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A Method to Solve Non-homogeneous Strongly Coupled Mixed Parabolic Boundary Value Systems with Non-homogeneous Boundary Conditions

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Abstract

In this paper, a method to construct the solution of non-homogeneous parabolic coupled systems with non-homogeneous boundary conditions of the type $u_t - Au_{xx} = G(x,t)$, $A_1u(0,t)+B_1u_x(0,t) = P(t)$, $A_2u(l,t)+B_2u_x(l,t) = Q(t)$, $0 < x < 1$, $t > 0$, $u(x,0) = f(x)$, where A is a positive stable matrix and A_1, A_2, B_1, B_2 are arbitrary matrices for which the block matrix $\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix}$ is non-singular, is proposed. Two illustrative examples of the method are given.

Mathematics Subject Classification: 35K50

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1 Introduction

Coupled partial differential systems with coupled boundary-value conditions are frequent in different areas of science and technology. Recently, an exact series solution for the homogeneous initial-value problem

$$\begin{aligned} u_t(x,t) - Au_{xx}(x,t) &= 0\ ,\ 0 < x < 1\ ,\ t > 0 && (1)\\ A_1u(0,t) + B_1u_x(0,t) &= 0\ ,\ t > 0 && (2)\\ A_2u(1,t) + B_2u_x(1,t) &= 0\ ,\ t > 0 && (3)\\ u(x,0) &= f(x)\ ,\ 0 \leq x \leq 1\ , && (4) \end{aligned}$$

where $u = (u_1, u_2, \ldots, u_m)^T$ and $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))^T$ are a $m-$dimensional vectors, was constructed under the following hypotheses and notation:

1. The matrix coefficient A is a matrix which satisfies the following condition

$$\mathrm{Re}(z) > 0\ ,\ \forall z \in \sigma(A), \tag{5}$$

where $\sigma(C)$ denotes the set of all the eigenvalues of a matrix C in $\mathbb{C}^{m\times m}$. Thus A is a positive stable matrix (where $\mathrm{Re}(z)$ denotes the real part of $z \in \mathbb{C}$).

2. Matrices $A_i, B_i, i = 1, 2$, are $m \times m$ complex matrices, and we assume that the block matrix

$$\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} \ \textit{is regular}\ , \tag{6}$$

and also that the matrix pencil

$$A_1 + \rho B_1\ \ \textit{is regular}. \tag{7}$$

Condition (7) is well known in the literature of singular systems of differential equations, see [1], and involves the existence of some $\rho_0 \in \mathbb{C}$ so that matrix $A_1 + \rho_0 B_1$ is invertible. In this case, matrix $A_1 + \rho B_1$ is invertible with the possible exception of at most a finite number of complex numbers ρ. In particular, we may assume that $\rho_0 \in \mathbb{R}$.

Using condition (7) we can introduce the following matrices $\widetilde{A}_1$ and $\widetilde{B}_1$ defined by

$$\widetilde{A}_1 = (A_1 + \rho_0 B_1)^{-1} A_1 \ , \ \widetilde{B}_1 = (A_1 + \rho_0 B_1)^{-1} B_1, \tag{8}$$

which satisfy the condition $\widetilde{A}_1 + \rho_0 \widetilde{B}_1 = I$, where matrix I denotes, as usual, the identity matrix. Under hypothesis (6), is it easy to show that matrix $B_2 - (A_2 + \rho_0 B_2)\widetilde{B}_1$ is regular and we can introduce matrices $\widetilde{A}_2$ and $\widetilde{B}_2$ defined by

$$\widetilde{A}_2 = \left[B_2 - (A_2 + \rho_0 B_2)\widetilde{B}_1\right]^{-1} A_2 \ , \ \widetilde{B}_2 = \left[B_2 - (A_2 + \rho_0 B_2)\widetilde{B}_1\right]^{-1} B_2, \tag{9}$$

that satisfy the conditions $\widetilde{B}_2 - \left(\widetilde{A}_2 + \rho_0 \widetilde{B}_2\right)\widetilde{B}_1 = I, \widetilde{B}_2\widetilde{A}_1 - \widetilde{A}_2\widetilde{B}_1 = I$.

