\prod_{j=1}^{n} \ell^u_j, \ell^{u_n+1}\right).$$

It follows from theorem 7 that $\mathbf{N}_r(T_m - T_s) = \mathbf{I}_r(T_m - T_s)$ for $m, s \in \mathbb{N}$ and using (42), it turns out that $\{ T_m \}_{m=1}^{\infty}$ is a Cauchy sequence in $\mathcal{N}_r\left(\prod_{j=1}^{n} \ell^u_j, \ell^{u_n+1}\right)$. Then $T \in \mathcal{N}_r\left(\prod_{j=1}^{n} \ell^u_j, \ell^{u_n+1}\right)$ and by theorem 15 $\left(\bigotimes_{i=1}^{n} \ell^u_j, \alpha_r\right)$ is reflexive.

Case S2). Let $1 \leq j_0 \neq j_1 \leq n+1$ such that $u_j' > 2, 1 \leq j \neq j_0 \leq n+1, r_j' \geq 2, 1 \leq j \neq j_1 \leq n+1$ and (37) holds. In a first step we are going to see that we can assume $r_{j_1}' \geq 2$ too.
Consider the case that \( r'_{j_1} < 2 \). In such a case we have \( u'_{j_1} > 2 \) because \( j_0 \neq j_1 \). If \( j_1 = n + 1 \), defining \( s'_{n+1} = 2, s'_{j} := r'_{j}, 1 \leq j \leq n \) and \( \frac{1}{s_0} := \frac{1}{r_0} + \frac{1}{r_{n+1}} - \frac{1}{2} \) we obtain an admissible \((n + 2)\)-pla \( s = (s_j)_{j=0}^{n+1} \) verifying (37) still and such that, \( \ell^{u_{n+1}} \) having cotype 2, by corollary 4, we have \( (\bigotimes_{j=1}^{n+1} \ell^{u_j}, \alpha_r) \approx (\bigotimes_{j=1}^{n+1} \ell^{u_j}, \alpha_s) \). If \( 1 \leq j_1 \leq n \), a transposition \( j_1 \rightarrow n+1, n+1 \rightarrow j_1 \) would reduce the situation to the just considered case. So, in the formulation of S1) we can assume that \( r'_{j} \geq 2, 1 \leq j \leq n + 1 \).

After the eventual transposition \( j_0 \rightarrow n + 1, n + 1 \rightarrow j_0 \) we can assume that \( u'_{j} > 2 \) for every \( 1 \leq j \leq n \), \( r'_{j} \geq 2 \) for every \( 1 \leq j \leq n + 1 \) and (37) holds for \( j_0 = n + 1 \). Using (5) this last condition can be written in the way

\[
\frac{1}{r_0} + \sum_{j \mid r'_j < u'_j} \left( \frac{1}{r'_j} - \frac{1}{u'_j} \right) > \sum_{j \mid r'_j \geq u'_j} \left( \frac{1}{u'_j} - \frac{1}{r'_j} \right).
\]

For every \( 1 \leq j \leq n \) such that \( r'_{j} \geq u'_{j} \), choose \( 2 \leq t'_j < u'_j \) close enough to \( u'_j \) in order that

\[
\frac{1}{t_0} := \frac{1}{r_0} + \sum_{j \mid r'_j < u'_j} \left( \frac{1}{r'_j} - \frac{1}{u'_j} \right) - \sum_{j \mid r'_j \geq u'_j} \left( \frac{1}{u'_j} - \frac{1}{r'_j} \right) > 0.
\]

Now define \( t'_j := r'_j \) if \( r'_j < u'_j \), \( 1 \leq j \leq n \) and \( t_{n+1} := r_{n+1} \). By (2) we have

\[
\frac{1}{t_{n+1}} = \sum_{j=1}^{n} \frac{1}{t'_j} + \sum_{j \mid r'_j < u'_j} \left( \frac{1}{r'_j} - \frac{1}{r'_j} \right) + \sum_{j \mid r'_j \geq u'_j} \left( \frac{1}{r'_j} - \frac{1}{u'_j} \right) + \frac{1}{r_0}
\]

and it turns out that \( \mathbf{t} = (t_j)_{j=0}^{n+1} \) is an admissible \((n+2)\)-pla such that \( 2 \leq t'_j < u'_j \) and \( t'_j \leq r'_j \) for every \( 1 \leq j \leq n \) and moreover, by corollary 5 we have \( (\bigotimes_{j=1}^{n+1} \ell^{u_j}, \alpha_r) \approx (\bigotimes_{j=1}^{n+1} \ell^{u_j}, \alpha_s) \). Hence by case S1), \( (\bigotimes_{j=1}^{n+1} \ell^{u_j}, \alpha_r) \) is reflexive.

