

## Coreflectively modified continuous duality applied to classical product theorems

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**ABSTRACT.** Several (old and new) results on stability of quotients (of various types) under product, on sequentiality of product of sequential spaces, on relationships between a topology and the upper Kuratowski convergence on its closed sets are derived from a general mechanism of duality that uses the continuous convergence. Coreflectively modified biduals with respect to the continuous convergence lead to new reflectors which are of fundamental interest in this quest.

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**Keywords:** Product topology and product maps; Quotient, hereditarily quotient, countably biquotient, biquotient maps; Sequential, strongly sequential, Fréchet, strongly Fréchet topology; Convergence, product convergence, Antoine convergence, continuous convergence; Cartesian-closedness, exponential objects.

### 1. INTRODUCTION

In this paper, I present a general mechanism of continuous duality together with various applications concerning sequentiality of products of sequential convergences, quotientness of products of quotient maps and relations between a convergence and the upper Kuratowski convergence on its closed sets (homeomorphically Scott convergence on the complete lattice of its open sets). Even if one is only interested in the corollaries for topologies, this is the framework of general convergences that enables to develop a unified theory.

A convergence  $\xi$  on a set  $X$  is a relation between points and filters on  $X$  denoted

$$\mathcal{F} \xrightarrow[\xi]{} x \text{ or } x \in \lim_{\xi} \mathcal{F},$$

that fulfills<sup>1</sup>

$$\begin{aligned} \mathcal{F} \leq \mathcal{G} \text{ and } \mathcal{F} \rightarrow x &\implies \mathcal{G} \rightarrow x; \\ (x) \rightarrow x; \\ \mathcal{F} \rightarrow x \text{ and } \mathcal{G} \rightarrow x &\implies \mathcal{F} \wedge \mathcal{G} \rightarrow x, \end{aligned}$$

for each  $x \in X$ , where  $(x)$  denotes the principal ultrafilter generated by  $x$ . Every topology can be considered as a convergence and there exists important naturally defined non topological convergences. If  $X$  is a convergence space, let  $|X|$  denote the underlying set. A map  $f : X \rightarrow Y$  between two convergence spaces is *continuous* if

$$f(\mathcal{F} \rightarrow f(x)) \text{ whenever } \mathcal{F} \rightarrow x \text{ for every } x \in |X|.$$

I write  $X \geq Y$  if  $|X| = |Y|$  and if the identity carried map  $\text{Id}_{X,Y} : X \rightarrow Y$  is continuous. The category **Conv** whose objects are the convergence spaces and with continuous maps as morphisms is a topological category [1] (I refer to this book for every undefined categorical notion). In other words, for every sink  $(f_i : X_i \rightarrow |Y|)_{i \in I}$  there exists the finest convergence on  $|Y|$  making every  $f_i$  continuous. This convergence is called the *final convergence associated to*  $(f_i : X_i \rightarrow |Y|)_{i \in I}$ . Dually, for every source  $(f_i : |X| \rightarrow Y_i)_{i \in I}$  there exists the coarsest convergence on  $|X|$  making every  $f_i$  continuous. I denote by  $\vec{f}X$  the final convergence space associated to  $f : X \rightarrow |Y|$  and by  $\overleftarrow{f}Y$  the initial convergence space associated to  $f : |X| \rightarrow Y$ . Hence usual categorical constructions such as products, coproducts, subspaces and so on always exist in **Conv**. Moreover, every such construction in the sequel is assumed to be performed in **Conv** if no contrary mention is given. In particular,  $X \times Y$  denotes the **Conv**-product of  $X$  and  $Y$ , that is

$$\mathcal{M} \xrightarrow{X \times Y} (x, y) \text{ if and only if } \mathcal{M} \geq \mathcal{F} \times \mathcal{G} \text{ with } \mathcal{F} \xrightarrow{X} x \text{ and } \mathcal{G} \xrightarrow{Y} y.$$

Notice that in case  $X$  and  $Y$  are topological,  $X \times Y$  coincide with the usual topological product. Indeed, the category **T** of topological spaces with continuous maps is a full concretely reflective subcategory of **Conv**. This means that for every convergence space  $X$  there exists a topological space  $TX$ , called *topological reflection of  $X$* , such that  $TX \leq X$  and every continuous map  $f : X \rightarrow Y$  where  $Y$  is a topological space underlies a continuous map from  $TX$  to  $Y$ . A reflective subcategory of **Conv** is closed under initial constructions (in particular product) performed in **Conv**. The map  $T$  is called a (concrete)<sup>2</sup> reflector.

All the considered categories are subcategories of **Conv** and they are denoted by bold capitals. If a subcategory is (co)reflective, the associated (co)reflector will be denoted by the same (non bold) capital letter. For example if **J** is a reflective subcategory of **Conv**, the associated reflector is  $J : \mathbf{Conv} \rightarrow \mathbf{J}$ .

<sup>1</sup>the subscript  $\xi$  is omitted when no confusion is possible.

<sup>2</sup>In this paper, every (co)reflector is concrete in the sense of [1], that is a convergence and its (co)reflection have the same underlying set. Such (co)reflectors are also called bi(co)reflectors [36] or rigid (co)reflectors [35].

In [10], B. J. Day and G. M. Kelly investigated the two following dual questions:

- (1) Characterize topological spaces  $X$  for which

$$\text{Id}_X \times f : X \times W \rightarrow X \times Z$$

is quotient for every quotient map  $f : W \rightarrow Z$ .

- (2) Characterize the continuous surjections  $f : X \rightarrow W$  between topological spaces for which

$$f \times \text{Id}_Y : X \times Y \rightarrow W \times Y$$

is quotient, for every topological space  $Y$ .

Let me give a convergence-theoretic rephrasing of these two problems. By definition, a map  $f : X \rightarrow Y$  between topological spaces is quotient if it is final in the full subcategory  $\mathbf{T}$  of  $\mathbf{Conv}$ . As  $\mathbf{T}$  is reflective in  $\mathbf{Conv}$ , this means that  $Y$  is the topological reflection of the final convergence space  $\overrightarrow{f}X$ . In other words,  $f : X \rightarrow Y$  is quotient if and only if

$$(1.1) \quad Y \geq T(\overrightarrow{f}X).$$

This is meaningful for arbitrary convergences and a continuous surjection  $f : X \rightarrow Y$  in  $\mathbf{Conv}$  is called *quotient* or *T-quotient* if (1.1) is satisfied<sup>3</sup>. As  $\mathbf{Conv}$  is cartesian-closed,

$$\overrightarrow{f \times g}(X \times Y) = \overrightarrow{f}X \times \overrightarrow{g}Y,$$

so that the Day-Kelly questions extended to  $\mathbf{Conv}$  can be rephrased as follows<sup>4</sup>:

- (1) Characterize convergence (topological) spaces  $X$  for which

$$(1.2) \quad \forall Y \in \text{Ob}(\mathbf{Conv}), \quad X \times TY \geq T(X \times Y).$$

- (2) Characterize maps  $f : X \rightarrow W$  such

$$(1.3) \quad \forall Y \in \text{Ob}(\mathbf{Conv}), \quad W \times Y \geq T(\overrightarrow{f}X \times Y).$$

In order to deal with both (1.2) and (1.3) simultaneously, we have to investigate conditions (on  $X$  and  $W$  with the same underlying set) for the following to hold

$$(1.4) \quad \forall Y \in \text{Ob}(\mathbf{Conv}), \quad W \times JY \geq T(X \times Y),$$

where  $J$  is a reflector, which can be either  $T$  (first case) or the identity functor (second case), but which can also be something else. Moreover, I expect the above scheme to also handle relativizations like the following variant, due to E. Michael, of the classical Whitehead theorem.

**Theorem 1.1.** [32, Theorem 4.1] *A regular (Hausdorff) topological space is locally countably compact if and only if the product of its identity with every quotient map from a sequential topological space is quotient.*

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<sup>3</sup>Of course the notion coincide with the usual categorical notion of quotient in  $\mathbf{T}$  if both  $X$  and  $Y$  are topological spaces. With the definition (1.1) a map need not be domained and codomained in  $\mathbf{T}$ -objects to be  $\mathbf{T}$ p-quotient.

<sup>4</sup>take  $Y = \overrightarrow{f}W$  in (1.2).

Recall that a topology is sequential if sequentially closed and closed sets coincide. This can be rephrased in convergence-theoretic terms [11]. A convergence is *sequentially based* if  $\mathcal{F} \rightarrow x$  implies that there exists a sequence  $(x_n)_{\mathbb{N}}$  such that  $\mathcal{F} \geq (x_n)_{\mathbb{N}}$  and  $(x_n)_{\mathbb{N}} \rightarrow x$ . Analogously, a convergence is *first-countable* if  $\mathcal{F} \rightarrow x$  implies that there exists a countably based filter  $\mathcal{H}$  such that  $\mathcal{F} \geq \mathcal{H}$  and  $\mathcal{H} \rightarrow x$ . Both categories **Seq** of sequentially based convergence spaces and **First** of first-countable convergence spaces are (concretely) coreflective subcategories of **Conv**. Obviously, a topological space  $X$  is sequential if and only if  $X \geq T \text{Seq } X$ . It is moreover equivalent to

$$(1.5) \quad X \geq T \text{First } X.$$

This is meaningful for arbitrary convergence spaces and allows to deal with sequentiality in the general context of **Conv**.