Under the above assumptions, the homogeneous problem (1)–(4) was solved in [2, 3] under two different cases:

(a) We can consider the following hypothesis:

$$\begin{array}{c} \textit{exist } b_1 \in \sigma\left(\widetilde{B}_1\right) - \{0\}, b_2 \in \sigma\left(\widetilde{B}_2\right), \textit{ and } v \in \mathbb{C}^m - \{0\}, \\ \textit{such that } \left(\widetilde{B}_1 - b_1 I\right) v = \left(\widetilde{B}_2 - b_2 I\right) v = 0 \ . \end{array} \tag{10}$$

Then, if the vector valued function $f(x)$ satisfies hypotheses

$$\left.\begin{array}{c} f \in \mathcal{C}^2([0,1]) \\ (1 - \rho_0 b_1) f(0) + b_1 f'(0) = 0 \\ -\left(\dfrac{1 - b_2 + \rho_0 b_1 b_2}{b_1}\right) f(1) + b_2 f'(1) = 0 \end{array}\right\}, \tag{11}$$

with the additional condition:

$$\begin{array}{c} f(x) \in Ker\left(\widetilde{B}_1 - b_1 I\right) \bigcap Ker\left(\widetilde{B}_2 - b_2 I\right) \ , \ 0 \le x \le 1 \\ \textit{and} \\ Ker\left(\widetilde{B}_1 - b_1 I\right) \bigcap Ker\left(\widetilde{B}_2 - b_2 I\right) \textit{ is an invariant subspace with respect to matrix } A, \end{array} \tag{12}$$

where a subspace E of $\mathbb{C}^m$ is invariant by the matrix $A \in \mathbb{C}^{m\times m}$, if $A(E) \subset E$, we can construct an exact series solution $u(x,t)$ of homogeneous problem (1)–(4). This construction was made in Ref. [2].

(b) We can consider the following hypothesis:

$$0 \in \sigma\left(\widetilde{B}_1\right),\ a_2 \in \sigma\left(\widetilde{A}_2\right),\ \textit{and we have}\ w \in \mathbb{C}^m - \{0\}, \\ \textit{so that}\ \widetilde{B}_1 w = \left(\widetilde{A}_2 - a_2 I\right) w = 0\ . \tag{13}$$

Then, if the vector valued function $f(x)$ satisfies the hypotheses

$$\left.\begin{array}{c} f \in \mathcal{C}^2\left([0,1]\right) \\ \\ f(0) = 0 \\ \\ a_2 f(1) + f'(1) = 0 \end{array}\right\}, \tag{14}$$

under the additional condition:

$$\begin{array}{c} f(x) \in Ker\left(\widetilde{B}_1\right) \cap Ker\left(\widetilde{A}_2 - a_2 I\right)\ ,\ 0 \leq x \leq 1 \\ \textit{and} \\ Ker\left(\widetilde{B}_1\right) \cap Ker\left(\widetilde{A}_2 - a_2 I\right)\quad \textit{is an invariant subspace respect to matrix } A, \end{array} \tag{15}$$

then we can construct an exact series solution $u(x,t)$ of homogeneous problem (1)–(4). This construction was made in Ref. [3].

By other hand, the solution of the non-homogeneous problem

$$\begin{aligned} u_t(x,t) - A u_{xx}(x,t) &= G(x,t)\ ,\ 0 < x < 1\ ,\ t > 0 && (16) \\ A_1 u(0,t) + B_1 u_x(0,t) &= 0\ ,\ t > 0 && (17) \\ A_2 u(1,t) + B_2 u_x(1,t) &= 0\ ,\ t > 0 && (18) \\ u(x,0) &= f(x)\ ,\ 0 \leq x \leq 1\ , && (19) \end{aligned}$$

was made in Ref. [4] under the two different hypotheses (a) and (b).

