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New results on topological dynamics of antitriangular maps

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Abstract. We present some results concerning the topological dynamics of antitriangular maps, $F : X^2 \to X^2$ with the form $F(x,y) = (g(y), f(x))$, where (X,d) is a compact metric space and $f, g : X \to X$ are continuous maps. We make an special analysis in the case of $X = [0,1]$.

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1. Introduction

Let (X,d) be a compact metric space and let $\varphi : X \to X$ be a continuous map, $\varphi \in C(X,X)$. The pair (X,φ) is called a *discrete dynamical system,* whose orbits are given by the sequence $\{\varphi^n(x)\}_{n=0}^{\infty}$, $x \in X$, where $\varphi^n = \varphi \circ \varphi^{n-1}$, $n \geq 1$ and $\varphi^0 =$ Identity. In general, the full knowledge of all orbits of the system is a difficult problem and it is only known in some particular cases. Nevertheless, good approximations can be given. These approaches can be probabilistic (invariant measures, metric entropy, ...) or topological (periodic structure, topological entropy, ...). In this paper we will follow this last approach.

A point $x \in X$ is *periodic* when, for some $n > 0$, is $\varphi^n(x) = x$. If $n = 1$, the periodic point is called *fixed point.* The *order* or *period* of a periodic point is precisely the smallest of the values m for which $\varphi^m(x) = x$. We denote by $\mathrm{Per}(\varphi)$ the set of periods that the continuous map φ has.

We use $\mathcal{A}(\varphi)$ to denote one of the following sets: the set of *periodic points,* $\mathrm{P}(\varphi)$; the set $\mathrm{AP}(\varphi)$ of *almost periodic points,* that is, the points $x \in X$ such that for any neighborhood $V = V(x)$ of x, there is $N = N(V) \in \mathbb{N}$ such that $\varphi^{kN}(x) \in V$, for every $k \geq 0$; the set $\mathrm{UR}(\varphi)$ of *uniformly recurrent points*, $x \in X$ such that for any neighborhood $V = V(x)$ of x, there is $N = N(V)$ such that

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for all $q > 0$ it holds $\varphi^r(x) \in V$ for some $q \leq r < q+N$; $\mathrm{R}(\varphi) = \{x \in X : x \in \omega_\varphi(x)\}$ is the set of *recurrent points*, with $\omega_\varphi(x)$ the *omega-limit set of the point* x, that is, the points $y \in X$ such that there is a subsequence $\{n_i\}_i \subset \mathbb{N}$ with $\varphi^{n_i}(x) \to y$ as $n_i \to \infty$; $\mathrm{C}(\varphi) = \overline{\mathrm{R}(\varphi)}$ is the *centre* of φ, where $\overline{A}$ denotes the closure of a set of $A \subseteq X$; $\omega(\varphi)$ is the (global) ω*-limit set*, $\omega(\varphi) = \bigcup_{x\in X}\omega_\varphi(x)$; $\Omega(\varphi)$ is the set of *non-wandering points,* those points $x \in X$ such that for any neighborhood $U = U(x)$ of x there exists $N = N(U) \in \mathbb{N}$ in a way that $\varphi^N(U) \cap U \neq \varnothing$; and finally the set $\mathrm{CR}(\varphi)$ of *chain-recurrent points,* the points $x \in X$ for which given any $\varepsilon > 0$, there is $\{x_i\}_{i=0}^n \subset X$ such that $x_0 = x$, $d(x_{i+1}, \varphi(x_i)) < \varepsilon$ for $i = 0, 1, ..., n-1$ and $d(x_0, \varphi(x_n)) < \varepsilon$.

It follows from the definitions that

$$\mathrm{P}(\varphi) \subseteq \mathrm{AP}(\varphi) \subseteq \mathrm{UR}(\varphi) \subseteq \mathrm{R}(\varphi) \subseteq \mathrm{C}(\varphi) \tag{1.1}$$

and

$$\omega(\varphi) \subseteq \Omega(\varphi) \subseteq \mathrm{CR}(\varphi). \tag{1.2}$$

In this paper, we are devoted to topological dynamics of antitriangular maps, that is, continuous maps $F : X \times X \to X \times X$ of the form

$$F(x,y) = (g(y), f(x)), \tag{1.3}$$

with $(x,y) \in X \times X$.

Antitriangular maps appear in some economical models, particularly with the so–called *Cournot duopoly* (see [12] or [8]). The Cournot duopoly consists in an economy in which two firms are competitors in the same sector. This situation is modelled by a map F having the form of (1.3) and such that $X = I = [0,1]$.

From (1.3) it is clear that

$$F^{2n}(x,y) = ((g\circ f)^n(x), (f\circ g)^n(y)) \tag{1.4}$$

and

$$F^{2n+1}(x,y) = (g\circ (f\circ g)^n(y), f\circ (g\circ f)^n(x)) \tag{1.5}$$

for any $(x,y) \in X \times X$ and for any $n \in \mathbb{N}$. So, it is natural to expect the dynamics of F to be strongly connected to the dynamics of $g\circ f$ and $f \circ g$.