It turns out (see [17] for details) that spaces  $X$  for which

$$\forall Y \in \text{Ob}(\mathbf{First}), X \times TY \geq T(X \times Y),$$

are those that verify  $\text{Id}_X \times f$  is quotient for every quotient map  $f$  with sequential domain. In other words, (1.4) should be coreflectively relativized.

This approach initiated in [17] is systematized. In this paper, given two (concretely) reflective subcategories **L** and **J** of **Conv** and a (concretely) coreflective subcategory **E**, I investigate the general problem

$$(1.6) \quad \forall Y \in \text{Ob}(\mathbf{E}), W \times JY \geq L(X \times Y),$$

where  $X$  and  $W$  are two (possibly equal) convergence spaces with the same underlying set. I obtain general results on (1.6) which turn out to be particularly efficient to derive corollaries on product of quotient maps, when **J**, **L**, **E** are particularized. Indeed, it is known from the works of D. C. Kent [26], of H.L Bentley, H. Herrlich and R. Lowen [4] and of S. Dolecki [11] that quotient, hereditarily quotient, countably biquotient and biquotient maps are  $J$ -quotient maps where **J** stands for the reflective subcategory (of **Conv**) **T** of topologies, **P** of pretopologies, **P<sub>ω</sub>** of paratopologies and **S** of pseudotopologies respectively. The following gathers most of such corollaries that are obtained in this paper. In the two following tables, the parenthesis stand for “equivalently”, while conditions written in italic are supplementary assumptions.

	for every $g$	$f \times g$ is	iff $f$ is
1	quotient	quotient	$A$ -quotient with core-compact <i>topological range</i>
2	quotient with sequential domain		$A_\omega$ -quotient with core-contour(First)-compact <i>topological range</i>
3	hereditarily quotient		$A$ -quotient with $T$ -core-compact range
4	hereditarily quotient with Fréchet domain		$A_\omega$ -quotient with $T$ -core-countably compact range
5	countably biquotient		$A$ -quotient with core-bi- $k$ range
6	countably biquotient with strongly Fréchet domain		$A_\omega$ -quotient with core-bi-quasi- $k$ range
7	$A$ -quotient (identity)	quotient ( $A$ -quotient)	$A$ -quotient
8	$A$ -quotient with quasi-bisequential domain (identity of metrizable topology)	quotient ( $A_\omega$ -quotient)	$A_\omega$ -quotient
9	hereditarily quotient	hereditarily quotient	biquotient with finitely generated range
10	biquotient with finitely generated range		hereditarily quotient
11	hereditarily quotient with Fréchet domain		countably biquotient with finitely generated range
12	countably biquotient	hereditarily quotient (countably biquotient)	biquotient with bisequential range
13	countably biquotient with strongly Fréchet domain		countably biquotient with bisequential range
14	biquotient with bisequential range (identity of metrizable topology)	countably biquotient	countably biquotient
15	biquotient (identity)	hereditarily quotient (biquotient)	biquotient

$A$ -quotient and  $A_\omega$ -quotient maps are the quotient maps in the categories of Antoine convergences and of countably Antoine convergences respectively. In particular results 1 to 4 and 7 and 8 were proved in [17] while the results 5, 6

and 9 to 13 seem to be new. Row 14 is [33, Proposition 4.4] and row 15 can be found in [4] or deduced from [39, Theorem 3].

Given two convergence spaces  $X$  and  $Y$ , the *continuous convergence*  $[X, Y]$  is the coarsest convergence space on the set  $C(X, Y)$  of continuous functions from  $X$  to  $Y$  that makes the evaluation (jointly) continuous.  $[X, Y]$  is called the (continuous) *dual of  $X$*  (with respect to  $Y$ ) and  $[[X, Y], Y]$  is called the *bidual of  $X$* . Convergences initially determined by their biduals will play a key role in the study of (1.6). Such convergences have been intensively studied in convergence theory. Most important examples of this type are the pseudotopologies of Choquet [7], the  $c$ -spaces of Binz and the Antoine convergences [8]. In contrast, even if coreflectively modified biduals

$$[E[X, Y], Y],$$

where  $\mathbf{E}$  denotes a (concretely) coreflective subcategory of  $\mathbf{Conv}$  were used by D. C. Kent and G. Richardson in [28] and by D. C. Kent and R. Frič in [21] in the study of sequential envelopes<sup>5</sup>, they seem to be used here for the first time in the context of general convergences. This approach enabled the author to solve in [34] a problem of Y. Tanaka of characterizing topologies whose product with every first-countable topology is sequential. This result corresponds to the first row of the following table that gathers the corollaries of the general mechanism in terms of product of sequential spaces. The details concerning the first row can be found in [34], while the results of rows 2 and 3 are detailed and proved in [17], but they follow from a single general result, just like row 4 [33, Proposition 4.D.4, Proposition 4.D.5] and rows 5 and 6 which seem to be new.

	for every $Y$	$X \times Y$ is	iff $X$ is
1	bisequential (metrizable)	sequential	strongly sequential
2	Fréchet		$T$ -core-countably compact strongly sequential
3	sequential		when $X$ is a topology core-contour(First)-compact strongly sequential
4	bisequential (metrizable)	Fréchet (strongly	strongly Fréchet
5	Fréchet	FrÚchet)	finitely generated
	for every $Y$	$X \times Y$ is	whenever $X$ is
6	strongly Fréchet	sequential	core-bi-quasi- $k$ and strongly sequential

<sup>5</sup>In [28] and [21], the framework is that of the category  $\mathbf{E}$  and such modification of the biduals arise from this particular context.

$T$ -core-countable-compactness and core-contour(First)-compactness are relativizations (for convergences) of the classical topological local countable compactness. Analogously, core-bi-quasi- $k$ -ness relativizes the classical topological notions of bi-quasi- $k$ -ness.

The above results (of both tables) are all corollaries of the same principle, given in the following Theorem 1.2 and in full generality in Theorem 3.1.

**Theorem 1.2.** *Let  $\mathbf{L} \subset \mathbf{J}$  denote two (concretely) reflective subcategories of  $\mathbf{Conv}$  and let  $\mathbf{E}$  be a (concretely) coreflective subcategory. The following are equivalent:*

- (1) For every  $Y \geq JEY$

$$X \times JY \geq L(X \times Y);$$

- (2)  $\text{Id}_X \times f$  is  $L$ -quotient for every  $J$ -quotient map  $f$  with  $JE$ -domain;  
 (3)  $JE[X, Z] \geq [X, Z]$  for every  $\mathbf{L}$ -object  $Z$ ;  
 (4)  $JE[X, Z] \geq [X, Z]$  for every  $Z$  in an initially dense subclass of  $\mathbf{L}$ .

This applies to relationships between a convergence space and its continuous duals. In particular, given a convergence space  $X$ , the upper Kuratowski convergence on its closed sets (equivalently the Scott convergence on the complete lattice of its open sets) can be identified with the (continuous) dual  $[X, \$]$  of  $X$  with respect to the Sierpiński topology  $\$$ . Since  $\$$  is initially dense in  $\mathbf{T}$ , Theorem 1.2 can be rephrased as follow in case  $\mathbf{L}=\mathbf{T}$ .

**Theorem 1.3.** *Let  $\mathbf{J}$  be a reflective subcategory of  $\mathbf{Conv}$  that contains  $\mathbf{T}$  and let  $\mathbf{E}$  be a coreflective subcategory of  $\mathbf{Conv}$ . The following are equivalent:*

- (1) For every  $Y \geq JEY$

$$X \times JY \geq T(X \times Y);$$

- (2)  $\text{Id}_X \times f$  is quotient for every  $J$ -quotient map  $f$  with  $JE$ -domain;  
 (3)  $JE[X, Z] \geq [X, Z]$  for every topological space  $Z$ ;  
 (4)  $JE[X, \$] \geq [X, \$]$ .

By use of coreflectively modified biduals I characterize internally the convergences  $X$  that verify the properties of Theorem 1.3, for various  $\mathbf{J}$  and  $\mathbf{E}$ .

For example the pretopologicity and the paratopologicity of the upper Kuratowski convergence (Scott convergence) are characterized. Such new items of information might lead to a better understanding of some aspects of the lattice theory or of consonance.

## 2. CONVERGENCES

The *adherence* of a filter  $\mathcal{F}$  is the union of the limits of all filters that are finer than  $\mathcal{F}$ :

$$\text{adh}_X \mathcal{F} = \bigcup_{\mathcal{G} \supset \mathcal{F}} \lim_X \mathcal{G}.$$

The *adherence*  $\text{adh}_X A$  of a subset  $A$  of  $X$  is the adherence of the principal filter of  $A$ . A set  $V$  is a  $X$ -vicinity of  $x$  whenever  $x \notin \text{adh}_X V^c$ . I denote  $\mathcal{V}_X(x)$

the set of all the vicinities of  $x$ . I depart here from the usual terminology of convergence theory, where the adherence is called the “closure” and a vicinity is called a “neighborhood”. I reserve the latter terms for analogous notions related to those of closed and open sets. A subset  $A$  of  $X$  is  $X$ -closed whenever for every filter  $\mathcal{F}$  with  $A \in \mathcal{F}$ , one has  $\lim_X \mathcal{F} \subset A$ . A set is  $X$ -open if its complement is  $X$ -closed. The closure  $\text{cl}_X A$  is the least closed set that includes  $A$ . A set  $V$  is a neighborhood of  $x$  if and only if  $x \notin \text{cl}_X V^c$ . The set of all the neighborhoods is denoted by  $\mathcal{N}_X(x)$ .