This paper deals a method to construct the exact solution of the non-homogeneous problem with non-homogeneous conditions

$$\begin{aligned} u_t(x,t) - A u_{xx}(x,t) &= G(x,t)\ ,\ 0 < x < 1\ ,\ t > 0 \\ A_1 u(0,t) + B_1 u_x(0,t) &= P(t)\ ,\ t > 0 \\ A_2 u(1,t) + B_2 u_x(1,t) &= Q(t)\ ,\ t > 0 \\ u(x,0) &= f(x)\ ,\ 0 \leq x \leq 1\ , \end{aligned}$$

in term of solutions of problems of the type (16)-(19). Throughout this paper we will assume the results and nomenclature given in [2, 3, 4]. This paper is organized as follows: In section 2 a method to construct a solution of (16)–(19) is obtained. In section 3 an algorithm and two illustrative examples are given. Conclusion are presented in section 4.

2 The proposed method

We consider the non-homogeneous problem with non-homogeneous conditions

$$\begin{aligned} u_t(x,t) - Au_{xx}(x,t) &= G(x,t)\ ,\ 0 < x < 1\ ,\ t > 0 && (20)\\ A_1u(0,t) + B_1u_x(0,t) &= P(t)\ ,\ t > 0 && (21)\\ A_2u(1,t) + B_2u_x(1,t) &= Q(t)\ ,\ t > 0 && (22)\\ u(x,0) &= f(x)\ ,\ 0 \le x \le 1\ , && (23) \end{aligned}$$

where $u(x,t)$, $G(x,t)$, $P(t)$, $Q(t)$ and $f(x)$ are vectors in $\mathbb{C}^m$, and matrices $A_1, A_2, B_1, B_2 \in \mathbb{C}^{m\times m}$ satisfying the conditions (5) and (6)–(7).

We are looking for a solution of (20)–(23) in the form

$$u(x,t) = w(x,t) + v(x,t), \tag{24}$$

where function $v(x,t)$ satisfies the conditions

$$\left.\begin{aligned} A_1v(0,t) + B_1v_x(0,t) &= P(t)\ ,\ t > 0,\\ A_2v(1,t) + B_2v_x(1,t) &= Q(t)\ ,\ t > 0. \end{aligned}\right\}. \tag{25}$$

Thus, we can define the function $G_1(x,t)$ as

$$G_1(x,t) = v_t(x,t) - Av_{xx}(x,t) \tag{26}$$

then $v(x,t)$ satisfies:

$$\left.\begin{aligned} v_t(x,t) - Av_{xx}(x,t) &= G_1(x,t)\ ,\ 0 < x < 1\ \ ,\ \ t > 0\\ A_1v(0,t) + B_1v_x(0,t) &= P(t)\ ,\ t > 0\\ A_2v(1,t) + B_2v_x(1,t) &= Q(t)\ ,\ t > 0 \end{aligned}\right\}$$

which implies that $w(x,t)$ must satisfy

$$\begin{aligned} w_t(x,t) - Aw_{xx}(x,t) &= G(x,t) - G_1(x,t) \\ &= \widetilde{G}(x,t)\ ,\ 0 < x < 1\ ,\ t > 0 \end{aligned}$$

with the homogeneous conditions:

$$\left.\begin{aligned} A_1 w(0,t) + B_1 w_x(0,t) &= 0\ ,\ t > 0, \\ A_2 w(1,t) + B_2 w_x(1,t) &= 0\ ,\ t > 0. \end{aligned}\right\} .$$

and the initial condition:

$$\begin{aligned} w(x,0) &= f(x) - v(x,0) \\ &= \widetilde{f}(x)\ ,\ 0 \le x \le 1. \end{aligned}$$

Then, function $u(x,t)$ defined by (24) satisfy:

$$\begin{aligned} u_t(x,t) - Au_{xx}(x,t) &= v_t(x,t) - Av_{xx}(x,t) + w_t(x,t) - Aw_{xx}(x,t) \\ &= G_1(x,t) + G(x,t) - G_1(x,t) \\ &= G(x,t), \end{aligned}$$

with the boundary conditions (21)-(22) and the initial condition (23), so it is the desired solution of our problem (20)–(23).