Case S3). Once again after the transposition \( j_0 \rightarrow n + 1, n + 1 \rightarrow j_0 \) we can assume that \( r'_{n+1} \leq 2, u'_{j} > 2 \) for every \( 1 \leq j \leq n + 1 \) and (38) holds for \( j_0 = n + 1 \), or in an equivalent way (by (2)),

\[
\frac{1}{r_0} + \frac{1}{r'_{n+1}} - \frac{1}{2} + \sum_{j \mid r'_j < u'_j} \left( \frac{1}{r'_j} - \frac{1}{u'_j} \right) > \sum_{j \mid r'_j \geq u'_j} \left( \frac{1}{u'_j} - \frac{1}{r'_j} \right).
\]

Remark that, by (2) we have necessarily \( r'_{j} \geq 2, 1 \leq j \leq n \). Since \( \ell^{u_{n+1}} \) has cotype 2, by corollary 4 there exists an \((n + 2)\)-pla \( s = (s_j)_{j=0}^{n+1} \) such that \( s'_{n+1} = 2, s'_{j} := r'_{j}, 1 \leq j \leq n \) and \( \frac{1}{s_0} := \frac{1}{r_0} + \frac{1}{r_{n+1}} - \frac{1}{2} \) and \( (\bigotimes_{j=1}^{n+1} \ell^{u_j}, \alpha_s) \approx (\bigotimes_{j=1}^{n+1} \ell^{u_j}, \alpha_r) \). Then \( (\bigotimes_{j=1}^{n+1} \ell^{u_j}, \alpha_s) \) is reflexive by the case S2) and so \( (\bigotimes_{j=1}^{n+1} \ell^{u_j}, \alpha_r) \) does.
Case S4). Assume the existence of \( 1 \leq j_0 \leq n + 1 \) such that \( u'_j = 2, r'_{j_0} \leq 2, u'_j > 2 \) for every \( 1 \leq j \neq j_0 \leq n + 1 \) and (39) holds. Consider the admissible \((n + 2)\)-pla \( s = (s_j)_{j=0}^{n+1} \) such that \( s_{j_0} = 2, s_j := r_j \) for every \( 1 \leq j \neq j_0 \leq n + 1 \) and \( \frac{1}{r_0} := \frac{1}{r_0} + \frac{1}{r'_{j_0}} - \frac{1}{2} \). We obtain from Kwapień’s generalized theorem and Pietsch’s inclusion theorem that \( \mathfrak{H}_r(\prod_{j=1}^{n+1} \ell^u_j, \ell^{u'_0} + 1) \subset \mathfrak{H}_s(\prod_{j=1}^{n+1} \ell^u_j, \ell^{u'_0} + 1) \). The reverse inclusion is true by Kwapień’s factorization theorem and Maurey’s theorem [2, corollary 3, 31.6] because \( \ell^{u_0} = \ell^2 \) has cotype 2 and \( r'_{j_0} \leq 2 \) give \( \mathfrak{H}_2(\ell^2, M) = \mathfrak{H}_0(\ell^2, M) \) for every Banach space \( M \). Then \((\otimes_{j=1}^{n+1} \ell^u_j, \alpha_r) \approx (\otimes_{j=1}^{n+1} \ell^u_j, \alpha_s) \) is reflexive by (39) and the case S2).

**Necessary conditions.** We are going to see that \((\otimes_{j=1}^{n+1} \ell^u_j, \alpha_r) \) is not reflexive if none of the previous conditions holds. It is enough to consider the following cases.

**Case N1.** Assume there exist \( 1 \leq j_0 \leq n \) such that \( u'_j \leq 2 \) and \( 1 \leq j_0 \neq j_1 \leq n + 1 \) such that \( u_j \geq 2 \). After the transposition \( j_1 \rightarrow n + 1, n + 1 \rightarrow j_1 \) on \( \{1, 2, \ldots, n + 1\} \) if necessary, we can assume that \( j_1 = n + 1 \), i.e. \( u_{n+1} \geq 2 \).