**2.1. Reflectors and coreflectors.** A convergence space  $X$  is *topological* (or the convergence is a topology) if  $\mathcal{F} \rightarrow x$  amounts to  $\mathcal{F} \geq \mathcal{N}(x)$ ; *pretopological* if  $\mathcal{F} \rightarrow x$  amounts to  $\mathcal{F} \geq \mathcal{V}(x)$ ; *pseudotopological* if and only if

$$\lim_X \mathcal{F} = \bigcap_{\mathcal{U} \in \beta(\mathcal{F})} \lim_X \mathcal{U},$$

where  $\beta(\mathcal{F})$  denotes the set of all the ultrafilters finer than  $\mathcal{F}$ .

All these classes are closed for arbitrary suprema in the complete lattice of convergences. Moreover, the initial convergence of a topology (resp., pretopology, pseudotopology) is a topology (resp., pretopology, pseudotopology). In terms of the category theory, the above classes (together with continuous maps) are concretely reflective subcategories of the category **Conv**. The map  $J$  that associates to every convergence space  $X$  the finest convergence space coarser than  $X$  from such a class is a contractive and idempotent functor. Such a (concrete) functor is a *reflector*. Concrete reflectors are exactly contractive and idempotent concrete functors. Actually, functors should be defined on morphisms. However, concrete endofunctors (that is functors  $F : \mathbf{Conv} \rightarrow \mathbf{Conv}$  such that  $|\cdot| \circ F = |\cdot|$ ) can be characterized objectwise because **Conv** is a topological category.

**Proposition 2.1.** [18] *Let  $(\mathbf{A}, |\cdot|)$  be a topological category (over **Set**). A map  $F : \text{Ob}(\mathbf{A}) \rightarrow \text{Ob}(\mathbf{A})$  such that  $|\cdot| \circ F = |\cdot|$  is the restriction of a concrete functor to the objects of  $\mathbf{A}$  if and only if  $F$  is order-preserving and*

$$(2.7) \quad \overrightarrow{f}(FA) \geq F(\overrightarrow{f}A),$$

for each  $\mathbf{A}$ -object  $A$  and each **Set**-morphism  $f : |A| \rightarrow Y$ ; if and only if  $F$  is order-preserving and

$$(2.8) \quad F\overleftarrow{f}(B) \geq \overleftarrow{f}(FB),$$

for each  $\mathbf{A}$ -object  $B$  and each **Set**-morphism  $f : X \rightarrow |B|$ .

In the sequel every (co)reflector  $F$  (and more generally every functor if not specified differently) is supposed to be a concrete endofunctor of **Conv**, that is,  $F : \mathbf{Conv} \rightarrow \mathbf{F}$  and a convergence space and its (co)reflection have the same underlying set.

Two families of subsets  $\mathcal{A}$  and  $\mathcal{B}$  *mesh* ( $\mathcal{A}\#\mathcal{B}$ ) if  $A \cap B \neq \emptyset$  for each  $A \in \mathcal{A}$  and each  $B \in \mathcal{B}$ . The elements of a class  $\mathfrak{F}$  of filters are called  $\mathfrak{F}$ -filters. A class  $\mathfrak{F}$  of filters is said to be *composable* if it contains principal filters and if  $\mathcal{HG}$ , the



filter generated by  $\{HG : H \in \mathcal{H}, G \in \mathcal{G}\}$ <sup>6</sup>, is a (possibly degenerate)  $\mathfrak{J}$ -filter on  $Y$  whenever  $\mathcal{H}$  is a  $\mathfrak{J}$ -filter on  $X \times Y$  and  $\mathcal{G}$  a  $\mathfrak{J}$ -filter on  $X$ . For example, the classes of principal filters and of countably based filters are composable, while that of sequential filters is not.

The map  $\text{Adh}_{\mathfrak{J}}$  given by

$$(2.9) \quad \lim_{\text{Adh}_{\mathfrak{J}} X} \mathcal{F} = \bigcap_{\mathfrak{J} \ni \mathcal{H} \# \mathcal{F}} \text{adh}_X \mathcal{H}$$

is a (concrete) reflector if  $\mathfrak{J}$  is a composable class of filters.

If  $\mathfrak{J}$  is the class of all filters, then  $\text{Adh}_{\mathfrak{J}} = S$ ; the class of principal filters, then  $\text{Adh}_{\mathfrak{J}} = P$ . If we take  $\mathfrak{J}$  to be the class of countably based filters, then (2.9) defines the *paratopological modification*  $P_{\omega}X$  of  $X$  [11].

Dually, a class  $\mathbf{E}$  of convergence spaces closed for arbitrary infima and for final convergences is a (concretely) coreflective subcategory of  $\mathbf{Conv}$ . The corresponding *coreflector*  $E$  associates to each convergence space  $X$  the coarsest  $\mathbf{E}$ -object finer than  $X$ . The map  $E$  is order-preserving, expansive and idempotent. Moreover

$$(2.10) \quad \overrightarrow{f}(EX) \geq E(\overrightarrow{f}X),$$

for each map  $f : X \rightarrow Y$ . The latter amounts to  $E(\overleftarrow{f}Y) \geq \overleftarrow{f}(EY)$ . The concrete coreflectors in the category  $\mathbf{Conv}$  are characterized (modulo Proposition 2.1) by the above properties.

If  $\mathfrak{J}$  denotes a class of filters, the coreflector  $\text{Base}_{\mathfrak{J}}$  on  $\mathfrak{J}$ -based convergences is defined by

$$(2.11) \quad \lim_{\text{Base}_{\mathfrak{J}} X} \mathcal{F} = \bigcup_{\mathcal{G} \leq \mathcal{F}, \mathcal{G} \in \mathfrak{J}} \lim_X \mathcal{G}.$$

For example, if  $\mathfrak{J}$  is the class of principal filters  $\text{Base}_{\mathfrak{J}}$  is denoted  $\text{Fin}$ , the coreflector on countably based convergences is denoted  $\text{First}$ , while the coreflector on convergences based in filters generated by sequences is denoted  $\text{Seq}$ . The coreflector on discrete convergences, denoted  $\text{Dis}$  is the coreflector on convergences based in principal ultrafilters. I call a coreflector  $E$  *finitely productive* whenever

$$E(X \times Y) = EX \times EY,$$

for every  $X$  and  $Y$ . Notice that  $E(X \times Y) \geq EX \times EY$  holds for every coreflector.

**Lemma 2.2.** *If  $\mathfrak{J}$  is a composable class of filters, then  $\text{Base}_{\mathfrak{J}}$  is a finitely productive coreflector.*

*Proof.* It suffices to show that  $\mathcal{F} \times \mathcal{G}$  is a  $\mathfrak{J}$ -filter on  $X \times Y$  if  $\mathcal{F}$  is a  $\mathfrak{J}$ -filter on  $X$  and  $\mathcal{G}$  is a  $\mathfrak{J}$ -filter on  $Y$ . Both  $\mathcal{F} \times Y$  and  $X \times \mathcal{G}$  are  $\mathfrak{J}$ -filters on  $X \times Y$ , as images of  $\mathfrak{J}$ -filters under the relations  $\{(x, (x, y)) : x \in X, y \in Y\} \subset X \times (X \times Y)$  and  $\{((x, y), y) : x \in X, y \in Y\} \subset (X \times Y) \times Y$  respectively. Now  $\mathcal{F} \times \mathcal{G}$  is the image of  $\mathcal{F} \times Y$  under the  $\mathfrak{J}$ -filter  $X \times \mathcal{G}$  on the diagonal of  $(X \times Y)^2$ .  $\square$

<sup>6</sup> $HG = \{y : \exists x \in G(x, y) \in H\}$

On the other hand, it is known [11] that numerous classes of convergence spaces (and in particular of topological spaces)  $Y$  can be characterized by inequalities of the type

$$(2.12) \quad Y \geq J E Y,$$

where  $J$  is a reflector and  $E$  is a coreflector. A convergence space fulfilling (2.12) is called a *JE-convergence space*. For example, if  $E = \text{First}$ , then, by choosing  $J$  to be respectively the topologizer, pretopologizer, paratopologizer, pseudotopologizer and identity, (2.12) characterizes sequential, FrÚchet, strongly FrÚchet, bisequential and first-countable convergences; if  $E = K$ , the compact localizer<sup>7</sup>, then (2.12) characterizes  $k, k'$ , strongly  $k'$ , locally compact and, once again locally compact convergences respectively.