Summarizing, the following theorem has been proved:

Theorem 2.1 *Let be consider the problem (20)–(23). Let $v(x,t)$ be a vector valued function satisfying conditions (25). We define the vector valued functions*

$$\widetilde{G}(x,t) = G(x,t) - G_1(x,t)\ ,\ \widetilde{f(x)} = f(x) - v(x,0),$$

where $G_1(x,t)$ is given by (26). We consider the non-homogeneous problem with homogeneous conditions

$$\begin{aligned} w_t(x,t) - Aw_{xx}(x,t) &= \widetilde{G}(x,t)\ ,\ 0 < x < 1\ ,\ t > 0 && (27) \\ A_1 w(0,t) + B_1 w_x(0,t) &= 0\ ,\ t > 0 && (28) \\ A_2 w(1,t) + B_2 w_x(1,t) &= 0\ ,\ t > 0 && (29) \\ w(x,0) &= \widetilde{f(x)}\ ,\ 0 \le x \le 1\ , && (30) \end{aligned}$$

which solution $w(x,t)$ can be obtain using Theorem 2.1 of Ref. [4] if conditions established in this theorem holds. Then, $u(x,t) = v(x,t) + w(x,t)$ is a solution of problem (20)-(23).

3 Algorithm and Examples

We can establish the following algorithm to solve problem (20)–(23):

Algorithm 1 Solution of problem (20)-(23).
Input data: Matrices $A, A_1, A_2, B_1, B_2 \in \mathbb{C}^{m\times m}$, vectors $G(x), f(x) \in \mathbb{C}^m$.
Result obtained: If the stated assumptions are met, the series solution $u(x,t)$.

1: Determine a vector valued function $v(x,t)$ satisfying conditions (25).
2: Determine $\widetilde{G}(x,t) = G(x,t) - G_1(x,t)$ and $\widetilde{f(x)} = f(x) - v(x,0)$.
3: Using the Algorithm given in Ref. [4] determine, if it is possible, a solution $w(x,t)$ of problem (27)–(30).
4: Determine the solution of problem (20)–(23) as $u(x,t) = w(x,t) + v(x,t)$.

Of course, the choice of the function $v(x,t)$ determine the choice of the functions $\widetilde{G}(x,t)$ and $\widetilde{f}(x)$, which must satisfy the hypotheses of Theorem 2.1 of Ref. [4], and depend on the nature of the given function $G(x,t)$. Here we present two different examples.

Example 3.1 *We consider problem (20)–(23) where function $G(x,t)$ is a linear combination of functions* $\sin(\pi x)$ *and* $\cos(\pi x)$*. Then, we will look for a function $v(x,t)$ which is also a linear combination of functions* $\sin(\pi x)$ *and* $\cos(\pi x)$ *with coefficients are functions of variable t. Thus, we look for a solution of (25) given in the form*

$$v(x,t) = R_1(t)\sin(\pi x) + R_2(t)\cos(\pi x)\ , \tag{31}$$

where vector-valued functions $R_i(t) \in \mathcal{C}^1[0,+\infty), i = 1,2$ must be determinate. This solution (31) must to satisfy boundary conditions (25):

$$\left.\begin{array}{rclcrcl} A_1 v(0,t) + B_1 v_x(0,t) &=& P(t) & \Longrightarrow A_1 R_2(t) + \pi B_1 R_1(t) &=& P(t) \\ A_2 v(1,t) + B_2 v_x(1,t) &=& Q(t) & \Longrightarrow -A_2 R_2(t) - \pi B_2 R_1(t) &=& Q(t) \end{array}\right\} \tag{32}$$

Writing (32) in matrix form:

$$\begin{pmatrix} A_1 & B_1 \\ -A_2 & -B_2 \end{pmatrix} \begin{pmatrix} R_2(t) \\ \pi R_1(t) \end{pmatrix} = \begin{pmatrix} P(t) \\ Q(t) \end{pmatrix} . \tag{33}$$

Premultiplying (33) by the invertible matrix $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ *one gets*

$$\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} \begin{pmatrix} R_2(t) \\ \pi R_1(t) \end{pmatrix} = \begin{pmatrix} P(t) \\ -Q(t) \end{pmatrix} . \tag{34}$$