For every \( 1 < p < \infty \), let \( \{R_{p,h}\}_{h=1}^\infty \) be the sequence of Rademacher functions in \( L^p([0, 1]) \). It is well known that the sequence \( \{R_{p,h}\}_{h=1}^\infty \) is equivalent to the standard unit basis of \( \ell^2 \) and its closed linear span \( X_p \) is complemented in \( L^p([0, 1]) \) (Khinchine’s inequality and [12, proposition 5]).

Let \( P_{n+1} \in \mathcal{L}(L^{n+1}([0, 1]), X_{n+1}) \) be a projection. Let \( S_{j_0} : \ell^{u'_0} \rightarrow X_{r'_{j_0}} \) be the continuous linear map such that \( S_{j_0}(e_h) = R_{r'_{j_0}, h} \). On the other hand, for every \( 1 \leq j \neq j_0 \leq n \) fix a sequence \( (\alpha_{j,h})_{h=1}^\infty \in \ell^2 \) such that \( \alpha_{j,1} = 1 \) and denote by \( S_j : \ell^{u_j} \rightarrow X_{r'_j} \) the continuous linear map such that \( S_j(e_h) = \alpha_{j,h} R_{r'_j, h} \) (remark that

\[
\|S_j((\beta_h))\| \leq C_j \|((\alpha_{j,h})_h)\|_{\ell^2} \leq C_j \|((\alpha_{j,h})_h)\|_{\ell^\infty} \leq C_j \|((\alpha_{j,h})_h)\|_{\ell^2} \|((\beta_h))\|_{\ell^{u_j}}
\]

for some \( C_j > 0 \) by Khinchine’s inequality).

Take \( g := \prod_{j=1,j \neq j_0}^{n} R_{r'_j, 1} \in L^n([0, 1]) \), and consider the well defined map \( T_{n+1} \in \mathcal{L}(X_{r_{n+1}}, \ell^{u_{n+1}}) \) such that \( T_{n+1}(R_{r_{n+1}, h}) = e_h \) for \( h \in \mathbb{N} \). Then

\[
T := T_{n+1} \circ P_{n+1} \circ D_y \circ \left( S_j \right)_{j=1}^{n}
\]

is \( r \)-integral by theorem 14. Let \( \{ z_{j,h} \}_{h=1}^\infty := \{ (a_{1,h}, a_{2,h}, \ldots, a_{n,h}) \}_{h=1}^\infty \subset \prod_{j=1}^{n} \ell^{u_j} \) such that \( a_{j,h} = e_1 \) if \( j \neq j_0 \) and \( a_{j,h} = e_h \) for every \( h \in \mathbb{N} \). We obtain \( T(z_{j,h}) = e_h \) for every \( h \in \mathbb{N} \) and so \( T \) is not compact. If \( r_0 \neq \infty \), by the remark after theorem 9 we have \( T \notin \mathfrak{H}_r(\prod_{j=1}^{n+1} \ell^{u_j}, \ell^{u_{n+1}}) \) and by theorem 15, \((\otimes_{j=1}^{n+1} \ell^{u_j}, \alpha_r) \) is not reflexive.

In the case \( r_0 = \infty \) we need to consider several possibilities. First assume that there are \( 1 \leq j_2 \neq j_0 \leq n + 1 \) and \( 1 \leq j_3 \neq j_2 \leq n + 1 \) such that \( r'_{j_2} \geq 2 \) and \( r'_{j_3} \geq 2 \). By corollary 6 there is an admissible \((n + 2)\)-pla \( s = (s_j)_{j=0}^{n+1} \) such that \( s_0 \neq \infty \) and
\( \bigotimes_{j=1}^{n+1} \ell^u_j, \alpha_r \) \( \approx \bigotimes_{j=1}^{n+1} \ell^u_j, \alpha_s \). Then by the previous case with \( r_0 \neq \infty \), we see that \( \bigotimes_{j=1}^{n+1} \ell^u_j, \alpha_r \) is not reflexive.

Finally, having (2) in mind, it remains to consider the case that \( r'_{j_0} \leq 2 \) and \( n = 1 \).