In [10] a program is developed for *triangular maps*, that is, continuous maps $T : I^2 \to I^2$ of the form $T(x,y) = (f(x), g(x,y))$. This is made investigating the relationship between the sets $\mathcal{A}(T) \subseteq I^2$ and $\mathcal{A}(f) \subseteq I$, where

$$\mathcal{A}(\cdot) \in \{\mathrm{P}(\cdot), \mathrm{AP}(\cdot), \mathrm{UR}(\cdot), \mathrm{R}(\cdot), \mathrm{C}(\cdot), \omega(\cdot), \Omega(\cdot), \mathrm{CR}(\cdot)\}.$$

When $X = I$, a first step to follow a similar program for antitriangular maps is to do the same with the sets $\mathcal{A}(F) \subseteq I^2$ and $\mathcal{A}(g\circ f), \mathcal{A}(f\circ g) \subseteq I$.

For $\mathcal{A}(\cdot) \in \{\mathrm{P}(\cdot), \mathrm{AP}(\cdot), \mathrm{C}(\cdot), \mathrm{CR}(\cdot)\}$ we see that $\mathcal{A}(F) = \mathcal{A}(g\circ f) \times \mathcal{A}(f\circ g)$ and when $\mathcal{A}(F) = \mathrm{R}(F)$ the two situations, $\mathrm{R}(F) = \mathrm{R}(g\circ f) \times \mathrm{R}(f\circ g)$ and $\mathrm{R}(F) \subsetneqq \mathrm{R}(g\circ f) \times \mathrm{R}(f\circ g)$ are possible (see [3]). We also see that in the case of the set of uniformly recurrent points the same result is true. For $\mathcal{A}(F) = \Omega(F)$ the situation is more complicated and the case $\Omega(F) \nsubseteq \Omega(g\circ f) \times \Omega(f\circ g)$ can happen. It remains open what is the situation when $\mathcal{A}(F) = \omega(F)$. We conjecture that the cases $\omega(F) = \omega(g\circ f)\times\omega(f\circ g)$ and $\omega(F) \subsetneqq \omega(g\circ f)\times\omega(f\circ g)$ could be found.

We underline that similarly to the interval case in antitriangular maps on I^2 it is held $\mathrm{C}(F) = \overline{\mathrm{P}(F)}$, which is not true in general in the triangular case (see [10]).

Similarly to the interval case, we construct examples proving that the following chain is possible

$$\mathrm{P}(F) \neq \mathrm{AP}(F) \neq \mathrm{UR}(F) \neq \mathrm{R}(F) \neq \mathrm{C}(F) \neq \omega(F) \neq \Omega(F) \neq \mathrm{CR}(F). \tag{1.6}$$

The paper is organized as follows. In the next section, we study the relationship between the sets $\mathcal{A}(F)$ and $\mathcal{A}(g\circ f)\times\mathcal{A}(f\circ g)$. The last section is concerned with the introduction of the chain (1.6).

2. Projection of the topological dynamics

If (X,d) is a compact metric space, we denote the product space by $(X\times X,\rho)$, where

$$\rho((x_1,y_1),(x_2,y_2)) = \max\{d(x_1,x_2),d(y_1,y_2)\}$$

for all $(x_1,y_1),(x_2,y_2)\in X\times X$. If $f,g\in C(X,X)$ we define the product map $f\times g : X\times X\to X\times X$ by $(f\times g)(x,y) = (f(x),g(y))$ for all $(x,y)\in X\times X$. So, if $F(x,y) = (g(y),f(x))$ is an antitriangular map, we obtain that $F^2 = (g\circ f)\times(f\circ g)$.

In this section we consider an antitriangular map $F : X\times X\to X\times X$ and we study if the equality $\mathcal{A}(F) = \mathcal{A}(g\circ f)\times\mathcal{A}(f\circ g)$ holds, where $\mathcal{A}(\cdot)$ denotes one of the subsets $\mathrm{P}(\cdot)$, $\mathrm{AP}(\cdot)$, $\mathrm{UR}(\cdot)$, $\mathrm{R}(\cdot)$, $\mathrm{C}(\cdot)$, $\omega(\cdot)$, $\Omega(\cdot)$ or $\mathrm{CR}(\cdot)$. Before studying this problem, we need the following result.