Under hypothesis (6) this system has an unique solution. Thus, we have shown that we can determine a vector valued function $v(x,t)$ satisfying conditions (25) and defined by expression (31). Thus, we have now that

$$\begin{aligned} G_1(x,t) &= v_t(x,t) - Av_{xx}(x,t) \\ &= R_1'(t)\sin(\pi x) + R_2'(t)\cos(\pi x) + \pi^2 Av(x,t), \end{aligned} \tag{35}$$

and we can apply Theorem 2.1. We will consider a concrete numerical example. Consider problem (20)-(23) where matrix $A \in \mathbb{C}^{4\times 4}$ is given by

$$A = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 1 & 2 & 0 & -2 \\ -1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{36}$$

and the 4×4 matrices $A_i, B_i, i \in \{1,2\}$, are

$$\begin{aligned} A_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \quad A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ B_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad B_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{37}$$

The vectorial valued function $f(x)$ is defined by

$$f(x) = \begin{pmatrix} 0 \\ 0 \\ x^2 - 2x \\ 0 \end{pmatrix}, \tag{38}$$

function $G(x,t)$ is

$$G(x,t) = \begin{pmatrix} -\cos(\pi x)\left(\cos(t) + 2\left(t + \pi^2 t^2 + \pi^2 \sin(t)\right)\right) + \dfrac{\sin(\pi x)}{\pi}\left(2t\left(1+\pi^2 t\right)\right) \\ -\cos(\pi x)\left(t\left(2 + 3\pi^2 t\right) + \pi^2 \sin(t)\right) + \pi t^2 \sin(\pi x) \\ e^{-t}(-1+x)^2 x + \pi^2 \cos(\pi x)\left(t^2 + \sin(t)\right) - \pi t^2 \sin(\pi x) \\ 0 \end{pmatrix}, \tag{39}$$

and functions $P(t)$ *and* $Q(t)$ *are defined by*

$$P(t) = \begin{pmatrix} t^2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \;,\; Q(t) = \begin{pmatrix} 0 \\ \sin(t) \\ 0 \\ 0 \end{pmatrix}. \tag{40}$$

We follow the Algorithm 1 step by step

1. *We will determine a vector valued function* $v(x,t)$ *fulfilling conditions (25). As any of the components of the vector valued function* $G(x,t)$ *are combinations of functions* $\sin(\pi x)$ *and* $\cos(\pi x)$*, we will look for* $v(x,t)$ *in the form given by (31). To do this, from (34) we obtain*

$$R_1(t) = \begin{pmatrix} t^2/\pi \\ 0 \\ 0 \\ 0 \end{pmatrix} \;,\; R_2(t) = \begin{pmatrix} -t^2 - \sin(t) \\ -t^2 \\ 0 \\ 0 \end{pmatrix},$$

and thus determine the function $v(x,t)$ *defined by*

$$v(x,t) = \begin{pmatrix} -t^2\cos(\pi x) - \cos(\pi x)\sin(t) + \dfrac{t^2\sin(\pi x)}{\pi} \\ -t^2\cos(\pi x) \\ 0 \\ 0 \end{pmatrix},$$

where replacing in (35) one gets

$$G_1(x,t) = \begin{pmatrix} -\cos(\pi x)\left(\cos(t) + 2\left(t + \pi^2 t^2 + \pi^2\sin(t)\right)\right) + \dfrac{2t\,(1+\pi^2 t)\sin(\pi x)}{\pi} \\ -\cos(\pi x)\,(t\,(2+3\pi^2 t) + \pi^2\sin(t)) + \pi t^2\sin(\pi x) \\ \pi\,(\pi\cos(\pi x)\,(t^2+\sin(t)) - t^2\sin(\pi x)) \\ 0 \end{pmatrix}.$$

Thus, vector valued function $v(x,t)$ *verifies trivially (25).*

2. *From the definition of* $v(x,t)$ *we determine* $\widetilde{G}(x,t)$ *and* $\widetilde{f(x)}$*:*

$$\widetilde{G}(x,t) = \begin{pmatrix} 0 \\ 0 \\ (x-1)^2 x e^{-t} \\ 0 \end{pmatrix},$$

$$\widetilde{f}(x) = f(x) = \begin{pmatrix} 0 \\ 0 \\ x^2 - 2x \\ 0 \end{pmatrix}.$$