We are dealing with \( \ell^u_1 \otimes_{\alpha_r} \ell^u_2 \) where \( u'_1 \leq 2, r'_1 \leq 2 \) and \( u_2 \geq 2 \). By theorems 2 and 7 we have \( \ell^u_1 \otimes_{\alpha_r} \ell^u_2 \) as \( \bigotimes_{j=1}^{n+1} \ell^u_j, \alpha_r \) is bounded. If \( \ell^u_1 \otimes_{\alpha_r} \ell^u_2 \) were reflexive, \( \ell^u_1 \otimes_{\alpha_r} \ell^u_2 \) would be reflexive too and by Smul'yan’s theorem, switching to a suitable subsequence if necessary, we would assume that \( \{ e_i \otimes e_i \}_{i=1}^{\infty} \) is weakly convergent to some \( z \in \ell^u_1 \otimes_{\alpha_r} \ell^u_2 \). It follows from boundedness of \( K \) and the density of \( \{ e_h \}_{h=1}^{\infty} \otimes [e_h]_{h=1}^{\infty} \) in \( \ell^u_1 \otimes_{\alpha_r} \ell^u_2 \) that given \( T \in \ell^u_1 \otimes_{\alpha_r} \ell^u_2 \) and \( \rho > 0 \), there exist \( w \in \bigcup_{k=N}^{\infty} [e_h]_{h=1}^{k} \) such that

\[
\forall \ m \geq m_0 \quad | \langle T, z \rangle | \leq | \langle T, z - e_m \otimes e_m \rangle | + | \langle T - w, e_m \otimes e_m \rangle | + | \langle w, e_m \otimes e_m \rangle | \leq \rho
\]

because \( \langle w, e_m \otimes e_m \rangle = 0 \) if \( m \) is large enough. Then \( z = 0 \). But we are assuming that \( J_r(\ell^{u_1}, \ell^{u_2}) = (\ell^u_1 \otimes_{\alpha_r} \ell^u_2)' = \ell^u_1 \otimes_{\alpha_r} \ell^u_2 \) and so, by the construction made in the case \( r_0 \neq \infty \) there is \( T \in \ell^u_1 \otimes_{\alpha_r} \ell^u_2 \) such that \( \langle T(e_i), e_i \rangle = \langle e_i, e_i \rangle = 1 \) for every \( i \in \mathbb{N} \), a contradiction. Then \( \ell^u_1 \otimes_{\alpha_r} \ell^u_2 \) is not reflexive.

**Case N2).** Assume that \( u'_1 \geq 2 \) for every \( 1 \leq j \leq n \), \( r'_j \geq 2 \) for every \( 1 \leq j \leq n + 1 \), \( u'_{n+1} \leq r'_{n+1} \) and \( \frac{1}{r_0} \leq \sum_{j=1}^{n+1} \frac{1}{u'_j} \), or equivalently (by (5))

\[
\frac{1}{r_0} + \sum_{j=1}^{n+1} \left( \frac{1}{r_j'} - 1 \right) \leq \sum_{j=1}^{n+1} \left( \frac{1}{u'_j} - 1 \right) \quad (45)
\]

Given \( 1 \leq j \leq n \), if \( r'_j < u'_j \) and \( t'_j \in [u'_j, \infty[ \) it turns out that we have

\[
\frac{1}{r_0} + \sum_{j \in \{ j : r'_j < u'_j \} \} \left( \frac{1}{r'_j} - 1 \right) \leq \left[ \frac{1}{r_0} + \sum_{j \in \{ j : r'_j < u'_j \} \} \left( \frac{1}{r'_j} - 1 \right) \right] , \frac{1}{r_0} + \sum_{j \in \{ j : r'_j < u'_j \} \} \frac{1}{r'_j} \leq \left[ \frac{1}{r_0} + \sum_{j \in \{ j : r'_j < u'_j \} \} \left( \frac{1}{r'_j} - 1 \right) \right] .
\]

On the other hand, if \( r'_j \geq u'_j \) and \( t'_j \in [u'_j, r'_j) \) we have