Proposition 2.1. *Let (X,d) be a compact metric space. Let $f,g\in C(X,X)$. Then*

(a) $\mathrm{P}(f^2) = \mathrm{P}(f)$ *and* $\mathrm{P}(f\times g) = \mathrm{P}(f)\times\mathrm{P}(g)$.
(b) $\mathrm{AP}(f^2) = \mathrm{AP}(f)$ *and* $\mathrm{AP}(f\times g) = \mathrm{AP}(f)\times\mathrm{AP}(g)$.
(c) $\mathrm{UR}(f^2) = \mathrm{UR}(f)$ *and* $\mathrm{UR}(f\times g)\subseteq\mathrm{UR}(f)\times\mathrm{UR}(g)$.
(d) $\mathrm{R}(f^2) = \mathrm{R}(f)$ *and* $\mathrm{R}(f\times g)\subseteq\mathrm{R}(f)\times\mathrm{R}(g)$.
(e) $\mathrm{C}(f^2) = \mathrm{C}(f)$ *and* $\mathrm{C}(f\times g)\subseteq\mathrm{C}(f)\times\mathrm{C}(g)$.
(f) $\omega(f^2) = \omega(f)$ *and* $\omega(f\times g)\subseteq\omega(f)\times\omega(g)$.
(g) $\Omega(f^2)\subseteq\Omega(f)$ *and* $\Omega(f\times g)\subseteq\Omega(f)\times\Omega(g)$.
(h) $\mathrm{CR}(f^2) = \mathrm{CR}(f)$ *and* $\mathrm{CR}(f\times g) = \mathrm{CR}(f)\times\mathrm{CR}(g)$.

Additionally, if $X = [0,1]$, then $\mathrm{C}(f\times g) = \mathrm{C}(f)\times\mathrm{C}(g)$.

Proof. For the first part of (a)-(h) see [4]. The second part of properties (a)–(h) follow from definitions. For instance, we prove here the equality $\mathrm{CR}(f\times g) = \mathrm{CR}(f)\times\mathrm{CR}(g)$.

Let $(x_0,y_0)\in\mathrm{CR}(f\times g)$. Given an arbitrary $\varepsilon>0$, we must prove that there is an ε-chain for x and f, and an ε-chain for y and g. For $\varepsilon>0$, there is an ε-chain for (x_0,y_0) and $f\times g$,

$$(x_0,y_0),(x_1,y_1),\ldots,(x_n,y_n),(x_{n+1},y_{n+1}) = (x_0,y_0),$$

such that $d((f\times g)(x_i,y_i),(x_{i+1},y_{i+1}))<\varepsilon$ for $i=0,1,...,n$. Clearly

$$x_0,x_1,...,x_n,x_{n+1}=x_0$$

is an ε–chain for x_0 and f and

$$y_0,y_1,...,y_n,y_{n+1}=y_0$$

is an ε–chain for y_0 and g. So $\mathrm{CR}(f\times g)\subseteq \mathrm{CR}(f)\times\mathrm{CR}(g)$.

Now, let $x_0\in\mathrm{CR}(f)$, $y_0\in\mathrm{CR}(g)$ and prove that $(x_0,y_0)\in\mathrm{CR}(f\times g)$. Fix $\varepsilon>0$ and let $x_0,x_1,...,x_n,x_{n+1}=x_0$ be an ε–chain for x_0 and f, and $y_0,y_1,...,y_m,y_{m+1}=y_0$ an ε–chain for y_0 and g. We can clearly assume that $n=m$ (repeating the chains if necessary). Then,

$$(x_0,y_0),(x_1,y_1),...,(x_n,y_n),...,(x_{n+1},y_{n+1})=(x_0,y_0)$$

is an ε–chain for (x_0,y_0) and $f\times g$. Hence $(x_0,y_0)\in\mathrm{CR}(f\times g)$.

To finish the proof, assume that $X=[0,1]$ and prove that $\mathrm{C}(f\times g)=\mathrm{C}(f)\times\mathrm{C}(g)$. Since $\mathrm{C}(f)=\overline{\mathrm{P}(f)}=\overline{\mathrm{R}(f)}$ (cf. [6]), we obtain that

$$\overline{\mathrm{R}(f)}\times\overline{\mathrm{R}(g)}=\overline{\mathrm{P}(f)}\times\overline{\mathrm{P}(g)}=\overline{\mathrm{P}(f)\times\mathrm{P}(g)}=\overline{\mathrm{P}(f\times g)}\subseteq\mathrm{C}(f\times g),$$

and jointly with (e) we conclude the proof. □

Theorem 2.2. *Let (X,d) be a compact metric space. Consider $f,g\in C(X,X)$ and let $F(x,y)=(g(y),f(x))$. Then*

(a) $\mathrm{P}(F)=\mathrm{P}(g\circ f)\times\mathrm{P}(f\circ g)$.
(b) $\mathrm{AP}(F)=\mathrm{AP}(g\circ f)\times\mathrm{AP}(f\circ g)$.
(c) $\mathrm{UR}(F)\subseteq\mathrm{UR}(g\circ f)\times\mathrm{UR}(f\circ g)$.
(d) $\mathrm{R}(F)\subseteq\mathrm{R}(g\circ f)\times\mathrm{R}(f\circ g)$.
(e) $\mathrm{C}(F)\subseteq\mathrm{C}(g\circ f)\times\mathrm{C}(f\circ g)$.
(f) $\omega(F)\subseteq\omega(g\circ f)\times\omega(f\circ g)$.
(g) $\Omega(F^2)\subseteq\Omega(g\circ f)\times\Omega(f\circ g)$.
(h) $\mathrm{CR}(F)=\mathrm{CR}(g\circ f)\times\mathrm{CR}(f\circ g)$.