3. *Using the algorithm given in Ref. [4] we can construct a solution* $w(x,t)$ *of problem (27)-(30) with these date. Observe that this problem is precisely the non-homogeneous problem with homogeneous conditions which was solved in the Example 3.2 of Ref. [4], whose exact solution is given by the series*

$$w(x,t) =$$
$$\left(\sum_{n\geq 0} -\frac{32e^{-\frac{1}{2}(\pi+2n\pi)^2 t}\sin\left(\frac{1}{2}(1+2k)\pi x\right)}{\pi^3(2k+1)^3}\right)\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$
$$-\left(\sum_{n\geq 0}\frac{3072(-1)^n e^{-\frac{(2n+1)^2\pi^2 t}{2}}\left(e^{\frac{\left(-2+(2n+1)^2\pi^2\right)t}{2}}-1\right)\left((2n+1)^2\pi^2-10\right)\sin\left(\frac{(2n+1)\pi x}{2}\right)}{(2n+1)^6\pi^6\left(-2+(2n+1)^2\pi^2\right)}\right)\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

4. *The solution of problem (20)-(23) is given by* $u(x,t) = w(x,t) + v(x,t)$*, i.e., by the expression:*

$$u(x,t) =$$
$$\left(\sum_{n\geq 0} -\frac{32e^{-\frac{1}{2}(\pi+2n\pi)^2 t}\sin\left(\frac{1}{2}(1+2k)\pi x\right)}{\pi^3(2k+1)^3}\right)\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$
$$-\left(\sum_{n\geq 0}\frac{3072(-1)^n e^{-\frac{(2n+1)^2\pi^2 t}{2}}\left(e^{\frac{\left(-2+(2n+1)^2\pi^2\right)t}{2}}-1\right)\left((2n+1)^2\pi^2-10\right)\sin\left(\frac{(2n+1)\pi x}{2}\right)}{(2n+1)^6\pi^6\left(-2+(2n+1)^2\pi^2\right)}\right)\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$
$$+\begin{pmatrix} -t^2\cos(\pi x) - \cos(\pi x)\sin(t) + \frac{t^2\sin(\pi x)}{\pi} \\ -t^2\cos(\pi x) \\ 0 \\ 0 \end{pmatrix}.$$

Example 3.2 *We consider problem (20)–(23). Suppose that $G(x,t)$ is a polynomial in x, with coefficients are functions of the variable t. Thus, we look for a vector valued function $v(x,t)$ which is also a polynomial in x (cubic, for example), whose coefficients are functions of the variable t, in the form*

$$v(x,t) = R_3(t)x^3 + R_2(t)x^2 + R_1(t)x + R_0(t) \ , \tag{41}$$

where functions $R_i(t) \in \mathcal{C}^1[0,+\infty), i = 0,1,2,3$ must be determinate. This function (41) satisfy the boundary conditions (25), i.e.

$$\left.\begin{array}{rcl} A_1 R_0(t) + B_1 R_1(t) & = & P(t) \\ A_2\left(R_3(t) + R_2(t) + R_1(t) + R_0(t)\right) + B_2\left(3R_3(t) + 2R_2(t) + R_1(t)\right) & = & Q(t). \end{array}\right\}$$

we can write the above system in matrix form:

$$\begin{pmatrix} A_1 & B_1 & 0 & 0 \\ A_2 & A_2 + B_2 & A_2 + 2B_2 & A_2 + 3B_2 \end{pmatrix} \begin{pmatrix} R_0(t) \\ R_1(t) \\ R_2(t) \\ R_3(t) \end{pmatrix} = \begin{pmatrix} P(t) \\ Q(t) \end{pmatrix} . \tag{42}$$