\[
\sum_{j \in \{ j : r'_j \geq u'_j \} \} \left( \frac{1}{t'_j} - 1 \right) \in \left[ 0, \sum_{j \in \{ j : r'_j \geq u'_j \} \} \left( \frac{1}{u'_j} - 1 \right) \right] .
\]
Then it follows from (45) that we can choose \( t_j' \geq u_j' \) for every \( 1 \leq j \leq n \) such that \( r_j' < u_j' \) and \( u_j' \leq t_j' \leq r_j' \) for every \( 1 \leq j \leq n \) which verifies \( u_j' \leq r_j' \) in order that
\[
\frac{1}{r_0} + \sum_{\{j \mid r_j' < u_j'\}} \left( \frac{1}{r_j'} - \frac{1}{u_j'} \right) = \sum_{\{j \mid r_j' \geq u_j'\}} \left( \frac{1}{r_j'} - \frac{1}{u_j'} \right).
\]
By (2) we have
\[
\frac{1}{r_{n+1}} = \sum_{j=1}^{n} \frac{1}{t_j'} + \sum_{\{j \mid r_j' < u_j'\}} \left( \frac{1}{r_j'} - \frac{1}{t_j'} \right) + \sum_{\{j \mid r_j' \geq u_j'\}} \left( \frac{1}{r_j'} - \frac{1}{t_j'} \right) + \frac{1}{r_0} = \sum_{j=1}^{n} \frac{1}{t_j'}.
\]
Taking \( t_0 = \infty \) and \( t_{n+1} = r_{n+1} \) we obtain an admissible \((n+2)\)-pla \( t = (t_j')_{j=0}^{n+2} \) such that \( t_j' \geq u_j' \geq 2 \) for every \( 1 \leq j \leq n \). By corollary 5 we have \((\mathcal{X}_{j=1}^{n+1} \ell^{u_j'}, \alpha_t) \approx (\mathcal{X}_{j=1}^{n+1} \ell^{u_j'}, \alpha_t)\) and so \( \mathcal{J}_t(\prod_{j=1}^{n} \ell^{u_j'}, \ell^{u_{n+1}}) = \mathcal{J}_r(\prod_{j=1}^{n} \ell^{u_j'}, \ell^{u_{n+1}}) \). But by lemma 16 there is a non compact map \( S \in \mathcal{J}_t(\prod_{j=1}^{n} \ell^{u_j'}, \ell^{u_{n+1}}) \). Now we take \( s_j' = t_j' \) if \( 1 \leq j \leq n \), \( s_{n+1}' > t_{n+1}' \) and define \( s_0 < \infty \) such that \( \frac{1}{s_0} := \frac{1}{t_0} - \frac{1}{r_{n+1}} \). Then \( S = (s_j')_{j=0}^{n+1} \) is another admissible \((n+2)\)-pla verifying \((\mathcal{X}_{j=1}^{n+1} \ell^{u_j'}, \alpha_s) \approx (\mathcal{X}_{j=1}^{n+1} \ell^{u_j'}, \alpha_t)\) corollary 6 and \( S \in \mathcal{J}_s(\prod_{j=1}^{n} \ell^{u_j'}, \ell^{u_{n+1}}) \). By remark after theorem 9 we have \( S \notin \mathcal{G}_{n+1}(\prod_{j=1}^{n} \ell^{u_j'}, \ell^{u_{n+1}}) \) and by theorem 15 \((\mathcal{X}_{j=1}^{n+1} \ell^{u_j'}, \alpha_t) \approx (\mathcal{X}_{j=1}^{n+1} \ell^{u_j'}, \alpha_s)\) turns out to be not reflexive.

**Case N3.** Assume that \( u_j' \geq 2 \) for every \( 1 \leq j \leq n+1 \), \( r_j' \leq 1 \) and \( \frac{1}{2} < \sum_{j=1}^{n} \frac{1}{u_j'} \), or, in an equivalent form (by (2))
\[
\frac{1}{r_0} + \frac{1}{r_{n+1}} - \frac{1}{2} + \sum_{\{j \mid r_j' < u_j'\}} \left( \frac{1}{r_j'} - \frac{1}{u_j'} \right) \leq \sum_{\{j \mid r_j' \geq u_j'\}} \left( \frac{1}{r_j'} - \frac{1}{u_j'} \right).
\]
By (2) we have \( r_j' \geq 2, 1 \leq j \leq n \). Defining \( \frac{1}{s_0} := \frac{1}{r_0} + \frac{1}{r_{n+1}} - \frac{1}{2}, s_j' := r_j', 1 \leq j \leq n \) and \( s_{n+1} := 2 \) we obtain an admissible \((n+2)\)-pla \( s = (s_j')_{j=0}^{n+1} \) such that, \( \ell^{u_{n+1}} \) having cotype 2, by corollary 4 one has \((\mathcal{X}_{j=1}^{n+1} \ell^{u_j'}, \alpha_s) \approx (\mathcal{X}_{j=1}^{n+1} \ell^{u_j'}, \alpha_t)\). Then \((\mathcal{X}_{j=1}^{n+1} \ell^{u_j'}, \alpha_s)\) turns out to be not reflexive by the case N2), obtaining the desired conclusion by isomorphism.