**2.2. Classes of quotient.** It is known that almost open, biquotient [26], countably biquotient [11], hereditarily quotient [26] and quotient maps  $f : X \rightarrow Y$  can be characterized as continuous surjections that fulfill

$$(2.13) \quad Y \geq J(\vec{f}X),$$

where  $J$  is respectively the identity, pseudotopologizer, paratopologizer, pretopologizer and topologizer. A continuous surjection  $f$  that fulfills (2.13) is called *J-quotient*. A map is a *J-quotient map* onto a **J**-object if and only if

$$(2.14) \quad Y = J(\vec{f}X).$$

If  $J$  is a reflector and  $E$  a coreflector and if  $X$  is a *JE-convergence space*, then  $\vec{f}X$  is also a *JE-convergence space*, because of (2.7). Moreover, each *J-quotient image* of a *JE-convergence* is a *JE-convergence* [11, Theorem 4.2].

Recall the following classical characterization of a quotient map.

**Proposition 2.3.** *A continuous surjection  $f : X \rightarrow Y$  is a quotient map if and only if for each topological space  $W$ , and each map  $g : Y \rightarrow W$ , the continuity of  $g \circ f$  implies the continuity of  $g$ .*

I need an analogous characterization of a *J-quotient map* with a *JE-convergence space* as domain.

**Proposition 2.4.** *Let **J** be a reflective subcategory and let **E** be a coreflective subcategory of **Conv**. If  $f : X \rightarrow Y$  is a continuous surjection with  $X \geq J E X$ , then  $f$  is *J-quotient* if and only if for each convergence space  $W \leq J E W$  (equivalently  $W \in \text{Ob}(\mathbf{J})$ ) and each map  $g : Y \rightarrow W$ , the continuity of  $(g \circ f)$  implies the continuity of  $g$ .*

*Proof.* Assume that  $f$  is *J-quotient*, i.e.,  $f : X \rightarrow Y \geq J(\vec{f}X)$ . Let  $g : Y \rightarrow W \leq J E W$ . If  $g \circ f$  is continuous, then  $\vec{g}(\vec{f}X) \geq W$ . Since  $Y \geq J(\vec{f}X)$ , one has  $\vec{g}Y \geq \vec{g}(J\vec{f}X) \geq \vec{g}(J E \vec{f}X)$  because  $X \geq J E X$  implies  $\vec{f}X \geq J E \vec{f}X$ . Since  $J E$  is a concrete functor,  $\vec{g}(J E \vec{f}X) \geq J E \vec{g}(\vec{f}X)$  by Proposition 2.1. Consequently,  $\vec{g}Y \geq J E W \geq W$  and  $g$  is continuous.

<sup>7</sup> $x \in \lim_{KX} \mathcal{F}$  if and only if  $x \in \lim_X \mathcal{F}$  and  $\mathcal{F}$  contains a compact set.

Conversely, assume that for each  $W \leq JEW$  and each  $g : Y \rightarrow W$ ,  $g \circ f$  continuous implies  $g$  continuous. Taking  $W = J(\overrightarrow{f}X)$  and  $g = \text{Id}_{Y,W}$ , the map  $g \circ f : X \rightarrow W \leq \overrightarrow{f}X$  is continuous, so that  $g$  is also continuous. Consequently,  $Y \geq J(\overrightarrow{f}X)$ .  $\square$

In particular,  $f : X \rightarrow Y$  is  $J$ -quotient if and only if the continuity of  $g \circ f$  implies that of  $g$ , for every map  $g : Y \rightarrow W$  with  $W \in \text{Ob}(\mathbf{J})$ .

**2.3. Continuous convergence.** The *continuous convergence*  $[X, Z]$  of  $X$  (with respect to a *coupling convergence*  $Z$ ) is the coarsest convergence space  $Y$  on the set of continuous maps from  $X$  to  $Z$  for which the evaluation map  $ev : X \times Y \rightarrow Z$  is continuous, that is, the coarsest such that

$$(2.15) \quad X \times Y \geq \overleftarrow{ev}Z.$$

The reason why continuous convergence appears naturally in many problems that involve products is the exponential law:

$$(2.16) \quad [X \times Y, Z] = [Y, [X, Z]],$$

for every convergence spaces  $X, Y, Z$ . Here the equality means the homeomorphism via the *transposition map*  ${}^t : |[X \times Y, Z]| \rightarrow |[Y, [X, Z]]|$  defined by  ${}^t f(y)(x) = f(x, y)$ . Notice that if  $f : X \rightarrow Y$  is continuous then  $f^* : [Y, Z] \rightarrow [X, Z]$  defined by  $f^*(h) = h \circ f$  is continuous (for every  $Z$ ).

In the particular case where the coupling convergence space is the *Sierpiński space*  $\$$  defined on  $\{0, 1\}$  by the  $\$$ -open sets  $\emptyset, \{0\}$  and  $\{0, 1\}$ , then the continuous convergence  $[X, \$]$  is the *upper Kuratowski convergence* on the set of  $X$ -closed sets. Indeed, the continuous functions for the Sierpiński topology are precisely the characteristic functions<sup>8</sup> of closed sets. The upper Kuratowski convergence (homeomorphically the Scott convergence in the lattice of open sets) plays a crucial role in the study of upper semicontinuity or of closedness of graphs [5] and also in consonance [16]. In Section 6, I give new results on the relationships between a convergence space  $X$  and  $[X, \$]$ .

### 3. GENERAL MECHANISM

A subclass  $\mathbf{D}$  of  $\text{Ob}(\mathbf{C})$  is *initially dense* in  $\mathbf{C}$  if for each  $\mathbf{C}$ -object  $W$ , there exists an initial source  $(f_i : W \rightarrow Y_i)$ , where  $Y_i \in \mathbf{D}$ <sup>9</sup>. Dually, A subclass  $\mathbf{D}$  of  $\text{Ob}(\mathbf{C})$  is *finally dense* in  $\mathbf{C}$  if for each  $\mathbf{C}$ -object  $W$ , there exists a final sink  $(f_i : Y_i \rightarrow W)$ , where  $Y_i \in \mathbf{D}$ <sup>10</sup>.

A convergence space is *atomic* if its all but one points are isolated.

Let  $\mathbf{L}$  be a reflective subcategory of  $\mathbf{Conv}$ . Convergence spaces initially determined by their biduals have been classically used in convergence theory, in particular in studies of duality. Classical examples are the  $c$ -spaces of Binz

<sup>8</sup>If  $A \subset X$ , the characteristic function is  $1_A : X \rightarrow \$$  that takes the value 1 on  $A$  and 0 on  $A^c$ .

<sup>9</sup>In other words,  $W = \bigvee \overleftarrow{f_i} Y_i$ .

<sup>10</sup>In other words,  $W = \bigwedge \overrightarrow{f_i} Y_i$ .

[6] and the Antoine (or epitopological) spaces [2]. To deal with such objects, given a convergence space  $Z$ , I define for every  $X$  the convergence spaces

$$(3.17) \quad \text{Epi}^Z X = \overleftarrow{i} [[X, Z], Z],$$

where  $i : |X| \rightarrow |[X, Z], Z|$  is the canonical map from  $X$  to its bidual with respect to  $Z$ ; and

$$(3.18) \quad \text{Epi}^L X = \bigvee_{Z \in \text{Ob}(\mathbf{L})} \text{Epi}^Z X.$$

*Antoine spaces* are characterized by  $X = \text{Epi}^S X$  or  $X = \text{Epi}^T X$  and the *c-spaces* are characterized by  $X = \text{Epi}^R X$  or  $X = \text{Epi}^O X$  where  $\mathbf{O}$  denotes the category of completely regular topological spaces. These two examples are used in “classical” duality. The new technique of “modified” duality introduced in this paper requires more general objects. Precisely, given a (concrete endo)functor  $F$  of  $\mathbf{Conv}$  for which  $FZ \geq Z$  (for every  $\mathbf{L}$ -object  $Z$ ), I define for every  $X$

$$(3.19) \quad \text{Epi}_F^Z X = \overleftarrow{i} [F[X, Z], Z],$$

and

$$(3.20) \quad \text{Epi}_F^L X = \bigvee_{Z \in \text{Ob}(\mathbf{L})} \text{Epi}_F^Z X.$$

Notice that  $FZ \geq Z$  ensures that  $i(x) : F[X, Z] \rightarrow Z$  is continuous for every  $x \in |X|$ <sup>11</sup>.

Analogously, for every  $\mathbf{L}$ -object  $Z$ ,

$$\bigvee_{f \in C(X, Z)} \overleftarrow{f} Z$$

is a  $\mathbf{L}$ -object coarser than  $X$ , so that  $LX \geq \bigvee_{f \in C(X, Z)} \overleftarrow{f} Z$ . Moreover,

$$(3.21) \quad LX = \bigvee_{Z \in \text{Ob}(\mathbf{L})} \bigvee_{f \in C(X, Z)} \overleftarrow{f} Z.$$

If  $\mathbf{D}$  is an initially dense subclass of  $\mathbf{L}$ , then,

$$(3.22) \quad \text{Epi}_F^L X = \bigvee_{Z \in \mathbf{D}} \text{Epi}_F^Z X;$$

and, by definition of initial density,

$$(3.23) \quad LX = \bigvee_{Z \in \mathbf{D}} \bigvee_{f \in C(X, Z)} \overleftarrow{f} Z.$$

**Theorem 3.1.** *Let  $X \geq W$  be two convergence spaces. Let  $\mathbf{E}$  be a coreflective subcategory and let  $\mathbf{L} \subset \mathbf{J}$  be two reflective subcategories of  $\mathbf{Conv}$ . The following are equivalent:*

<sup>11</sup>I am indebted to Mark Nauwelaerts (Antwerp) for pointing out that such a condition was missing in a preliminary version of this paper.