If in addition $X=I$, then $\mathrm{C}(F)=\mathrm{C}(g\circ f)\times\mathrm{C}(f\circ g)$.

Proof. Just notice that $F^2=(g\circ f)\times(f\circ g)$ and apply Proposition 2.1. □

Now we fix $X=I$. It was proved in [3] that the inclusion (d) of Theorem 2.2 can be strict, that is, there is an antitriangular map F holding

$$\mathrm{R}(F)\subsetneqq\mathrm{R}(g\circ f)\times\mathrm{R}(f\circ g). \tag{2.7}$$

We are able to give an example showing that

$$\mathrm{UR}(F)\subsetneqq\mathrm{UR}(g\circ f)\times\mathrm{UR}(f\circ g), \tag{2.8}$$

that is, the inclusion (c) of Theorem 2.2 can be strict. To this end, consider the trapezoidal tent map $f(x)=\max\{1-|2x-1|,\mu\}$ ($\mu=0.8249...$) from [11]. The idea for constructing the example is the following: any infinite ω–limit set of f is contained in a solenoidal structure. This structure can be labelled by codes which characterizes the elements of infinite ω–limit sets (see below). We take two uniformly recurrent points x_0,y_0 belonging to the same infinite

ω–limit set and such that x_0, y_0 are labelled by the same code. We prove that $(x_0, y_0) \notin \mathrm{UR}(F)$ for the map $F(x,y) = (y, f(x))$.

Now, we need some definitions. For any $Z \subseteq \mathbb{Z}$ let $Z^\infty = \{\alpha = (\alpha_i)_{i=1}^\infty : \alpha_i \in Z,\ i \in \mathbb{N}\}$. For $n \in \mathbb{N}$ let $Z^n = \{(\alpha_1, \alpha_2, ..., \alpha_n) : \alpha_i \in Z,\ 1 \leq i \leq n\}$. If $\theta \in \mathbb{Z}^n$ and $\vartheta \in \mathbb{Z}^m$, $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{\infty\}$, then $\theta * \vartheta \in \mathbb{Z}^{n+m}$ (where $n + \infty$ means ∞) will denote the sequence λ defined by $\lambda_i = \theta_i$ if $1 \leq i \leq n$ and $\lambda_i = \vartheta_{i-n}$ for any $i > n$. In what follows we denote $\mathbf{0} = (0, 0, \ldots, 0, \ldots)$ and $\mathbf{1} = (1, 1, \ldots, 1, \ldots)$, while if $\alpha \in \mathbb{Z}^\infty$ then $\alpha|_n \in \mathbb{Z}^n$ is defined by $\alpha|_n = (\alpha_1, \alpha_2, \ldots, \alpha_n)$.

Proposition 2.3. *Let f be the trapezoidal map defined above. Consider the antitriangular map $F(x,y) = (y, f(x))$. Then*

$$\mathrm{UR}(F) \subsetneqq \mathrm{UR}(f) \times \mathrm{UR}(f).$$

Proof. By [11], f has periodic points of periods 2^n, $n \in \mathbb{N} \cup \{0\}$. By [9, Proposition 1], there is a family $\{K_\alpha\}_{\alpha \in \mathbb{Z}^\infty}$ of pairwise disjoint (possibly degenerate) compact subintervals of $[0,1]$ satisfying the following properties.

(P1) The interval $K_{\mathbf{0}}$ contains all absolute maxima of f.

(P2) Define in $\mathbb{Z}^\infty$ the following total ordering: if $\alpha, \beta \in \mathbb{Z}^\infty$, $\alpha \neq \beta$ and k is the first integer such that $\alpha_k \neq \beta_k$ then $\alpha < \beta$ if either $\mathrm{Card}\{1 \leq i < k : \alpha_i \leq 0\}$ is even and $\alpha_k < \beta_k$ or $\mathrm{Card}\{1 \leq i < k : \theta_i \leq 0\}$ is odd and $\beta_k < \alpha_k$. Then $\alpha < \beta$ if and only if $K_\alpha < K_\beta$ (that is, $x < y$ for all $x \in K_\alpha$, $y \in K_\beta$).

(P3) Let $\alpha \in \mathbb{Z}^\infty$, $\alpha \neq \mathbf{0}$, and let k be the first integer such that $\alpha_k \neq 0$. Define $\beta \in \mathbb{Z}^\infty$ by $\beta_i = 1$ for $1 \leq i \leq k-1$, $\beta_k = 1 - |\alpha_k|$ and $\beta_i = \alpha_i$ for $i > k$. Then $f(K_\alpha) = K_\beta$. Also $f(K_{\mathbf{0}}) \subset K_{\mathbf{1}}$.

(P4) For any n and $\theta \in \mathbb{Z}^n$, let K_θ be the least interval including all intervals K_α, $\alpha \in \mathbb{Z}^\infty$, such that $\alpha|_n = \theta$. Then, for any $\alpha \in \mathbb{Z}^\infty$, $K_\alpha = \bigcap_{n=1}^\infty K_{\alpha|_n}$.

