A mixed derivative terms removing method in multi-asset option pricing problems

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Abstract

The challenge of removing the mixed derivative terms of a second order multidimensional partial differential equation is addressed in this paper. The proposed method, which is based on proper algebraic factorization of the so-called diffusion matrix, depends on the semidefinite or indefinite character of this matrix. Computational cost of the transformed equation is considerably reduced and well-known numerical drawbacks are avoided.

Keywords: Multiasset option pricing; multidimensional partial differential equations; mixed derivative terms; LDL^T factorization; Bunch-Kaufman factorization.
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1. Introduction

Multiasset option pricing problems have an increasing interest because they are natural and frequent in the real market practice. In the multi-dimensional Black-Scholes model the asset prices follow a geometric Brownian motion

$$dS_i(t) = (\mu_i - q_i)S_i dt + \sigma_i S_i(t) dW_i(t), \quad t \geq 0, \tag{1}$$

where S_i is the i-th underlying asset having an expected return of μ_i, a continuous divided of q_i, and the volatility of σ_i, for $i = 1, 2, \ldots, M$ and $M \in \mathbb{N}$. The Wiener processes are correlated with $\rho_{ij} dt =< dW_i, dW_j >$, for $1 \leq i, j \leq M$, $i \neq j$. Using Martingale strategies and Itô's calculus, one gets that option price $V(S, \tau) = V(S_1, \ldots, S_M, \tau)$ satisfies [23]:

$$\frac{\partial V}{\partial \tau} = \frac{1}{2} \sum_{i,j=1}^{M} \rho_{ij} \sigma_i \sigma_j S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i=1}^{M} (r - q_i) S_i \frac{\partial V}{\partial S_i} - rV, \tag{2}$$

where $\tau = T - t$, T is the time of maturity, $\rho_{ii} = 1$, $\rho_{ij} = \rho_{ji}$, $i \neq j$, and $|\rho_{ij}| \leq 1$. The mixed derivative terms appearing in (2) show the correlation among the prices of the assets S_i.

If the curse of dimensionality is a very significant problem within the pricing techniques due to the exponential growth of unknowns and complexity, the mixed derivative terms are a source of numerical drawbacks. If not accurately discretized, they may generate oscillations, spurious solutions and other instabilities [27]. Furthermore, standard finite difference discretization schemes involve stencils with a considerable greater number of nodes.

It is important to point out that multidimensional partial differential equations with mixed derivative terms also appear in many different engineering problems [15, 17]. It is well known that the presence of the mixed derivative terms may cause instability and inaccuracies, which complicates numerical schemes as splitting methods [26] and references therein.

Authors overcome these mixed derivative drawbacks in two different ways. Some of them construct special schemes to reduce the number of stencil nodes [3, 4, 14] or propose high order compact schemes [6]. The second approach, developed for two asset problems in [12] and for stochastic volatility models in [5, 7], uses different transformation techniques to remove the mixed derivative terms. In [20], the authors remove the cross derivative terms using the orthogonal diagonalization of the covariance matrix. Such approach requires the use of iterative methods, [8, chapter 8] and has the drawback of the

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reliable computation of eigenvectors [24, chapter 5]. In this paper, we show that, in fact, for the general multiasset option pricing problem the cross derivative terms of (2) can be removed by means of an easy to implement transformation based on Gaussian elimination and pivoting strategies.

Sample covariance and correlation matrices from real data may lack definiteness, see [19], [25, pp. 189-190]. In such cases, authors use to transform the original problem into another close one where the correlation matrix becomes positive definite [10, 22].

Our strategy is based on the LDL^T transformation of the symmetric positive semi-definite correlation matrix

$$R = (\rho_{ij})_{1\le i,j\le M}, \tag{3}$$

(see e.g. [16, p. 540] and [21, p. 369]). The organization of the paper is as follows. Section 2 contains the proposed general method to remove the $\frac{M(M-1)}{2}$ mixed derivative terms of equation (2). Thinking of applications in other fields where the real symmetric matrix A involving the coefficients of second order derivative terms, so-called the diffusion matrix, becomes indefinite, it is important to point out that in such cases the factorization $A = LDL^T$ may not exist, and if it exists, its computation may be unstable (see e.g., [11, p. 214]). Then, in Section 3 we provide an alternative to remove only a part of the mixed derivative terms based on the Bunch-Kaufman factorization of the matrix R (see e.g. [1], [8, p. 192] or [11, p. 217]). A conclusion section finishes the work.

2. Removing mixed derivative terms transformation

For the sake of clarity in the presentation, we recall some algebraic definitions and results about real symmetric matrices that are relevant in the following. We begin with a well-known definition.