- (1) For every  $Y \geq JEY$
- (3.24)  $W \times JY \geq L(X \times Y);$
- (2) (3.24) holds for every  $Y \in \text{Ob}(\mathbf{E});$
- (3)  $\text{Id}_{X,W} \times f$  is  $L$ -quotient for every  $J$ -quotient map  $f$  with  $JE$ -domain;
- (4)  $C(W \times JY, Z) = C(X \times Y, Z)$  for every  $Y \in \text{Ob}(\mathbf{E})$  and every  $Z \in \text{Ob}(\mathbf{L});$
- (5)  $C(W \times JY, Z) = C(X \times Y, Z)$  for every  $Y \in \text{Ob}(\mathbf{E})$  and every  $Z$  in an initially dense subclass of  $\mathbf{L};$
- (6)  $JE[X, Z] \geq [W, Z]$  for every  $Z \in \text{Ob}(\mathbf{L}).$
- (7)  $JE([X, Z]) \geq [W, Z]$  for every convergence space  $Z$  in an initially dense subclass of  $\mathbf{L};$
- (8)  $W \geq \text{Epi}_{JE}^L X.$

The condition  $\mathbf{L} \subset \mathbf{J}$  ensures that  $JEZ \geq Z$  for every  $\mathbf{L}$ -object  $Z$ , so that  $\text{Epi}_{JE}^L$  is well-defined.

Since there is a unique convergence structure on a singleton, this (unique) convergence is fixed by every reflector and every coreflector. Thus, if  $W$  verifies (3.24) for every  $\mathbf{E}$ -object  $Y$ , it does in particular if  $Y$  is a singleton. Consequently  $W \geq LX$ . Since  $L$  is a reflector,  $C(X, Z) = C(LX, Z)$  for every  $Z \in \text{Ob}(\mathbf{L})$ , so that  $C(X, Z) = C(W, Z)$ , because  $X \geq W \geq LX$ .

*Proof.* 1  $\implies$  2 is easy.

2  $\implies$  3: Consider  $f : X_1 \rightarrow Y_1 \geq J(\vec{f} X_1)$  with  $X_1 \geq JEX_1$ . Then  $\vec{f} X_1 \geq JE(\vec{f} X_1)$  because  $JE$  is a concrete functor. Applying 2 with  $Y = E(\vec{f} X_1)$  we get  $W \times JE(\vec{f} X_1) \geq L(X \times E\vec{f} X_1) \geq L(\overline{\text{Id}_{X,W} \times \vec{f}}(X \times X_1))$ . Since  $f$  is  $J$ -quotient,  $Y_1 \geq JE(\vec{f} X_1)$  so that  $W \times Y_1 \geq L(\overline{\text{Id}_{X,W} \times \vec{f}}(X \times X_1))$ . In view of (2.13),  $\text{Id}_{X,W} \times f$  is  $L$ -quotient.

3  $\implies$  4:  $X \times Y \geq W \times JY$  because  $X \geq W$ , so that  $C(W \times JY, Z) \subset C(X \times Y, Z)$ . Consider  $g \in C(X \times Y, Z)$ . Let  $\bar{g}$  denote the map  $g$  considered from  $W \times JY$  to  $Z$ . By definition,  $g = \bar{g} \circ (\text{Id}_{X,W} \times \text{Id}_{Y,JY})$ . The map  $\text{Id}_{Y,JY}$  is  $J$ -quotient with  $JE$ -domain, so that, by 3,  $\text{Id}_{X,W} \times \text{Id}_{Y,JY}$  is  $L$ -quotient. In view of Proposition 2.4,  $\bar{g}$  is continuous because  $g$  is continuous. Thus,  $C(X \times Y, Z) \subset C(W \times JY, Z)$ .

4  $\iff$  5 is obvious in view of the definition of initial density.

4  $\implies$  6: For each  $\mathbf{L}$ -object  $Z$ , let  $Y = E([X, Z])$ . By 4,  $C(X \times E[X, Z], Z) \subset C(W \times JE[X, Z], Z)$ . Since the evaluation  $ev$  is continuous from  $X \times E[X, Z]$  to  $Z$ , it is continuous from  $W \times JE[X, Z]$  to  $Z$ . Hence,  $JE[X, Z] \geq [W, Z]$ , by definition (2.15) of  $[W, Z]$ .

6  $\iff$  7 follows from the equivalence between 4 and 5.

6  $\implies$  8: Assume that  $W \not\geq \text{Epi}_{JE}^L X$ . Then there exist  $x \in \lim_W \mathcal{F}$ ,  $Z \in \text{Ob}(\mathbf{L})$  and a filter  $\mathcal{G}$  such that  $f \in \lim_{JE[X, Z]} \mathcal{G}$  but  $f(x) \notin \lim_Z ev(\mathcal{F} \times \mathcal{G})$ . Hence  $f \notin \lim_{[W, Z]} \mathcal{G}$  so that  $JE[X, Z] \not\geq [W, Z]$ .

8  $\implies$  1: Let  $(x_0, y_0) \in \lim_{\text{Epi}_{JE}^L X \times JY} (\mathcal{F} \times \mathcal{G})$ , let  $Z \in \text{Ob}(\mathbf{L})$  and let  $f : X \times Y \rightarrow Z$  be a continuous map. In view of (3.21), it suffices to show that

$f(x_0, y_0) \in \lim_Z f(\mathcal{F} \times \mathcal{G})$ . By (2.16), the map  ${}^t f : Y \rightarrow [X, Z]$  is continuous. Since  $JE$  is a concrete functor,  ${}^t f : JEY \rightarrow JE[X, Z]$  is also continuous, so that  ${}^t f(y_0) \in \lim_{JE[X, Z]} {}^t f(\mathcal{G})$ . By definition of  $\text{Epi}_{JE}^L X$ , I conclude that  ${}^t f(y_0)(x_0) \in \lim_Z \text{ev}(\mathcal{F} \times {}^t f(\mathcal{G}))$ , in other words  $f(x_0, y_0) \in \lim_Z f(\mathcal{F} \times \mathcal{G})$ .  $\square$

In categorical terms, a  $\mathbf{J}$ -object  $X$  for which there exists a  $\mathbf{J}$ -object on  $\mathcal{C}(X, Z)$  verifying (2.16) for each  $\mathbf{J}$ -object  $Z$  is called *exponential* in the category  $\mathbf{J}$ . F. Schwarz proved the equivalence between exponentiality of a convergence  $X$  in a finally dense reflective subcategory  $\mathbf{C}$  of  $\mathbf{Conv}$  and the fact that for every  $\mathbf{C}$ -object  $Z$ , (equivalently for every  $Z$  in an initially dense subclass of  $\mathbf{C}$ ) the continuous convergence  $[X, Z]$  is a  $\mathbf{C}$ -object. In this particular case of exponentiality, he proved all the other equivalence stated in Theorem 3.1, except 1, 2 and 8. However, the equivalence between the commutation of the reflector on a reflective subcategory  $\mathbf{C}$  of a cartesian closed category (such as the category of convergences) with finite product and the  $\mathbf{C}$ -quotientness of product maps is well-known from categorists. From a convergence-theoretic point of view rather than a categorical one, the assumption that an exponential object in a subcategory  $\mathbf{J}$  of  $\mathbf{Conv}$  must be a  $\mathbf{J}$ -object is not relevant, so that I call a convergence space *quasi-exponential in  $\mathbf{J}$*  if  $X \times JY \geq J(X \times Y)$  for every convergence space  $Y$ . The following is a rephrasing of the equivalent conditions of [37, Theorem 5.1] which are of interest for my purpose.

**Theorem 3.2.** *Let  $\mathbf{B}$  be an epireflective subcategory of  $\mathbf{Conv}$  containing a finite non-indiscrete space and closed under formation of coproducts in  $\mathbf{Conv}$ ,  $\mathbf{D}$  an initially dense subclass of  $\mathbf{B}$  and  $X \in \mathbf{B}$ . The following are equivalent:*

- (1)  $X$  is exponential in  $\mathbf{B}$ ;
- (2) For each  $Y \in \text{Ob}(\mathbf{B})$ ,  $[X, Y] \in \text{Ob}(\mathbf{B})$ ;
- (3) For each  $Y \in \mathbf{D}$ ,  $[X, Y] \in \text{Ob}(\mathbf{B})$ ;
- (4)  $X \times -$  preserves quotient maps in  $\mathbf{B}$ .