(P5) If $\omega(x,f)$ is an infinite ω-limit set of f, then $\omega(x,f) \subset \{0,1\}^\infty$, where $\{0,1\}^\infty$ denotes the set of infinite sequences $(\alpha_i)_{i=1}^\infty$ with $\alpha_i \in \{0,1\}$ for all $i \in \mathbb{N}$.

(P2) gives us information on the positions of $\{K_\alpha\}_{\alpha \in \mathbb{Z}^\infty}$ in $[0,1]$ while (P3) gives us information on the dynamics of the family of intervals $\{K_\alpha\}_{\alpha \in \mathbb{Z}^\infty}$. On the other hand, since f has an interval of absolute maxima, (P1) gives us that $K_{\mathbf{0}}$ is non–degenerate. Let $[x_0, y_0] = K_{\mathbf{0}}$. By (P2) it is straightforward to prove that

$$f^{2^n(1+2k)}(K_{\mathbf{0}}) \subseteq K_{\mathbf{0}|_n * (1)} \tag{2.9}$$

for all $n, k \in \mathbb{N}$. Then (P3) gives us that

$$f^{2^{2n-1}(1+2k)}(K_{\mathbf{0}}) < K_{\mathbf{0}} < f^{2^{2n}(1+2k)}(K_{\mathbf{0}}) \tag{2.10}$$

for $n, k \in \mathbb{N}$. We claim that $x_0, y_0 \in \mathrm{UR}(f)$. In order to see this fix $|K_{\mathbf{0}}| > \varepsilon > 0$ small enough and consider the open interval $(x_0 - \varepsilon, x_0 + \varepsilon)$. By (P4) $K_{\mathbf{0}|_{2n-1} * (1)} \subset (x_0 - \varepsilon, x_0 + \varepsilon)$ if $2n - 1 \geq m$ and $K_{\mathbf{0}|_{2n} * (1)} \cap (x_0 - \varepsilon, x_0 + \varepsilon) = \varnothing$ for all $n \in \mathbb{N}$. By (2.9) and (2.10), $f^{2^{2n-1}(1+2k)}(x_0) \in K_{\mathbf{0}|_{2n-1} * (1)} \subset (x_0 - \varepsilon, x_0 + \varepsilon)$

if $2n-1 \geq m$ for all $k \in \mathbb{N}$. This gives us $x_0 \in \mathrm{UR}(f)$ (similarly it can be proved that $y_0 \in \mathrm{UR}(f)$). Let

$$x_0(f) := \{n \in \mathbb{N} : f^{2^n}(x_0) < K_{\mathbf{0}}\}$$

and

$$y_0(f) := \{n \in \mathbb{N} : f^{2^n}(y_0) > K_{\mathbf{0}}\}.$$

Consider the antitriangular map $F(x,y) = (y, f(x))$. Then, it is clear that $F^2(x,y) = (f(x), f(y))$. By [5, Proposition 3.5], $(x_0, y_0) \in \mathrm{UR}(F)$ if and only if $x_0(f) \cap y_0(f)$ is infinite. However, by (2.10), $x_0(f) \cap y_0(f) = \varnothing$. Therefore $(x_0, y_0) \notin \mathrm{UR}(F)$ while $(x_0, y_0) \in \mathrm{UR}(f) \times \mathrm{UR}(f)$. This concludes the proof. $\square$

We are not able to say anything about the inclusions (f)–(g) of Theorem 2.2, but we are able to give an example of an antitriangular map F such that $\Omega(F) \not\subseteq \Omega(g \circ f) \times \Omega(f \circ g)$ (compare with (a)–(f)). To this end, we prove the following lemma.

Lemma 2.4. *Let $F(x,y) = (f(y), f(x))$ be an antitriangular map with $f \in C(I,I)$. Then $x \in \Omega(f^n)$ if and only if $(x,x) \in \Omega(F^n)$ for any integer $n \geq 1$.*

Proof. First assume that $x \in \Omega(f^n)$ and let $V \subset I^2$ be an open neighborhood of (x,x). Let $U \subset I$ be an open set such that $(x,x) \in U \times U \subset V$. Since $x \in \Omega(f^n)$, there is a positive integer m such that $(f^n)^m(U) \cap U \neq \varnothing$. Hence according to (1.4) and (1.5) we have

$$F^{nm}(U \times U) \cap (U \times U) \neq \varnothing,$$

which gives us $(x,x) \in \Omega(F^n)$. On the other hand, let $(x,x) \in \Omega(F^n)$. Let $U \subset I$ be an open set such that $x \in U$. Since $(x,x) \in \Omega(F^n)$, there is an $m \in \mathbb{N}$ such that $(F^n)^m(U \times U) \cap (U \times U) \neq \varnothing$. Then $f^{nm}(U) \cap U \neq \varnothing$ and $x \in \Omega(f^n)$. $\square$

Lemma 2.4 allows us to show that in general

$$\Omega(F) \not\subseteq \Omega(g \circ f) \times \Omega(f \circ g). \tag{2.11}$$