Definition 1. *[8] A matrix $L = (l_{ij})$ is said to be a unit lower triangular matrix if $l_{ii} = 1$, and $l_{ij} = 0$ for $1 \le i, j \le M$ and $j > i$. Also, a symmetric matrix $R \in \mathbb{R}^{M\times M}$ is said to be positive semidefinite if $x^T R x \ge 0$ for all vectors $x \in \mathbb{R}^M$.*

The next result is an adapted one from the results of Chapter 10 of [11] and Chapter 4 of [8].

Proposition 1. *[8] Let R be a symmetric positive semidefinite matrix in $\mathbb{R}^{M\times M}$. Then, there exists a unit lower triangular matrix L and a diagonal matrix $D = (d_{ij})$ in $\mathbb{R}^{M\times M}$ with $d_{ii} \ge 0$, $1 \le i \le M$, such that*

$$R = LDL^T. \tag{4}$$

The expression (4) is called LDL^T factorization of matrix R with the additional hypothesis of invertibility on R, i.e., the positive definite case, the above decomposition (4) is unique, but not if R is only positive semidefinite.

In order to guarantee the numerical stability, it is convenient to ensure that no large entries appear in the computed triangular factors of (4). This is performed by means of a permuted version of R [8, Chapters 3 and 4].

Taking advantage of a diagonal pivoting strategy in algorithm 4.2.2 of [8], one constructs a LDL^T decomposition of a symmetric semidefinite matrix R using a permutation matrix P such that $|l_{ij}| \le 1$ and

$$PRP^T = LDL^T, \tag{5}$$

with

$$d_{11} \ge d_{22} \ge \cdots \ge d_{nn} \ge 0. \tag{6}$$

Let us start from equation (2). It is well-known that logarithm transformation for spatial variables leads to constant coefficient partial differential equation [4]. Here, we use the substitution

$$x_i = \frac{\log S_i}{\sigma_i}, \qquad 1 \le i \le M, \tag{7}$$

with $V(S,\tau) = W(X,\tau)$, where $X = (x_1, x_2, \ldots, x_M)^T$, achieving

$$\frac{\partial W}{\partial \tau} = \frac{1}{2}\sum_{i,j=1}^{M} \rho_{ij} \frac{\partial^2 W}{\partial x_i \, \partial x_j} + \sum_{i=1}^{M} \left(r - q_i - \frac{\sigma_i^2}{2} \right) \frac{1}{\sigma_i} \frac{\partial W}{\partial x_i} - rW. \tag{8}$$

In order to explore the possibility of removing the mixed derivative terms of (8), let us propose a linear transformation

$$Y = CX, \qquad C = (c_{ij})_{1\leq i,j\leq M}, \tag{9}$$

where C is a nonsingular matrix to be determined later. Using (9), the equation (8) becomes $U(Y,\tau) = W(X,\tau)$ and

$$\frac{\partial U}{\partial \tau} = \frac{1}{2}\sum_{i,j=1}^{M} (\mathbf{c}_i R \mathbf{c}_j^T) \frac{\partial^2 U}{\partial y_i\, \partial y_j} + \sum_{i,j=1}^{M} \left(\frac{r - q_j - \sigma_j^2/2}{\sigma_j} \right) c_{ij} \frac{\partial U}{\partial y_i} - rU, \tag{10}$$

where $\mathbf{c}_i = (c_{i1}, c_{i2}, \cdots, c_{iM})$ is the ith row vector of matrix C. Note that mixed derivative terms disappear in (10) if row vectors of matrix C are orthogonal with respect to R. As R is symmetric positive semidefinite, from (5), we have

$$(L^{-1}P)R(L^{-1}P)^T = D. \tag{11}$$

Let us take $\mathbf{c}_i$ as the ith row of matrix $L^{-1}P$: $\mathbf{c}_i = (L^{-1}P)_i$. From (11), one gets

$$\mathbf{c}_i R \mathbf{c}_j^T = \begin{cases} 0, & i \neq j, \\ d_{ii}, & i = j. \end{cases} \tag{12}$$

Hence, equation (10) becomes:

$$\frac{\partial U}{\partial \tau} = \frac{1}{2}\sum_{i=1}^{M} (d_{ii}) \frac{\partial^2 U}{\partial y_i^2} + \sum_{i=1}^{M} \left(\sum_{j=1}^{M} \frac{(r - q_j - \sigma_j^2/2) c_{ij}}{\sigma_j} \right) \frac{\partial U}{\partial y_i} - rU. \tag{13}$$

Summarizing the following result has been established.