Schwarz's theorem corresponds to the case  $\mathbf{L}=\mathbf{J}$ ,  $\mathbf{E}=\mathbf{Conv}$  and  $W = X \in \text{Ob}(\mathbf{J})$  in Theorem 3.1. In this particular case, Theorem 3.1 follows from F. Schwarz's work. In particular, he proved that exponential objects in  $\mathbf{T}$  are core-compact topologies and recovered the whole circle of results (quotientness of the product of an identity map with a quotient map, topologicity of the upper Kuratowski convergence, continuity of the lattice of open sets...) related to core-compactness exposed in [17](see [37, Theorem 6.5]). Moreover, Theorem 3.2 applies to epireflective subcategory of  $\mathbf{Conv}$  and not only to concretely reflective ones. This allows him to derive corollaries on exponential objects in categories of convergences that verify certain separation axioms. Analogously, some of the equivalences of Theorem 3.1 could be extended in case  $\mathbf{L}$  and  $\mathbf{J}$  are epireflective rather than concretely reflective. However, this would not be relevant in the kind of applications I am looking for, so that I prefer to use only concrete reflectors, in order to use freely comparison between convergence spaces, like in (3.24).

In contrast, to handle relativizations of the circle of results (quotientness of the product of an identity map with a quotient map with sequential domain, convergence theoretic properties related to sequentiality of the upper Kuratowski convergence...) related to core-compactness (see in the introduction the results involving  $T$ -core countable compactness or core-contour(First)-compactness [17]), the classical notion of exponentiality is not relevant anymore, because a coreflective relativization is needed.

By analogy, if  $\mathbf{J}$  is a reflective and  $\mathbf{E}$  a coreflective subcategory of  $\mathbf{Conv}$ , I call a convergence space  $\mathbf{J}$ -quasi-exponential relatively to  $\mathbf{E}$  if it verifies

$$X \times JY \geq J(X \times Y)$$

for every  $\mathbf{E}$ -object  $Y$ . In view of Theorem 3.1, the situation is then rather different from the classical exponentiality. Indeed, the relativization of  $\mathbf{J}$ -exponentiality no longer ensures that the duals  $[X, Z]$  are  $\mathbf{J}$ -objects (but only that  $JE[X, Z] \geq [X, Z]$ , which is by the way another reflective property, but weaker than belonging to  $\text{Ob}(\mathbf{J})$ ) provided  $Z$  is a  $\mathbf{J}$ -object. However, the analogy stands in the formulation 3 of Theorem 3.1 in terms of quotient maps. The introduction of a coreflector  $E$  is one direction of generalization of Schwarz's theorem. When the two reflectors  $L$  and  $J$  are no longer the same, Theorem 3.1 generalizes Theorem 3.2 in another direction.

In each case, the missing and significant step to be done is an internal characterization of convergence spaces  $W$  verifying one of the equivalent conditions of Theorem 3.1. In view of 8 in Theorem 3.1, the study of  $\text{Epi}_{JE}^L$  is a key point.

#### 4. $\text{Epi}_F^L$ FUNCTORS

**Proposition 4.1.** *Let  $\mathbf{L}$  be a reflective subcategory of  $\mathbf{Conv}$  and let  $F$  be a concrete endofunctor of  $\mathbf{Conv}$  such that  $FZ \geq Z$  for every  $\mathbf{L}$ -object  $Z$ . Then  $\text{Epi}_F^Z : \mathbf{Conv} \rightarrow \mathbf{Conv}$  and  $\text{Epi}_F^L : \mathbf{Conv} \rightarrow \mathbf{Conv}$  are concrete functors.*

*Proof.*  $\text{Epi}_F^L$  is isotone because for each  $Z$ ,  $\text{Epi}_F^Z$  is isotone. In view of Proposition 2.1, it suffices to prove

$$\vec{f}(\text{Epi}_F^Z X) \geq \text{Epi}_F^Z(\vec{f} X),$$

for every  $\mathbf{L}$ -object  $Z$ . Let  $y \in \lim_{\vec{f}(\text{Epi}_F X)} \mathcal{F}$  and let  $h \in \lim_{F[\vec{f} X, Z]} \mathcal{G}$ . I need to show that  $h(y) \in \lim_Z \text{ev}(\mathcal{F} \times \mathcal{G})$ , where  $\text{ev} : \vec{f} X \times [\vec{f} X, Z] \rightarrow Z$  is the evaluation. Since  $f^* : [\vec{f} X, Z] \rightarrow [X, Z]$  is continuous,  $f^* : F[\vec{f} X, Z] \rightarrow F[X, Z]$  is also continuous because  $F$  is a concrete functor, so that  $f^*(h) \in \lim_{F[X, Z]} f^*(\mathcal{G})$ . On the other hand, there exists  $\mathcal{L}$  such that  $f(\mathcal{L}) = \mathcal{F}$  and  $x \in \lim_{\text{Epi}_F X} \mathcal{L}$  for some  $x \in f^{-1}y$ . Let  $\text{ev}' : X \times [X, Z] \rightarrow Z$  denote the evaluation. From  $x \in \lim_{\text{Epi}_F X} \mathcal{L}$  and  $f^*(h) \in \lim_{F[X, Z]} f^*(\mathcal{G})$ , I deduce  $f^*(h)(x) \in \lim_Z \text{ev}'(\mathcal{L} \times f^*(\mathcal{G}))$ . The result follows from the observations that  $\text{ev}'(\mathcal{L} \times f^*(\mathcal{G})) = \text{ev}(f(\mathcal{L}) \times \mathcal{G}) = \text{ev}(\mathcal{F} \times \mathcal{G})$  and that  $f^*(h)(x) = h(y)$ .  $\square$

However,  $\text{Epi}_F X$  is not even always comparable to  $X$ . The only observations of interest are that

$$(4.25) \quad \text{Epi}_{JE}^L X = \text{Epi}_{JE}^L (\text{Epi}_E^L X) \geq \text{Epi}_E^L X,$$

$$(4.26) \quad \text{Epi}^L (\text{Epi}_{JE}^L X) = \text{Epi}_{JE}^L X,$$

for every convergence space  $X$ . In contrast, in case of a coreflector  $E$ , the behavior of  $\text{Epi}_E^L$  is much more convenient. Since  $i : X \rightarrow [E[X, Z], Z]$  is continuous,  $X \geq \text{Epi}_E^Z X$  for every  $Z$ , so that  $X \geq \text{Epi}_E^L X$ . On the other hand,  $\text{Epi}_E^L$  is idempotent because for every  $Z$

$$(4.27) \quad E[\text{Epi}_E^L X, Z] = E[X, Z].$$

Indeed,  $X \geq \text{Epi}_E^L X$  so that  $[\text{Epi}_E^L X, Z] \geq [X, Z]$  and in view of Theorem 3.1,  $E[X, Z] \geq [E[\text{Epi}_E^L X, Z]]$ . Since  $X \geq \text{Epi}_E^L X \geq LX$  and since  $L$  is a reflector,  $C(X, Z) = C(\text{Epi}_E^L X, Z)$  for every  $\mathbf{L}$ -object  $Z$ . Hence  $\text{Epi}_E^L$  is idempotent, contractive and isotone. In view of Proposition 4.1,

**Proposition 4.2.** *Let  $\mathbf{E}$  be a coreflective subcategory and let  $\mathbf{L}$  be a reflective subcategory of  $\mathbf{Conv}$ . Then  $\text{Epi}_E^L$  is a (concrete) reflector.*

Such reflectors will play a key role in the sequel. Indeed, by Theorem 3.1,  $\text{Epi}_E^L X \times Y \geq L(X \times Y)$  for every  $\mathbf{E}$ -object  $Y$ , so that

$$(4.28) \quad L(X \times Y) = L(\text{Epi}_E^L X \times Y).$$

Although I do not have any general decomposition theorem, in all the concrete cases I know  $\text{Epi}_{JE}^L$  is of the form  $\text{Epi}^L C \text{Epi}_E^L$  where  $C$  stands for a coreflector; what is not very surprising, in view of (4.25), (4.26) and Proposition 4.1. Hence, the following will be instrumental in applications.

**Lemma 4.3.** *Assume that for two reflective subcategory  $\mathbf{J}$  and  $\mathbf{L}$  and a coreflective subcategory  $\mathbf{E}$  of  $\mathbf{Conv}$ , there exist a reflector  $R$  and a (concrete endo)functor  $C$  of  $\mathbf{Conv}$  such that*

$$(4.29) \quad \forall_{Y \in \text{Ob}(\mathbf{E})} W \times JY \geq L(X \times Y) \iff W \geq RC \text{Epi}_E^L X.$$

Let  $f : X \rightarrow Y$  be a continuous surjection and let  $Y = \text{Epi}_E^L Y$ .

Then  $f \times g$  is  $L$ -quotient for every  $J$ -quotient map  $g$  with  $JE$ -domain if and only if  $f$  is  $\text{Epi}_E^L$ -quotient with  $RC$ -range.

*Proof.* Let

$$f : X \rightarrow Y \geq \text{Epi}_E^L \overrightarrow{f} X$$

such that  $Y \geq RCY$ ; hence  $Y \geq RC \text{Epi}_E^L \overrightarrow{f} X$ . On the other hand, if  $g : X_1 \rightarrow Y_1$  is a  $J$ -quotient map with  $X_1 \geq JEX_1$ , then  $\overrightarrow{g} X_1 \geq JE \overrightarrow{g} X_1$  because  $JE$  is a concrete functor. Consequently, (4.29) applies with  $W = Y$ ,  $X = \overrightarrow{f} X$  and  $Y = \overrightarrow{g} X_1$  to the effect that  $Y \times Y_1 \geq Y \times J \overrightarrow{g} X_1 \geq L(\overrightarrow{f} X \times \overrightarrow{g} X_1)$ . Thus  $f \times g$  is  $L$ -quotient.