Taking block matrices

$$\hat{A} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & -3I & I & 0 \\ 0 & 2I & 0 & I \end{pmatrix}, \hat{B} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 3I & I & 0 \\ 0 & -2I & 0 & I \end{pmatrix} ,$$

which trivially satisfy that $\hat{A}\hat{B} = I$, (42) can be writen in the form

$$\begin{pmatrix} A_1 & B_1 & 0 & 0 \\ A_2 & A_2 + B_2 & A_2 + 2B_2 & A_2 + 3B_2 \end{pmatrix} \hat{A}\hat{B} \begin{pmatrix} R_0(t) \\ R_1(t) \\ R_2(t) \\ R_3(t) \end{pmatrix} = \begin{pmatrix} P(t) \\ Q(t) \end{pmatrix} ,$$

thus

$$\begin{pmatrix} A_1 & B_1 & 0 & 0 \\ A_2 & B_2 & A_2 + 2B_2 & A_2 + 3B_2 \end{pmatrix} \begin{pmatrix} R_0(t) \\ R_1(t) \\ 3R_1(t) + R_2(t) \\ R_3(t) - 2R_1(t) \end{pmatrix} = \begin{pmatrix} P(t) \\ Q(t) \end{pmatrix} . \tag{43}$$

We can rewrite (43) in the form

$$\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix}\begin{pmatrix} R_0(t) \\ R_1(t) \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ A_2+2B_2 & A_2+3B_2 \end{pmatrix}\begin{pmatrix} R_2(t)+3R_1(t) \\ R_3(t)-2R_1(t) \end{pmatrix}=\begin{pmatrix} P(t) \\ Q(t) \end{pmatrix}. \quad (44)$$

If we impose the condition:

$$\begin{pmatrix} 0 & 0 \\ A_2+2B_2 & A_2+3B_2 \end{pmatrix}\begin{pmatrix} R_2(t)+3R_1(t) \\ R_3(t)-2R_1(t) \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or equivalently:

$$\left.\begin{array}{rcl} R_2(t) & = & -3R_1(t) \\ R_3(t) & = & 2R_1(t) \end{array}\right\}, \quad (45)$$

from (44) we have the matrix block system

$$\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix}\begin{pmatrix} R_0(t) \\ R_1(t) \end{pmatrix}=\begin{pmatrix} P(t) \\ Q(t) \end{pmatrix}. \quad (46)$$

Taking into account (6), system (46) have an unique solution. Thus, we have shown that we can determine a vector valued function $v(x,t)$ satisfying conditions (25) and defined by expression (31). Thus, we have now that

$$\begin{array}{rcl} G_1(x,t) & = & v_t(x,t)-Av_{xx}(x,t) \\ & = & R_3'(t)x^3+R_2'(t)x^2+R_1'(t)x+R_0'(t)-A\left(6R_3(t)x+2R_2(t)\right) \end{array} \quad (47)$$

and we can apply Theorem 2.1. We will consider a concrete numerical example. Consider problem (20)-(23) where matrix $A \in \mathbb{C}^{4\times 4}$ is given by

$$A=\begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 2 & 1 & -2 \\ -1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (48)$$

and the matrices $A_i, B_i, i \in \{1,2\}$ given by (37). Also, the vectorial valued functions $f(x)$ and $G(x,t)$ will be defined respectively as

$$f(x)=\begin{pmatrix} 0 \\ x^2-1 \\ 0 \\ 0 \end{pmatrix}, \quad (49)$$

and

$$G(x,t) = \begin{pmatrix} \cos(t) - 2t + 12t^2 + 2tx - 24t^2x - 6tx^2 + 4tx^3 \\ -2t + 6t^2 - 12t^2x + e^{-t}x^3 - 2e^{-t}x^4 + e^{-t}x^5 \\ -6t^2 + 12t^2x \\ 0 \end{pmatrix}, \tag{50}$$

and functions $P(t)$ and $Q(t)$ defined by (40).