**Case N4.** Assume there are \( 1 \leq j_0 \leq n \) and \( 1 \leq j_1 \neq j_0 \leq n+1 \) such that \( u_{j_0} < 2, r_{j_0} < 2 \) and \( r_{j_1} \leq w_{j_1} \).

a) First we consider the case that \( n \geq 2 \). By (2) necessarily exist \( 1 \leq j_2 \neq j_3 \leq n+1 \) such that \( r_{j_2}' \geq 2 \) and \( r_{j_3}' \geq 2 \) and so, by corollary 6 and eventually switching to an isomorphic tensor product \((\mathcal{X}_{j=1}^{n+1} \ell^{u_j'}, \alpha_s)\), we can suppose moreover, that \( r_0 < \infty \).
After the transposition \( j_1 \rightarrow n + 1, n + 1 \rightarrow j_1 \) if necessary we can assume that \( j_1 = n + 1 \), i.e. \( r_{n+1} \leq u_{n+1} \) indeed. If there exists \( 1 \leq j_4 \neq j_0 \leq n + 1 \) such that \( u_{j_4} \leq 2 \), the result follows from case N1). Hence we can assume \( u_j > 2 \) for every \( 1 \leq j \neq j_0 \leq n + 1 \).

Fix \( t < 2 \) such that \( r_{j_0} < t, u_{j_0} < t \) and \( u_{n+1} < t \). Let \( \{ \varphi_k \}_{k=1}^{\infty} \) be a sequence of standard independent identically distributed \( t \)-stable random variables in \([0, 1]\). It is known that the norm \( K_{t,p} := \| \varphi_k \|_{L^p([0,1]), k \in \mathbb{N}} \) is only dependent on \( t \) and \( p \) for every \( 1 \leq p < 2 \) and that \( \{ \Phi_{k,p} := \frac{\varphi_k}{K_{t,p}} \}_{k=1}^{\infty} \) is isometrically equivalent in \( L^p([0,1]) \), \( 1 \leq p < t \) to the canonical basis of \( \ell^t \) (see \([6], \) proposition IV.4.10 \) for example). Then \( \{ \Phi_{k,r_{n+1}} \}_{k=1}^{\infty} \) is a normalized basis in the reflexive subspace \( [\Phi_{k,r_{n+1}}]_{k=1}^{\infty} \approx \ell^t \) of \( L^r_{n+1}([0,1]) \) and thus it is weakly convergent to 0 in \( L^r_{n+1}([0,1]) \) (see \([7], \) footnote page 169 \) for instance). Switching to a suitable subsequence if necessary, by \([18], \) chapter III, theorem 1.8 \), the sequence \( \{ \Phi_k, r_{n+1} \}_{k=1}^{\infty} \) can be enlarged to obtain a normalized basis \( \mathcal{B} := \{ \Phi_{k,r_{n+1}} \}_{k=1}^{\infty} \cup \{ \psi_m \}_{m=1}^{\infty} \) in \( L^r_{n+1}([0,1]) \). By reflexivity the sequence \( \{ \Phi_{k,r_{n+1}} \}_{k=1}^{\infty} \cup \{ \psi_m \}_{m=1}^{\infty} \) of associated coefficient functionals to \( \mathcal{B} \) is a basis in \( L^{r_{n+1}}([0,1]) \). From \([18], \) chapter I, theorem 3.1 \) we find \( 1 \leq M \in \mathbb{R} \) such that \( 1 \leq \| \Phi_{k,r_{n+1}} \| \leq M \) and \( 1 \leq \| \psi_m \| \leq M \) for every \( k \in \mathbb{N} \). As above we obtain that \( \{ \Phi_{k,r_{n+1}} \}_{k=1}^{\infty} \) must be weakly convergent to 0. As \( r_{n+1} > 2 \), by the result \([7], \) corollary 5 \) of Kadec and Pečiński, switching to a subsequence again, it can be assumed that \( \{ \Phi_{k,r_{n+1}} \}_{k=1}^{\infty} \) is equivalent to the standard unit basis in \( \ell^{r_{n+1}} \) or to the standard unit basis in \( \ell^2 \). By \([7], \) corollary 1 \), the latter possibility would imply that \( \{ \Phi_{k,r_{n+1}} \}_{k=1}^{\infty} \) would be complemented in \( L^r_{n+1}([0,1]) \) and by reflexivity and duality, we would have the isomorphisms \( \big( \{ \Phi_{k,r_{n+1}} \}_{k=1}^{\infty} \big)' \approx \{ \Phi_{k,r_{n+1}} \}_{k=1}^{\infty} \approx \ell^t \approx \ell^2 \) which is not possible. Then \( \{ \Phi_{k,r_{n+1}} \}_{k=1}^{\infty} \) is equivalent to the standard basis of \( \ell^{r_{n+1}} \) and so, the map \( V \in \mathcal{L}(\ell^{r_{n+1}}, L^r_{n+1}([0,1])) \) such that \( V(\mathbf{e}_h) = \Phi_{h,r_{n+1}}, h \in \mathbb{N} \) is well defined.

