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Spectral properties with application to epidemic models

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Abstract. In this paper, we address our study to analyze diseases whose transmission occurs through a contaminated environment. Moreover, the individuals are organized in compartments by age range. This epidemiological process with indirect transmission of the disease has a mathematical representation by means of a nonlinear discrete-time invariant system. A strategy to act when the system is unstable is proposed. As a consequence of it, the new system is N-periodic and its stability is given by the monodromy matrix of the system. The above matrix analysis is applied to establish some conditions over the number of steps that we can be without action and being the epidemiological process asymptotically stable.

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1. Model description and results

In recent years, concern has grown about foodborne illness because food is the vehicle for the transmission of numerous diseases, such as salmonella [3, 4, 7, 8]. Therefore, the source of food contamination is becoming a growing public health problem in industrialized countries. This contamination can be endogenous (animals are fed on such a food) or exogenous (which takes place during processing, transport or storage), [9]. According to the annual report of Epidemiological Surveillance of Communicable Diseases issued by the National Epidemiological Surveillance Network; in 2011, 271 Salmonella outbreaks were reported in Spain, with 2861 sick people and 7 deaths. More than half of them were caused by salmonella (85% of outbreaks) followed by S. typhimurium (10%), [6]. Therefore, one of the strategies that can be raised is how to prevent the disease in the initial production stage

In [2, 4] mathematical models in order to analyze the salmonella disease in a henhouse are showed and, in [1] the amount of bacteria in the henhouse is also considered.

In this paper, we consider a discrete-time mathematical model where the variables $S(t)$ and $I(t)$ represent time dependent population densities of susceptible and infected individuals at time t, $t \in \mathbb{Z}$, $t \geq 0$, respectively. This model estimates the prevalence of salmonella infection in the henhouse. We analyze the epidemic process taking into account the salmonella density in the environment, being this density denoted by $C(t)$. Then, the behavior of the salmonella will be analyzed through the evolution of the infected individual and contaminant populations.

Then, the model is described by the following nonlinear system

$$x(t+1) = F(x(t))$$

where $x(t) = (S(t)\ I(t)\ C(t))^T$ and the parameters involved are the usual in this type of models. That is, the parameters $0 < p,\ q,\ s\ < 1$ shall represent density survival rate of $S(t),\ I(t)$ and $C(t)$, respectively, the parameter σ denotes the exposition rate, the death removal rate at time t is denoted by $\mu(t)$ and P is the population size. Moreover, $\mu(t)P$ and $\beta I(t)$, $0 < \beta < 1$, represent the replacement rate and the density of pathogen produced by individuals infected, respectively.

A linear model may arise from the linearization of above system around disease-free equilibrium point, and this approach can provide sufficient accuracy describing the evolution of the system.

Linearizing around of the disease-free equilibrium point $x^* = F(x^*)$ denoted by $P_f = (S_f, O, O)$ we obtain a linear discrete-time system

$$\hat{x}(t+1) = A\hat{x}(t) + b,$$

being $\hat{x}(t) = x(t) - x^\star$, $A = \begin{pmatrix} * & * \\ O & E \end{pmatrix}$ and we rewrite $E = T + F$. To analyze the behavior of the solution around the P_f, we only consider the subsystem containing the variables causing infection, i.e, variables that correspond to the infected and the contaminant. It is clear that this subsystem is not asymptotically stable if $\rho(E) > 1$. In our model, considering a population P, it easy to check that if $\rho(E) > 1$ the eigenvalues are

$$\lambda_{1,2} = \frac{q + s \pm \sqrt{(q-s)^2 + 4\beta\sigma P}}{2}$$

with $\lambda_1 > 1$ and the sign of the second eigenvalue, λ_2, depends on the size of the population.

So, if P satisfies $P < \dfrac{qs}{\beta\sigma}$, then $\lambda_2 > 0$, with $0 < \lambda_2 < 1 < \lambda_1$. And, if P satisfies $\dfrac{qs}{\beta\sigma} < P$, then $\lambda_2 < 0$, with $-1 < \lambda_2 < 0 < 1 < \lambda_1$.