We follow the Algorithm 1 step by step

1. *We will determine a vector valued function $v(x,t)$ fulfilling conditions (25). As any of the components of the vector valued function $G(x,t)$ are polynomials in the variable x, with coefficients are functions of the variable t, we will look for $v(x,t)$ in the form given by (41). To do this, from (46) we obtain*

$$R_0(t) = \begin{pmatrix} \sin(t) - t^2 \\ -t^2 \\ 0 \\ 0 \end{pmatrix}, \quad R_1(t) = \begin{pmatrix} t^2 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and from (45) we obtain

$$R_2(t) = \begin{pmatrix} -3t^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad R_3(t) = \begin{pmatrix} 2t^2 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and therefore we have the function

$$v(x,t) = \begin{pmatrix} -t^2 + t^2x - 3t^2x^2 + 2t^2x^3 + \sin(t) \\ -t^2 \\ 0 \\ 0 \end{pmatrix}.$$

From (47) one gets

$$G_1(x,t) = \begin{pmatrix} -2t + 12t^2 + 2tx - 24t^2x - 6tx^2 + 4tx^3 + \cos(t) \\ -2t + 6t^2 - 12t^2x \\ -6t^2 + 12t^2x \\ 0 \end{pmatrix}.$$

Thus, vector valued function $v(x,t)$ verifies trivially (25).

2. *From the definition of* $v(x,t)$ *we determine* $\widetilde{G}(x,t)$ *and* $\widetilde{f(x)}$*:*

$$\widetilde{G}(x,t) = \begin{pmatrix} 0 \\ (x-1)^2 x^3 e^{-t} \\ 0 \\ 0 \end{pmatrix},$$

$$\widetilde{f}(x) = f(x) = \begin{pmatrix} 0 \\ x^2 - 1 \\ 0 \\ 0 \end{pmatrix}.$$

3. *Using the algorithm given in Ref. [4] we can construct a solution* $w(x,t)$ *of problem (27)-(30) with these date. Observe that this problem is precisely the non-homogeneous problem with homogeneous conditions which was solved in the Example 3.1 of Ref. [4], whose exact solution is given by the series*

$$w(x,t) =$$

$$\left(\sum_{n\geq 0} -\frac{32(-1)^n e^{-\frac{1}{2}(\pi+2n\pi)^2 t}\cos\left(\frac{1}{2}(2n+1)\pi x\right)}{\pi^3(2n+1)^3}\right)\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$
$$+\left(\sum_{n\geq 0} -\frac{64 e^{-\frac{(2n+1)^2\pi^2 t}{2}}\left(e^{\frac{\left(-2+(2n+1)^2\pi^2\right)t}{2}}-1\right)\mathcal{A}(n)\cos\left(\frac{(2n+1)\pi x}{2}\right)}{(2n+1)^6\pi^6\left(-2+(2n+1)^2\pi^2\right)}\right)\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

where

$$\mathcal{A}(n) = (480+(2n+1)\pi(-144(-1)^n+(2n+1)\pi((-1)^n(2n+1)\pi-6))).$$

4. *The solution of problem (20)-(23) is given by* $u(x,t) = w(x,t) + v(x,t)$, *i.e., by the expression:*

$$u(x,t) =$$

$$\left(\sum_{n\geq 0} -\frac{32(-1)^n e^{-\frac{1}{2}(\pi+2n\pi)^2 t}\cos\left(\frac{1}{2}(2n+1)\pi x\right)}{\pi^3(2n+1)^3}\right)\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$+\left(\sum_{n\geq 0} -\frac{64e^{-\frac{(2n+1)^2\pi^2 t}{2}}\left(e^{\frac{\left(-2+(2n+1)^2\pi^2\right)t}{2}}-1\right)\mathcal{A}(n)\cos\left(\frac{(2n+1)\pi x}{2}\right)}{(2n+1)^6\pi^6\left(-2+(2n+1)^2\pi^2\right)}\right)\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$+\begin{pmatrix}-t^2+t^2x-3t^2x^2+2t^2x^3+\sin(t)\\-t^2\\0\\0\end{pmatrix},$$

where

$$\mathcal{A}(n)=(480+(2n+1)\pi(-144(-1)^n+(2n+1)\pi((-1)^n(2n+1)\pi-6))).$$

4 Conclusion

In this paper a method to solve non-homogeneous problem with non-homogeneous conditions of the type (20)-(23) in terms of the solution of a non-homogeneous with homogeneous conditions problem (16)-(19) with appropriate parameters, is developed. The computational process is outlined in Algorithm 1. The choose of the appropriate function $v(x,t)$ is illustrated in the examples 3.1 and 3.2.

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