Let \( S_j \in \mathcal{L}(\ell^{r_0}, L^{r_0}([0,1])) \), \( 1 \leq j \neq j_0 \leq n \) be defined as in previous case N1) and consider \( S_{j_0} \in \mathcal{L}(\ell^{r_{j_0}}, L^{r_{j_0}}([0,1])) \) such that \( S_{j_0}(\mathbf{e}_k) = \Phi_{k,r'} \) for every \( k \in \mathbb{N} \). Taking \( g \) as in case N1), the map \( T := V' \circ D_g \circ (S_j)_{j=1}^{n} \) is \( r \)-integral. However, for every \( k \in \mathbb{N} \) and every \( (\gamma_h)_{h \in \mathbb{N}} \in \ell^{r_{n+1}} \) we have

\[
\left( T(z_{j_0}, k), (\gamma_h) \right) = \left( \frac{K_{t,r_{n+1}}}{K_{t,r_{j_0}}} \Phi_{k,r_{n+1}} \sum_{h=1}^{\infty} \gamma_h \Phi_{h,r_{n+1}} \right) = \frac{K_{t,r_{n+1}}}{K_{t,r_{j_0}}} \gamma_k
\]

and so \( T(z_{j_0}) = \frac{K_{t,r_{n+1}}}{K_{t,r_{j_0}}} \mathbf{e}_k \) and \( T \) is not compact. By remark after theorem 9 we obtain \( T \notin \mathcal{N}(\prod_{j=1}^{n} \ell^{r_{j}}, \sigma_\tau \) and by theorem 15 (\( \bigotimes_{j=1}^{n+1} \ell^{r_{j}}, \sigma_\tau \) is not reflexive.

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b) Now we consider the case \( n = 1 \). If \( r_0 \neq \infty \) the previous argumentation can be used still and \( \ell^{u_1} \otimes_{\alpha_r} \ell^{u_2} \) is not reflexive. If \( r_0 = \infty \), after an eventual transposition, we will be dealing with the case \( u'_1 \leq 2, r'_1 < 2 \) and \( r_2 \leq u_2 \). If \( u_2 \geq 2 \) the result follows from \((N1)\). If \( u_2 < 2 \) and \( u'_1 = 2 \) we repeat the proof given in this case for \( n \geq 2 \) and \( \ell^{u_1} \otimes_{\alpha_r} \ell^{u_2} \) turns out to be non-reflexive. If \( u_2 < 2 \) and \( u'_1 < 2 \) the same construction just used in the case \( n \geq 2 \) show the existence of a map \( T \in \mathcal{J}_r(\ell^{u_1}, \ell^{u_2}) \) such that \( T(e_i) = \frac{k_{i,j}}{k_{i,j}} e_i \) for every \( i \in \mathbb{N} \). Then we can repeat the argumentation used in the last part of \((N1)\) with the set \( K := \{ e_i \otimes e_i, \ i \in \mathbb{N} \} \subset \ell^{u_1} \otimes \ell^{u_2} \) to conclude that \( \bigotimes_{j=1}^{n+1} \ell^{u_j}, \alpha_r \) is not reflexive.