To see this, let $\widetilde{f}$ be a continuous interval map such that $\Omega(\widetilde{f}) \setminus \Omega(\widetilde{f}^2) \neq \varnothing$ (see [7]) and define $\widetilde{F}(x,y) = (\widetilde{f}(y), \widetilde{f}(x))$ (in this case $g \circ f = f \circ g = \widetilde{f}^2$). Let $x \in \Omega(\widetilde{f}) \setminus \Omega(\widetilde{f}^2)$. By Lemma 2.4 we clearly obtain that $x \in \Omega(\widetilde{f})$ implies $(x,x) \in \Omega(F)$, while $(x,x) \notin \Omega(\widetilde{f}^2) \times \Omega(\widetilde{f}^2) = \Omega(g \circ f) \times \Omega(f \circ g)$. Additionally, we obtain that $(x,x) \in \Omega(F) \setminus \Omega(F^2)$.

In [7] it is shown that in the case of continuous interval maps every succession of equalities and strict inclusions is possible in the chain

$$\Omega(f) \supsetneqq \Omega(f^2) \supsetneqq \Omega(f^{2^2}) \supsetneqq \Omega(f^{2^3}) \supsetneqq \ldots \supsetneqq \Omega(f^{2^n}) \supsetneqq \Omega(f^{2^{n+1}}) \supsetneqq \ldots \tag{2.12}$$

According to Lemma 2.4, the same happens in the case of antitriangular maps for the chain of inclusions

$$\Omega(F) \supsetneqq \Omega(F^2) \supsetneqq \Omega(F^{2^2}) \supsetneqq \Omega(F^{2^3}) \supsetneqq \ldots \supsetneqq \Omega(F^{2^n}) \supsetneqq \Omega(F^{2^{n+1}}) \supsetneqq \ldots$$

It suffices to take $F(x,y) = (f(y), f(x))$ where f holds (2.12).

Again concerning the non-wandering set, (2.11) gives us that it can happen

$$\Omega(g \circ f) \subsetneqq \pi_1(\Omega(F)) \text{ and } \Omega(f \circ g) \subsetneqq \pi_2(\Omega(F)),$$

where π_i represents the canonical projection, $i = 1, 2$. Notice that it is straightforward to see that

$$\pi_1(\mathcal{A}(F)) = \mathcal{A}(g \circ f) \text{ and } \pi_2(\mathcal{A}(F)) = \mathcal{A}(f \circ g)$$

for $\mathcal{A}(\cdot) \in \{\mathrm{P}(\cdot), \mathrm{AP}(\cdot), \mathrm{UR}(\cdot), \mathrm{R}(\cdot), \mathrm{C}(\cdot), \omega(\cdot), \mathrm{CR}(\cdot)\}$.

Finally, we are able to prove that equalities are possible in Theorem 2.2 under some particular assumptions. We will see this in the next section.

3. Chain of inclusions

3.1. **General properties about the chain of inclusions.** Let $f : I \to I$ be continuous. Then [6] provides that $\mathrm{C}(f) \subseteq \omega(f)$ and the inclusions from (1.1) and (1.2) can be rewritten as follows:

$$\mathrm{P}(f) \subseteq \mathrm{AP}(f) \subseteq \mathrm{UR}(f) \subseteq \mathrm{R}(f) \subseteq \mathrm{C}(f) \subseteq \omega(f) \subseteq \Omega(f) \subseteq \mathrm{CR}(f). \tag{3.13}$$

Moreover the above inclusions can be strict.

Proposition 3.1. *There exists a continuous map $f_0 : I \to I$ such that*

$$\begin{aligned} \mathrm{P}(f_0) &\neq \mathrm{AP}(f_0) \neq \mathrm{UR}(f_0) \neq \mathrm{R}(f_0) \\ &\neq \mathrm{C}(f_0) \neq \omega(f_0) \neq \Omega(f_0) \neq \mathrm{CR}(f_0). \end{aligned}$$

Proof. In [14, Theorem 4.6] we can find an interval map $\widetilde{f}$ such that

$$\mathrm{P}(\widetilde{f}) \neq \mathrm{AP}(\widetilde{f}) \neq \mathrm{UR}(\widetilde{f}) = \mathrm{R}(\widetilde{f}) \neq \mathrm{C}(\widetilde{f}) \neq \omega(\widetilde{f}) \neq \Omega(\widetilde{f}) \neq \mathrm{CR}(\widetilde{f}).$$

Then we define a new continuous map by

$$f_0(x) = \begin{cases} \widetilde{f}(3x), \text{ if } x \in [0, \frac{1}{3}], \\ \text{affine in } [\frac{1}{3}, \frac{2}{3}], \\ f(3x-2), \text{ if } x \in [\frac{2}{3}, 1], \end{cases}$$

where $f : I \to I$ is a continuous map with positive topological entropy (see [1] for definition). Then $\mathrm{UR}(f) \neq \mathrm{R}(f)$ (see [14, Theorem 4.19]) and by an standard argument (see e.g. [7]) f_0 holds the statement. □

Here we investigate if (3.13) and Proposition 3.1 are true in the setting of antitriangular maps. From definitions it is clear that

$$\mathrm{C}(F) \subseteq \Omega(F), \tag{3.14}$$

but we are unable to say nothing about the inclusion

$$\mathrm{C}(F) \subseteq \omega(F). \tag{3.15}$$

For instance, this inclusion does not work for triangular maps, that is, two–dimensional maps with the form $T(x, y) = (f(x), g(x, y))$ (see [10]).