On the other hand, one of the most important parameters of epidemic modeling is the basic reproduction number $\mathcal{R}_0$ which is a measure or indicator to know whether the disease will disappear at all or not. In [5] the expression of $\mathcal{R}_0$ is given for an epidemiological model with indirect transmission. Thus, $\mathcal{R}_0^2 = \beta\sigma P(1-q)^{-1}(1-s)^{-1}$.

The disease will die out in the long run if $\mathcal{R}_0 < 1$ and the disease will be able to spread through a population if $\mathcal{R}_0 > 1$.

We consider the case $\mathcal{R}_0 > 1$, then the population P satisfies

$$P > K = \frac{(1-q)(1-s)}{\beta\sigma}$$

Note that, $K < \frac{qs}{\beta\sigma}$ if and only if $q + s > 1$, this occurs when the density survival rate of infected individual plus the density survival rate of pathogen is greater than one.

If we focus on the fact that the infection does not disappear in the henhouse, we want to find strategies for infection fade. Since the henhouse is usually not clean, one strategy may be to perform a periodic cleaning of the henhouse to eliminate the contaminant but we need to delay as much as possible this cleaning action. We want to find the minimum number of steps that can be uncleaned, keeping the asymptotically stable system. That is, if we let another step without cleaning the environment, then the disease grows and the system is unstable in a neighborhood of free-disease point.

Therefore, we start with the initial condition $(S(0), I(0), 0)$ and perform cleaning every N stages leaving the henhouse free of contaminant. Then, the process is given in the following system

$$\begin{array}{l} I(t+1) = qI(t) + \sigma(t)B(t)S(t) \\ B(N) = s(t)B(t) + \beta I(t) \end{array}$$

where $\sigma(t)$ y $s(t)$ are two N-periodic functions defined by

$$\sigma(t) = \begin{cases} 0 & t = 0 \\ \sigma & t = 1,,2,\ldots,N-1 \end{cases}, \quad s(t) = \begin{cases} 0 & t = 0 \\ s & t = 1,,2,\ldots,N-1 \end{cases},$$

$\sigma(t+N) = \sigma(t)$ y $s(t+N) = s(t)$.

Linearizing around of its disease-free equilibrium point, we have the following periodic system

$$\begin{pmatrix} I(t+1) \\ B(t+1) \end{pmatrix} = \begin{pmatrix} q & f(t) \\ \beta & s(t) \end{pmatrix} \begin{pmatrix} I(t) \\ B(t) \end{pmatrix} = E(t) \begin{pmatrix} I(t) \\ B(t) \end{pmatrix}, \tag{1}$$

where$f(t) = \sigma(t)S_f$ and $E(t) = T(t) + F(t)$ given by

$$T = \begin{pmatrix} q & 0 \\ 0 & s(t) \end{pmatrix}, \quad F(t) = \begin{pmatrix} 0 & f(t) \\ \beta & 0 \end{pmatrix}.$$

We obtain a parameter that quantifies the transmission potential of the disease by using the rates of transmissibility through of the pathogen produced by the infected individuals, taking into account the "monodromy matrix" associated with the periodic system. This matrix is defined as

$$\Phi_E(k, k_0) = \prod_{i=k_0}^{k-1} E(k_0 + k - 1 - i), \ k > k_0, E(k_0, k_0) = I_n.$$

In this case, it is easy to see that we can limit the study of the behavior of the spectral radius of the matrix of monodromy in $s = 0$. So, we want to get the greatest value of N so that $\rho(E_{N,0}) < 1$ y $\rho(E_{N+1,0}) > 1$.

The parameters q and s are important for controlling salmonella, and the population size in the henhouse plays an important role because it determines the prevalence of salmonella. Now, we study the case mentioned above.

If P satisfies that $K < P < \dfrac{qs}{\beta\sigma}$, then $\lambda_2 > 0$. With, $0 < \lambda_2 < 1 < \lambda_1$. We search for a period N such that

$$X_N = \frac{\lambda_1^N(q - \lambda_2) + \lambda_2^N(\lambda_1 - q)}{\lambda_1 - \lambda_2} < 1$$

and

$$X_{N+1} = \frac{\lambda_1^{N+1}(q - \lambda_2) + \lambda_2^{N+1}(\lambda_1 - q)}{\lambda_1 - \lambda_2} > 1$$

being $X_n = \rho(E_{N,0})$ and we denote

$$h(N) = \frac{\lambda_1 - \lambda_2 - (q - \lambda_2)\lambda_1^N}{\lambda_1^N(\lambda_1 - q)}.$$

Then, it follows the following result.