Conversely, if  $f \times g$  is  $L$ -quotient for every  $J$ -quotient map  $g$  with  $JE$ -domain, then, in particular,  $Y \times Y_1 \geq L(\overrightarrow{f} X \times Y_1)$  for every  $\mathbf{E}$ -object  $Y_1$  so that  $Y \geq$



$\text{Epi}_E^L \overrightarrow{f} X$ . Thus  $f$  is  $\text{Epi}_E^L$ -quotient. On the other hand,  $Y \times JY_1 \geq L(\overrightarrow{f} X \times Y_1)$  for every  $Y_1 \in \text{Ob}(\mathbf{E})$  (taking  $g = \text{Id}_{Y_1, JY_1}$ ) so that  $Y \geq RC\text{Epi}_E^L \overrightarrow{f} X$ , because of (4.29). Since  $f$  is  $\text{Epi}_E^L$ -quotient onto  $Y = \text{Epi}_E^L Y$ , I conclude that  $Y = \text{Epi}_E^L \overrightarrow{f} X$ , so that  $Y \geq RCY$ .  $\square$

#### 4.1. Commutation of the reflector $\text{Epi}_E^L$ with product.

**Theorem 4.4.** *Let  $E$  be a finitely productive coreflector in  $\mathbf{Conv}$  and let  $\mathbf{L}$  be a reflective subcategory of  $\mathbf{Conv}$ . Then*

$$(4.30) \quad \text{Epi}_E^L X \times EY \geq \text{Epi}_E^L(X \times Y),$$

for every convergence spaces  $X$  and  $Y$ .

*Proof.* Let  $y \in \lim_{EY} \mathcal{G}$ ,  $x \in \lim_{\text{Epi}_E^L X} \mathcal{F}$  and  $h \in \lim_{E[X \times Y, Z]} \mathcal{M}$ . Denote by  $ev : (X \times Y) \times [X \times Y, Z] \rightarrow Z$  the evaluation map. I need to show that  $h(x, y) \in \lim_Z ev((\mathcal{F} \times \mathcal{G}) \times \mathcal{M})$ . By the exponential law (2.16) and the coreflectivity of  $E$ ,  ${}^t : E[X \times Y, Z] \rightarrow E[Y, [X, Z]]$  is continuous, so that  ${}^t h \in \lim_{E([Y, [X, Z]])} {}^t \mathcal{M}$ . Let  $ev_1 : Y \times [Y, [X, Z]] \rightarrow [X, Z]$  be the evaluation map. Since  $E$  is a finitely productive coreflector,  $ev_1 : EY \times E([Y, [X, Z]]) \rightarrow E[X, Z]$  is a continuous map so that  ${}^t h(y) \in \lim_{E[X, Z]} \mathcal{H}$ , where  $\mathcal{H} = ev_1(\mathcal{G} \times {}^t \mathcal{M})$ . Since  $x \in \lim_{\text{Epi}_E^L X} \mathcal{F}$ , one has  ${}^t h(y)(x) \in \lim_Z ev_2(\mathcal{F} \times \mathcal{H}) = \lim_Z ev((\mathcal{F} \times \mathcal{G}) \times \mathcal{M})$ , where  $ev_2 : X \times [X, Z] \rightarrow Z$  is the evaluation map. Consequently,  $h(x, y) \in \lim_Z ev((\mathcal{F} \times \mathcal{G}) \times \mathcal{M})$ .  $\square$

Notice that, by definition of  $\text{Epi}_E^L$ ,

$$(4.31) \quad \text{Epi}^L \geq \text{Epi}_B^L \geq \text{Epi}_E^L \geq L = \text{Epi}_{\text{Dis}}^L,$$

whenever  $E$  and  $B$  are two coreflectors such that  $E \geq B$ .

**Corollary 4.5.** *Let  $E$  and  $B$  be two finitely productive coreflectors of  $\mathbf{Conv}$  such that  $E \geq B$  and let  $\mathbf{L}$  be a reflective subcategory of  $\mathbf{Conv}$ . Then*

$$(4.32) \quad \text{Epi}^L E \text{Epi}_B^L X \times \text{Epi}_E^L Y \geq \text{Epi}_E^L(X \times Y),$$

for every  $\mathbf{B}$ -object  $Y$ .

*Proof.* From Theorem 4.4, we have

$$\text{Epi}_B^L X \times Y \geq \text{Epi}_B^L(X \times Y),$$

for every  $\mathbf{B}$ -object  $Y$ . Moreover,  $\text{Epi}_B^L \geq \text{Epi}_E^L$  because  $E \geq B$ , so that  $\text{Epi}_E^L(E\text{Epi}_B^L X \times Y) \geq \text{Epi}_E^L(X \times Y)$ . Hence,

$$(4.33) \quad E\text{Epi}_B^L X \times \text{Epi}_E^L Y \geq \text{Epi}_E^L(X \times Y).$$

On the other hand,  $\text{Epi}^L$  commutes with finite products<sup>12</sup> so that applying  $\text{Epi}^L$  to (4.33) we get (4.32).  $\square$

<sup>12</sup>Apply (4.30) two times with  $\mathbf{E}=\mathbf{Conv}$ .

As observed in the above proof,

$$(4.34) \quad \mathbf{Epi}^L X \times \mathbf{Epi}^L Y \geq \mathbf{Epi}^L (X \times Y),$$

so that, by Theorem 3.1, the category  $\mathbf{Epi}^L$  is *cartesian closed*, i.e., every  $\mathbf{Epi}^L$ -object is exponential (in the category  $\mathbf{Epi}^L$ ). Moreover, it is the cartesian closed topological hull of the category  $\mathbf{L}$ , provided atomic topologies are  $\mathbf{L}$ -objects. Indeed, by [24] (see [23, Theorem 3.9]), the cartesian closed topological hull  $\mathbf{B}$  of a subcategory  $\mathbf{A}$  of  $\mathbf{Conv}$  is characterized as a cartesian closed category such that  $\mathbf{A}$  is finally dense in  $\mathbf{B}$  and such that  $\{[X, Z] : Z, X \text{ } \mathbf{A}\text{-objects}\}$  is initially dense in  $\mathbf{B}$ . If atomic topologies are  $\mathbf{L}$ -object, then  $\mathbf{L}$  is finally dense in  $\mathbf{Conv}$ , hence in  $\mathbf{Epi}^L$ . In [37], F. Schwarz call the atomic topologies *Fröhlich spaces* and remarks that the class of  $T_1$ -Fröhlich spaces is finally dense in  $\mathbf{Conv}$ . Moreover, he proves [37, Proposition 4.4], that an epireflective subcategory of  $\mathbf{Conv}$  contains this class if and only if it contains a finite non-indiscrete space. Notice that for every convergence space  $X$ , there exists a family  $(Y_i)_i$  of atomic topological spaces on  $|X|$  such that  $X = \wedge Y_i$ . I say that atomic topologies are *concretely finally dense* in  $\mathbf{Conv}$ .

On the other hand,

$$(4.35) \quad [\wedge \vec{f}_i X_i, Z] = \vee \overleftarrow{f}_i^* [X_i, Z],$$

for every convergence space  $Z$ , every family  $(X_i)_i$  of convergence spaces and every family of surjective maps  $f_i$  (see for example [30, Proposition 0.2]). Consequently,  $\{[X, Z] : X, Z \in \text{Ob}(\mathbf{L})\}$  is obviously initially dense in  $\mathbf{Epi}^L$ .

**Corollary 4.6.** *Let  $\mathbf{L}$  be a reflective subcategory of  $\mathbf{Conv}$ . The cartesian closed hull of  $\mathbf{L}$  is the category  $\mathbf{Epi}^L$ , provided  $\mathbf{L}$  is finally dense<sup>13</sup> in  $\mathbf{Epi}^L$ .*

Hence, Theorem 3.1 allows to describe both exponential object in reflective subcategories of  $\mathbf{Conv}$  and cartesian closed hulls of such subcategories. By analogy, I call *cartesian closed hull relatively to a coprojector  $E$*  of a finally dense subcategory  $\mathbf{C}$  of  $\mathbf{Conv}$  the smallest category  $\mathbf{HE}$  containing  $\mathbf{C}$  such that  $f \times \text{Id}_Y$  is  $\mathbf{HE}$ -quotient for every  $\mathbf{HE}$ -quotient map  $f$  and every  $\mathbf{E}$ -object  $Y$ . In this context, the cartesian closed hull relatively to  $E$  of a reflective subcategory  $\mathbf{L}$  of  $\mathbf{Conv}$  is the category  $\mathbf{Epi}_{\mathbf{E}}^L$  (see Theorem 4.8). Once again, the relativization of the classical concept leads to a rather different situation.

The two following theorems summarize the situations (when  $\mathbf{J} = \mathbf{Conv}$  in (3.24)) concerning the preservation of  $LE$ -properties under product on one hand, and concerning product of quotient maps on the other hand.