Clearly, $\mathrm{C}(F) = \overline{\mathrm{R}(F)} \subseteq \overline{\omega(F)}$. However, it is not known if $\omega(F)$ is closed. This would give us $\overline{\omega(F)} = \omega(F)$, which would prove (3.15).

It is well known that $\mathrm{C}(f) = \overline{\mathrm{R}(f)} = \overline{\mathrm{P}(f)}$ in the case of interval maps (see [6]). However this is false for triangular maps ([10]). Now in the case of antitriangular maps we obtain the following result.

Theorem 3.2. *Let $F(x,y) = (g(y), f(x))$ be an antitriangular map. Then* $\mathrm{C}(F) = \overline{\mathrm{P}(F)}$.

Proof. Since $\mathrm{P}(F) \subseteq \mathrm{R}(F)$, it is obvious that $\overline{\mathrm{P}(F)} \subseteq \overline{\mathrm{R}(F)}$. In order to prove the converse inclusion, we use [6] and Theorem 2.2 to write

$$\overline{\mathrm{R}(F)} \subseteq \overline{\mathrm{R}(g\circ f)} \times \overline{\mathrm{R}(f\circ g)} = \overline{\mathrm{P}(g\circ f)} \times \overline{\mathrm{P}(f\circ g)} = \overline{\mathrm{P}(F)},$$

which ends the proof. □

In order to prove more results, we need some additional hypothesis on f. A continuous interval map $f : I \to I$ is called a *piecewise monotone map* when there are $0 = a_1 < a_2 < \ldots < a_n = 1$ such that $f|_{[a_i,a_{i+1}]}$ is either decreasing or increasing for $1 \le i < n$. Then we can prove the following result.

Proposition 3.3. *Let $F(x,y) = (g(y), f(x))$ be an antitriangular map such that f and g are piecewise monotone maps. Then*

$$\omega(F) \subseteq \mathrm{C}(F) = \overline{\omega(F)}.$$

Proof. If f, g are piecewise monotone maps then $g\circ f$ and $f\circ g$ are also piecewise monotone maps. According to [4, Proposition 22, Chapter IV], $\overline{\mathrm{P}(g\circ f)} = \omega(g\circ f)$ and $\overline{\mathrm{P}(f\circ g)} = \omega(f\circ g)$. By Theorem 2.2 and [6]

$$\begin{aligned}\omega(F) &\subseteq \omega(g\circ f)\times\omega(f\circ g) = \overline{\mathrm{P}(g\circ f)} \times \overline{\mathrm{P}(f\circ g)} = \\ &= \overline{\mathrm{R}(g\circ f)} \times \overline{\mathrm{R}(f\circ g)} = \overline{\mathrm{R}(F)} = \mathrm{C}(F).\end{aligned}$$

On the other hand, since $\mathrm{P}(F) \subseteq \omega(F)$, by Theorem 3.2 we obtain

$$\begin{aligned}\mathrm{C}(F) &= \overline{\mathrm{P}(F)} \subseteq \overline{\omega(F)} \subseteq \overline{\omega(g\circ f)} \times \overline{\omega(f\circ g)} \\ &= \overline{\mathrm{P}(g\circ f)} \times \overline{\mathrm{P}(f\circ g)} = \overline{\mathrm{P}(F)} = \mathrm{C}(F).\end{aligned}$$

□

For antitriangular maps it is possible to find an interesting periodic structure, similar to the Šarkovskiĭ's ordering (see [13]). It is known that $\mathrm{Per}(g\circ f) = \mathrm{Per}(f\circ g)$ and either $\mathrm{Per}(F) = \wp$ or $\mathrm{Per}(F) = \wp \cup \{2\}$, with

$$\wp = 2\left(\mathrm{Per}(g\circ f)\setminus\{1\}\right) \cup \{k \in \mathrm{Per}(g\circ f) : k \text{ odd}, \, k \ge 1\},$$

where $2A = \{2a : a \in A\}$, for $A \subseteq \mathbb{N}$ (see [2]). Then we say that F has type less, equal or bigger than 2^∞ if the related one-dimensional map $g\circ f$ has the corresponding type. Then the following result makes sense.