Lemma 1 *Consider the spectrum of matrix E. If $\mathcal{R}_0 > 1$ with $P < \dfrac{qs}{\beta\sigma}$ then*

(i) $\lambda_2 < q < \lambda_1$

(ii) *$h(N)$ is a decreasing function and there exits $\tilde{N}$ such that $0 < h(N) < 1$ for all $N < \tilde{N}$.*

Proof. If $\mathcal{R}_0 > 1$ with P satisfying $P < \dfrac{qs}{\beta\sigma}$, then $\lambda_2 > 0$, with $0 < \lambda_2 < 1 < \lambda_1$.

(i) The proof is based on the following observation. $\lambda_1 + \lambda_2 = q + s$ and from the expression of the eigenvalues it is easy to see that $\lambda_1 - q > 0$ and $q - \lambda_2 > 0$.

(ii) The basic idea of the proof is to take

$$h(N) = \lambda_1^{-N}\frac{\lambda_1 - \lambda_2}{\lambda_1 - q} - \frac{q - \lambda_2}{\lambda_1 - q}.$$

By (i) it is clear that $\dfrac{\lambda_1 - \lambda_2}{\lambda_1 - q} > 0$ and $\dfrac{q - \lambda_2}{\lambda_1 - q}$ and λ_1^{-N} is a decreasing function, then $h(N)$ is also a decreasing function with $h(0) = 1$. According to the above expression of $h(N)$, we have

$$h(1) = \frac{\lambda_1(1 - q) + \lambda_2(\lambda_1 - 1)}{\lambda_1(\lambda_1 - q)} > 0.$$

Since $h(N)$ is a decreasing function there exists $\tilde{N}$ such that $h(\tilde{N}) > 0$ and $h(\tilde{N} + 1) < 0$. Moreover, $0 < h(N) < 1$ for all $N < \tilde{N}$. □

The behavior of the solution of the system around the free equilibrium point is now discussed. The prevalence of infection by excreting salmonella can be controlled using cleaning strategies and the desired value of N is obtained in the following result.

Proposition 1 *Consider that the trajectory of the N-periodic system given by (1) is in an environment of the disease free point with $\mathcal{R}_0 > 1$ and $P < \dfrac{qs}{\beta\sigma}$. Take $N \in \mathbb{N}$ satisfying $N < \tilde{N}$ and*

$$\frac{\ln(h(N))}{\ln(\frac{\lambda_2}{\lambda_1})} < N < N + 1 < \frac{\ln(h(N+1))}{\ln(\frac{\lambda_2}{\lambda_1})} \tag{2}$$

then N is the maximum of the period values such that the N-periodic system is asymptotically stable.

Proof. If N satisfies the condition given in the proposition and using the expression of the spectral radius we have $h(N) > (\dfrac{\lambda_2}{\lambda_1})^N > \dfrac{\lambda_1}{\lambda_2}h(N+1)$ and $\lambda_1 - \lambda_2 > \lambda_1^N((\frac{\lambda_2}{\lambda_1})^N(\lambda_1 - q) + (q - \lambda_2))$ then $\rho(E_{N,0}) < 1$.

On the other hand, $(\frac{\lambda_2}{\lambda_1})^{N+1} > h(N+1)$ and $\lambda_1^{N+1}((\frac{\lambda_2}{\lambda_1})^{N+1}(\lambda_1 - q) + (q - \lambda_2)) > \lambda_1 - \lambda_2$ then $\rho(E_{N+1,0}) > 1$.

Thus, N satisfies $\rho(E_{N,0}) < 1$ and $\rho(E_{N+1,0}) > 1$. □

The obtained results suggest that to reduce the level of salmonella in the henhouse is necessary to reduce the quantity of excretions in the henhouse because this decreases the transmission of salmonella. The given result shows the reduction in salmonella level by using an effective control strategy every N steps.

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