Finally we check that the proof of theorem 17 is complete. Assume that neither condition \((S1)\), \((S2)\), \((S3)\), \((S4)\) holds.

a) First case: assume there is \( 1 \leq j_0 \leq n + 1 \) such that \( u'_{j_0} \leq 2 \). After an eventual transposition with any \( 1 \leq k \neq j_0 \leq n + 1 \), we can take \( j_0 \leq n \). If there is some \( 1 \leq j_1 \neq j_0 \leq n + 1 \) such that \( u'_{j_1} \leq 2 \), by \((N1)\), \( \bigotimes_{j=1}^{n+1} \ell^{u_j}, \alpha_r \) is not reflexive. Then we can assume \( u'_{j_0} > 2, 1 \leq j \neq j_0 \leq n + 1 \). As \((S1)\) does not holds, there exists \( j_1 \neq j_0 \) such that \( r_{j_1} \leq u_{j_1} \). If it would be \( u'_{j_0} < 2 \) and \( r'_{j_0} < 2 \), \( \bigotimes_{j=1}^{n+1} \ell^{u_j}, \alpha_r \) would be not reflexive by \((N4)\). If \( u'_{j_0} = 2 \) and \( r'_{j_0} < 2 \), as \((S4)\) does not holds, after the transposition \( j_0 \to n + 1, n + 1 \to j_0 \), by \((N3)\) \( \bigotimes_{j=1}^{n+1} \ell^{u_j}, \alpha_r \) is not reflexive.

In the case \( r'_{j_0} \geq 2 \), by \((2)\) there is at most an unique \( 1 \leq j_2 \leq n + 1 \) such that \( r'_{j_2} < 2 \). Necessarily \( j_2 \neq j_0 \). As \((S2)\) does not holds, after an eventual transposition \( j_0 \to n + 1, n + 1 \to j_0 \), we see that \( u'_{n+1} \leq 2 \leq r'_{n+1} \) and by \((N2)\) \( \bigotimes_{j=1}^{n+1} \ell^{u_j}, \alpha_r \) is not reflexive.

b) Second case: assume that \( u'_j > 2, 1 \leq j \leq n + 1 \). As \((S1)\) does not holds, after an eventual transposition, it turns out that \( u'_{n+1} \leq r'_{n+1} \). But \((S3)\) is not verified. Then for every \( 1 \leq j_0 \leq n + 1 \) we have \( r'_{j_0} > 2 \) or \((38)\) does not holds. If it would be \( r'_j > 2 \) for every \( 1 \leq j \leq n + 1 \), as \((S2)\) is not verified, \( \bigotimes_{j=1}^{n+1} \ell^{u_j}, \alpha_r \) would be not reflexive by \((N3)\). If it would exists \( 1 \leq j_1 \leq n + 1 \) such that \( r'_{j_1} \leq 2 \), then \((38)\) would fails for this index \( j_1 \). After an evident transposition, by \((N3)\) \( \bigotimes_{j=1}^{n+1} \ell^{u_j}, \alpha_r \) would be not reflexive.

The application of theorem 17 to the case \( n = 1 \) gives the following characterization of reflexivity of classical Lapresté’s tensor products:

**Corollary 18** Let \( n = 1 \) and let \( r = (r_0, r_1, r_2) \) be an admissible triple. If \( 1 < u_1, u_2 < \infty, \ell^{u_1} \otimes_{\alpha_r} \ell^{u_2} \) is reflexive if and only if all of the following sets of conditions holds

1) \( u'_1 > 2, u'_1 > r'_1 \).
2) \( u'_2 > 2, u'_2 > r'_2 \).
3) \( u_1' > 2, r_2 \leq 2 \).
4) \( u_2' > 2, r_1 \leq 2 \).
5) \( u_1' \geq 2, u_2' > 2 \).
6) \( u_1' > 2, u_2' \geq 2 \).

**Proof.** By theorem 17, \( \ell^{u_1} \otimes_{\sigma_{ar}} \ell^{u_2} \) is reflexive if and only if one of the following sets of conditions holds

- a) \( u_1' > 2, u_1' > r_1' \).
- b) \( u_2' > 2, u_2' > r_2' \).
- c) \( u_1' > 2, u_1' > r_2, r_1' \geq 2 \).
- d) \( u_2' > 2, u_2' > r_1, r_2' \geq 2 \).
- e) \( u_1' > 2, u_2' > 2, r_1' \leq 2 \).
- f) \( u_1' > 2, u_2' > 2, r_2' \leq 2 \).
- g) \( u_1' = 2, u_2' > 2, r_1' \leq 2 \).
- h) \( u_2' = 2, u_1' > 2, r_2' \leq 2 \).

Clearly c) and 3) (resp. d) and 4) ) are equivalent. On the other hand, if 5) holds and \( r_1' \leq 2 \) then e) or g) holds. If 5) and \( r_1' > 2 \) are true we have \( r_1 < 2 < u_2' \) and d) is verified. The remaining of the proof is similar or trivial. 

**References**


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