Theorem 3.4. *Let $F(x,y) = (g(y), f(x))$ be an antitriangular map such that* $\mathrm{Per}(F) \subseteq \{2^n : n \in \mathbb{N}\cup\{0\}\}$. *Assume that* $\mathrm{P}(F)$ *is a closed set. If* $\mathcal{A}(\cdot), \mathcal{B}(\cdot) \in \{\mathrm{P}(\cdot), \mathrm{AP}(\cdot), \mathrm{UR}(\cdot), \mathrm{R}(\cdot), \mathrm{C}(\cdot), \omega(\cdot), \Omega(\cdot), \mathrm{CR}(\cdot)\}$ *it holds*

(a) $\mathcal{A}(F) = \mathcal{B}(F)$.

(b) $\mathcal{A}(F) = \mathcal{A}(g\circ f)\times\mathcal{A}(f\circ g)$. *Moreover* $\Omega(F^2) = \Omega(F)$.

Proof. If $\mathrm{P}(F)$ is closed, according to Theorem 2.2 we find $\overline{\mathrm{P}(F)} = \overline{\mathrm{P}(g \circ f)} \times \overline{\mathrm{P}(f \circ g)} = \mathrm{P}(g \circ f) \times \mathrm{P}(f \circ g) = \mathrm{P}(F)$, so $\mathrm{P}(g \circ f)$ and $\mathrm{P}(f \circ g)$ are closed, hence $\mathrm{P}(g \circ f) = \mathcal{A}(g \circ f)$, $\mathrm{P}(f \circ g) = \mathcal{A}(f \circ g)$, where $\mathcal{A}(.)$ represents one of the others seven sets ([14, Theorem 4.11]). Therefore

$$\mathrm{CR}(F) = \mathrm{CR}(g \circ f) \times \mathrm{CR}(f \circ g) = \mathrm{P}(g \circ f) \times \mathrm{P}(f \circ g) = \mathrm{P}(F).$$

Now (1.1), (1.2) and Theorem 2.2 end the proof. $\square$

3.2. **An example of strict inclusions.** Now we study if Proposition 3.1 holds in the case of antitriangular maps. To this end, first we prove the following lemma.

Lemma 3.5. *Let $F(x, y) = (y, f(x))$ be an antitriangular map defined on I^2. Let $x \in I$. Then $x \in \mathcal{A}(f)$ if and only if $(x, x) \in \mathcal{A}(F)$ where $\mathcal{A}(\cdot) \in \{\mathrm{P}(\cdot), \mathrm{AP}(\cdot), \mathrm{UR}(\cdot), \mathrm{R}(\cdot), \mathrm{C}(\cdot), \omega(\cdot), \Omega(\cdot), \mathrm{CR}(\cdot)\}$.*

Proof. The cases $\mathrm{P}(\cdot)$, $\mathrm{AP}(\cdot)$, $\mathrm{C}(\cdot)$, $\mathrm{CR}(\cdot)$ hold by Theorem 2.2 and $\mathrm{UR}(\cdot)$, $\mathrm{R}(\cdot)$, $\omega(\cdot)$ follow easily from definitions. So, we prove the case $\Omega(\cdot)$. First assume that $x \in \Omega(f)$ and let $V \subset I^2$ be an open neighborhood of (x, x). Let $U \subset I$ be an open set such that $(x, x) \in U \times U \subseteq V$. Since $x \in \Omega(f)$, there is a positive integer n such that $f^n(U) \cap U \neq \varnothing$. Then

$$F^{2n}(U \times U) \cap (U \times U) = (f^n(U) \times f^n(U)) \cap (U \times U) \neq \varnothing,$$

which provides $(x, x) \in \Omega(F)$. Second, assume that $(x, x) \in \Omega(F)$ and let $U \subset I$ be an open neighborhood of x. Since $(x, x) \in \Omega(F)$, there is a positive integer m such that $F^m(U \times U) \cap (U \times U) \neq \varnothing$. We have two possibilities: (1) $m = 2n$ for $n \in \mathbb{N}$ and (2) $m = 2n + 1$ for $n \in \mathbb{N}$. If (1) happens, then

$$F^{2n}(U \times U) \cap (U \times U) = (f^n(U) \times f^n(U)) \cap (U \times U) \neq \varnothing$$

and $f^n(U) \cap U \neq \varnothing$. If (2) happens, then

$$F^{2n+1}(U \times U) \cap (U \times U) = (f^n(U) \times f^{n+1}(U)) \cap (U \times U) \neq \varnothing,$$

and $f^n(U) \cap U \neq \varnothing$ and $f^{n+1}(U) \cap U \neq \varnothing$. In both cases $x \in \Omega(f)$, which ends the proof. $\square$

From Lemma 3.5 the following result follows.

Theorem 3.6. *There is an antitriangular map F_0 such that*

$$\begin{aligned} \mathrm{P}(F_0) &\neq \mathrm{AP}(F_0) \neq \mathrm{UR}(F_0) \neq \mathrm{R}(F_0) \\ &\neq \mathrm{C}(F_0) \neq \omega(F_0) \neq \Omega(F_0) \neq \mathrm{CR}(F_0). \end{aligned}$$

Proof. Just define $F_0(x, y) = (y, f_0(x))$, f_0 given by Proposition 3.1 and apply Lemma 3.5. $\square$

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