Theorem 1. *With previous notation, let R be the symmetric positive semi-definite matrix given in (3) and let L,P and D be matrices in $\mathbb{R}^{M\times M}$ satisfying (5). Then, under substitutions (7) and (9), where $C = L^{-1}P$, the equation (2) is transformed into equation (13) without mixed derivative terms.*

Remark 1. *Note that using the classic vector analysis notation where $\cdot$ denotes the Euclidean inner product and the gradient is represented by the operator $\nabla = \left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \ldots, \frac{\partial}{\partial y_M}\right)^T$, the transformed equation (13) of Theorem 1 can be written in the compact form*

$$\frac{\partial U}{\partial \tau} = \frac{1}{2}(D\nabla)\cdot\nabla U + (CQ)\cdot\nabla U - rU, \tag{14}$$

where $Q = (Q_1, Q_2, \ldots, Q_M)^T$ with $Q_i = \frac{r - q_i - \sigma_i^2/2}{\sigma_i}$, $1 \leq i \leq M$.

Remark 2. *It is important to point out that the transformation constructed in Theorem 1 has numerical advantage from the computational cost and stability points of view, but it is not the only way to eliminate mixed derivative terms. In fact, if one uses the standard orthogonal diagonalization of $R = FDF^{-1}$, with $F^{-1} = F^T$, the transformation $C = F^{-1}$ also transforms equation (2) into another without mixed derivative terms.*

In the next example, we apply the removing strategy to a 7-asset option pricing problem treated in [18, p. 18].

Example 1. *Consider equation (2) for $M = 7$ where the correlation positive definite matrix R is given by*

$$R = \begin{pmatrix} 1.00 & -0.65 & 0.25 & 0.2 & 0.25 & -0.05 & 0.05 \\ -0.65 & 1.00 & 0.5 & 0.1 & 0.25 & 0.11 & -0.016 \\ 0.25 & 0.5 & 1.00 & 0.37 & 0.25 & 0.21 & 0.076 \\ 0.2 & 0.1 & 0.37 & 1.00 & 0.25 & 0.27 & 0.13 \\ 0.25 & 0.25 & 0.25 & 0.25 & 1.00 & 0.14 & -0.04 \\ -0.05 & 0.11 & 0.21 & 0.27 & 0.14 & 1.00 & 0.19 \\ 0.05 & -0.016 & 0.076 & 0.13 & -0.04 & 0.19 & 1.00 \end{pmatrix}, \tag{15}$$

with the parameters $\sigma = (\sigma_1, \ldots, \sigma_7)$=(0.25, 0.35, 0.20,0.25, 0.20, 0.21, 0.27), $r = 0.045$, $T = 1$ *year,* $q = (q_1, \ldots, q_7)$= 0.05, 0.07, 0.04,0.06, 0.04, 0.03, 0.02. *With previous notation, using factorization* $PRP^T = LDL^T$ *and substitution (7) and (9), one gets* $D = diag(1.000, 0.998, 0.960, 0.907, 0.861, 0.787, 0.00786)$ *and*

$$C = L^{-1}P = \begin{pmatrix} 1.000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.050 & 0 & 0 & 0 & 0 & 1.000 & 0 \\ -0.060 & 0 & 0 & 0 & 0 & -0.190 & 1.000 \\ -0.260 & 0 & 0 & 0 & 1.000 & -0.170 & 0.085 \\ -0.210 & 0 & 1.000 & 0 & -0.170 & -0.190 & -0.036 \\ -0.110 & 0 & -0.270 & 1.000 & -0.130 & -0.190 & -0.074 \\ 0.900 & 1.000 & -0.680 & 0.021 & -0.330 & 0.120 & -0.017 \end{pmatrix}. \tag{16}$$

Now, the corresponding problem (2) is transformed into (13) and (14) as follows:

$$\frac{\partial U}{\partial \tau} = \frac{1}{2}(D\nabla) \cdot \nabla U + (CQ) \cdot \nabla U - rU, \tag{17}$$

where $U(Y, \tau) = V(S, \tau)$ *and* Q *appears in Remark 1. Note also that the original partial differential equation with 37 terms has been transferred into one with only 16 terms.*

3. Mixed derivative removing: The indefinite case

As it is pointed out in Section 1, the removing technique proposed in Section 2 is also applicable with changes to other second order partial differential equations where the diffusion matrix is symmetric possibly indefinite. Let us consider the equation

$$\sum_{i,j=1}^{M} a_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{M} b_i \frac{\partial v}{\partial x_i} + cv = 0, \tag{18}$$

where $A = (a_{ij})_{1 \le i,j \le M}$ is a real symmetric matrix, $\mathbf{b} = (b_1, \ldots, b_M)^T \in \mathbb{R}^M$ and $c \in \mathbb{R}$.

In the transformation of previous section, the positive semi-definite structure of matrix R was essential to guarantee the LDL^T factorization. As now the matrix A is allowed to be indefinite, we provide the Bunch-Kaufman factorization alternative. This approach not always provides a diagonal factorization of A, but only a block-diagonal matrix B with 1×1 or 2×2 diagonal blocks such that

$$PAP^T = LBL^T, \tag{19}$$

where the permutation matrix P provides a partial pivoting strategy. Thus, one gets a more efficient method than other diagonal pivoting strategies as complete pivoting [2], see chapter 10 of [11, p. 226]. Numerically stable computation of factorization (19) is given in [1], [11, p. 217], [8, p. 192] and [9].

The next example is related to multiasset cross currency option pricing [13, chapter 29] with indefinite sample correlation matrix.

Example 2. *[25, p. 189] Consider equation (2) for* $M = 3$*, with indefinite sample correlation matrix*

$$R = \begin{pmatrix} 1 & \frac{3}{10} & \frac{9}{10} \\ \frac{3}{10} & 1 & \frac{9}{10} \\ \frac{9}{10} & \frac{9}{10} & 1 \end{pmatrix}. \tag{20}$$

Using Bunch-Kaufman strategy, one gets the transformation matrix C *and the resulting matrix* B,

$$C = L^{-1}P = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{10} & 1 & 0 \\ -\frac{9}{13} & -\frac{9}{13} & 1 \end{pmatrix}, \qquad B = diag\,(1, 91/100, -16/65) = D. \tag{21}$$

Hence, the original partial differential equation is transformed into (14) without cross derivative terms.

Example 3. *Let us consider the 4-dimensional second order partial differential equation*

$$(A\nabla)\cdot\nabla v = 2\frac{\partial^2 v}{\partial x_1\partial x_2} + 4\frac{\partial^2 v}{\partial x_1\partial x_3} + 8\frac{\partial^2 v}{\partial x_2\partial x_3} + 6\frac{\partial^2 v}{\partial x_1\partial x_4} + 10\frac{\partial^2 v}{\partial x_2\partial x_4} + 12\frac{\partial^2 v}{\partial x_3\partial x_4} = 0, \tag{22}$$

and let A be the symmetric indefinite diffusion matrix

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 4 & 5 \\ 2 & 4 & 0 & 6 \\ 3 & 5 & 6 & 0 \end{pmatrix}. \tag{23}$$

Note that as all the diagonal entries of A are zeros the LDL^T factorization of PAP^T is not possible because diagonal pivoting interchanges among diagonal entries and no nonzero pivot is possible.

The Bunch-Kaufman algorithm provides the factorization (19) with

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/2 & 1/3 & 1 & 0 \\ 5/6 & 2/3 & 4/3 & 1 \end{pmatrix}, \qquad B = diag\left(\begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix}, -2, -\frac{28}{9}\right), \qquad P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{24}$$

Considering the substitution (9) where $C = L^{-1}P$ gives the transformed equation for $w(Y) = u(X)$:

$$(B\nabla)\cdot\nabla w = 12\frac{\partial^2 w}{\partial y_1\partial y_2} - 2\frac{\partial^2 w}{\partial y_3^2} - \frac{28}{9}\frac{\partial^2 w}{\partial y_4^2} = 0. \tag{25}$$

Remark 3. *Note that because of numerical stability requirements, mixed partial derivative still remains in (25). However, for the aim of solving the PDE, we can manage the 2×2 remaining block by standard eigenvalue diagonalization. Note that*

$$\begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix} = KD_1K^T, \qquad D_1 = \begin{pmatrix} 6 & 0 \\ 0 & -6 \end{pmatrix}, \qquad K = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{26}$$

and taking $C_1 = \left(diag(K^T, 1, 1)\right) L^{-1}P$, and $Z = C_1X$ one achieves finally $u(Z) = w(Y) = v(X)$:

$$6\frac{\partial^2 u}{\partial z_1^2} - 6\frac{\partial^2 u}{\partial z_2^2} - 2\frac{\partial^2 u}{\partial z_3^2} - \frac{28}{9}\frac{\partial^2 u}{\partial z_4^2} = 0. \tag{27}$$

This last transformation is always possible, when after applying the Bunch-Kaufman algorithm, some 2×2 blocks remain in the block-diagonal factorization of matrix A.

4. Conclusions

In this paper, a general numerically stable method is proposed to remove the mixed derivative terms of multidimensional second order partial differential equations with real symmetric diffusion matrix A. Although the cases where A is semidefinite or indefinite are treated separately, both removing techniques become numerically stable. In fact, the indefinite case may need in the last step, an eigenvalue diagonalization of a 2×2 block that is numerically stable.

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