**Theorem 4.7.** *Let  $E$  be a finitely productive (endo)coreflector of  $\mathbf{Conv}$  and let  $\mathbf{L}$  be a reflective subcategory of  $\mathbf{Conv}$ . The following are equivalent:*

- (1)  $W \times Y$  is a  $LE$ -convergence space for every  $Y$  in a concretely finally dense subclass of  $\mathbf{E}$ ;
- (2)  $W \times Y$  is a  $LE$ -convergence space for every  $\mathbf{E}$ -object  $Y$ ;

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<sup>13</sup>It suffices that there exists a finite non-indiscrete  $\mathbf{L}$ -object.

- (3)  $W \times Y$  is a  $\text{Epi}_E^L E$ -convergence space for every  $\text{Epi}^L E$ -convergence space  $Y$ ;
- (4)  $W$  is a  $\text{Epi}_E^L E$ -convergence space.

*Proof.*  $4 \implies 3$ : Theorem 4.5 applies with  $X = EY$  and  $Y = EW$  to the effect that  $\text{Epi}_E^L EW \times \text{Epi}^L EY \geq \text{Epi}_E^L(EW \times EY) \geq \text{Epi}_E^L E(W \times Y)$ . Hence,  $W \times Y \geq \text{Epi}_E^L E(W \times Y)$  because  $W \geq \text{Epi}_E^L EW$  and  $Y \geq \text{Epi}^L EY$ .

$3 \implies 2 \implies 1$  is obvious.

$1 \implies 4$  In view of Theorem 3.1 applied with  $\mathbf{J}=\mathbf{Conv}$ ,  $X = EW$ , it suffices to show that  $W \times Y \geq L(EX \times Y)$  for every  $\mathbf{E}$ -object  $Y$ . Therefore, consider a family  $(Y_i)_i$  such that  $Y = \wedge Y_i$  and  $W \times Y_i$  is a  $LE$ -convergence for every  $i$ . Then,  $W \times Y_i \geq L(EX \times Y_i) \geq L(EX \times Y)$  for every  $i$ , so that  $W \times Y = W \times \wedge Y_i \geq L(EX \times Y)$ . Hence  $W \times Y$  is a  $LE$ -convergence.  $\square$

**Theorem 4.8.** *Let  $E$  be a finitely productive (endo)coreflector of  $\mathbf{Conv}$  and let  $\mathbf{L}$  be a reflective subcategory of  $\mathbf{Conv}$ . Let  $f : X_1 \rightarrow Y_1$  be a continuous surjection. The following are equivalent:*

- (1)  $f$  is  $\text{Epi}_E^L$ -quotient;
- (2)  $f \times \text{Id}_Y$  is  $L$ -quotient for every  $Y$  in a concretely finally dense subclass of  $\mathbf{E}$ ;
- (3)  $f \times \text{Id}_Y$  is  $L$ -quotient for every  $\mathbf{E}$ -object  $Y$ ;
- (4)  $f \times g$  is  $\text{Epi}_E^L$ -quotient for every  $\text{Epi}^L$ -quotient map  $g$  with  $\text{Epi}^L E$ -range<sup>14</sup>.

*Proof.*  $4 \implies 3 \implies 2$  is obvious.

$2 \implies 1$  In view of Theorem 3.1 with  $\mathbf{J}=\mathbf{Conv}$ ,  $W = Y_1$  and  $X = \vec{f} X_1$ , it suffices to show that  $Y_1 \times Y \geq L(\vec{f} X_1 \times Y)$  if  $Y \in \text{Ob}(\mathbf{E})$ . Consider a family  $(W_i)$  in the concretely finally dense subclass of  $\mathbf{E}$  such that  $Y = \wedge W_i$ . For every  $i$ , the map  $f \times \text{Id}_{W_i}$  is  $L$ -quotient so that  $Y_1 \times W_i \geq L(\vec{f} X_1 \times W_i) \geq L(\vec{f} X_1 \times Y)$ . Thus,  $Y_1 \times Y = Y_1 \times \wedge W_i \geq L(\vec{f} X_1 \times Y)$ . By Theorem 3.1,  $Y_1 \geq \text{Epi}_E^L \vec{f} X_1$ .

$1 \implies 4$ : Let  $g : X_2 \rightarrow Y_2 \geq \text{Epi}^L \vec{g} X_2$  with  $Y_2 \geq \text{Epi}^L EY_2$ . Hence,  $Y_2 \geq \text{Epi}^L E\text{Epi}^L \vec{g} X_2$ . Since  $\text{Epi}^L \geq \text{Epi}_E^L$ , in view of (4.34),  $X \times \text{Epi}^L Y \geq \text{Epi}_E^L(X \times Y)$  for every convergence spaces  $X$  and  $Y$ , so that  $\text{Epi}_E^L(X \times Y) = \text{Epi}_E^L(X \times \text{Epi}^L Y)$ . Moreover, by Theorem 4.5,

$$(4.36) \quad \text{Epi}_E^L X \times \text{Epi}^L E\text{Epi}^L Y \geq \text{Epi}_E^L(X \times Y),$$

for every  $X$  and  $Y$ . Consequently,  $Y_1 \times Y_2 \geq \text{Epi}_E^L \vec{f} X_1 \times \text{Epi}^L E\text{Epi}^L \vec{g} X_2 \geq \text{Epi}_E^L(\vec{f} X_1 \times \vec{g} X_2)$ , assigning in (4.36)  $X = \vec{f} X_1$  and  $Y = \vec{g} X_2$ .  $\square$

Recall that, for example, atomic topological spaces are concretely finally dense in  $\mathbf{Conv}$  while metrizable atomic topological spaces are concretely finally dense in first countable convergence spaces.

<sup>14</sup>Notice that since  $\text{Epi}_E^L$  is a reflector while  $E$  is a coreflector, the range of a  $\text{Epi}_E^L$ -quotient map is a  $\text{Epi}_E^L E$ -convergence space whenever the domain is a  $\text{Epi}_E^L E$ -convergence space [11, Theorem 4.2].

Notice that in case  $\mathbf{E}=\mathbf{Conv}$  and  $\mathbf{L}$  is finally dense in  $\mathbf{Conv}$ , then, in view of Corollary 4.6, Theorem 4.8 states that a map is product-stable in  $\mathbf{L}$  in the sense of Schwarz [39]<sup>15</sup> if and only if it is quotient in the cartesian closed hull of  $\mathbf{L}$  [39, Theorem 3].

In view of Theorem 3.1 internal characterizations of convergence spaces  $W$  for which

$$W \times JY \geq L(X \times Y),$$

for every  $\mathbf{E}$ -object  $Y$  (for various instances coreflective subcategory  $\mathbf{E}$  and of reflective subcategories  $\mathbf{L}$  and  $\mathbf{J}$  of  $\mathbf{Conv}$ ) provides a large collection of applications. The challenging problem is to provide internal characterizations of  $\text{Epi}_{\mathbf{J}E}^L X$ . As said before  $\text{Epi}_E^L$  has a better structural behavior than  $\text{Epi}_{\mathbf{J}E}^L$ . Moreover, in view of Theorems 4.7 and 4.8, interesting results would be derived from internal characterizations of  $\text{Epi}_E^L$ -reflections. Consequently, I begin with  $\mathbf{J}=\mathbf{Conv}$ .

I primarily study concrete cases in which the category  $\mathbf{L}$  is *simple*, that is there exists a  $\mathbf{L}$ -object  $Z_0$  such that  $\{Z_0\}$  is initially dense in the category of  $\mathbf{L}$ .

Hence, when  $\mathbf{L}$  is simple, there exists  $Z_0$  such that for every convergence space  $X$ ,

$$(4.37) \quad LX = \bigvee_{f \in C(X, Z_0)} \overset{\leftarrow}{f} Z_0.$$

In the next sections, the category  $\mathbf{L}$  is either the (simple) category  $\mathbf{T}$  of topological spaces or the (simple) category  $\mathbf{P}$  of pretopological spaces. The corresponding initially dense convergence spaces  $Z_0$  are respectively the Sierpiński topology  $\$$ <sup>16</sup> and the pretopology  $\mathfrak{Y}$ <sup>17</sup>. Hence, for every coreflective subcategory  $\mathbf{E}$  of  $\mathbf{Conv}$ ,

$$(4.38) \quad TX = \bigvee_{f \in C(X, \$)} \overset{\leftarrow}{f} \$ \quad \text{and} \quad \text{Epi}_E^T = \text{Epi}_E^{\$};$$

$$(4.39) \quad PX = \bigvee_{f \in C(X, \mathfrak{Y})} \overset{\leftarrow}{f} \mathfrak{Y} \quad \text{and} \quad \text{Epi}_E^P = \text{Epi}_E^{\mathfrak{Y}}.$$

## 5. COREFLECTIVELY MODIFIED ANTOINE CONVERGENCES

I gave in [34] the following characterization of  $\text{Epi}_E^{\$}$ , in case  $E = \text{Base}_{\mathfrak{J}}$  for a composable class of filters  $\mathfrak{J}$ . In the present case ( $\mathbf{L}=\mathbf{T}$ ,  $Z_0 = \$$  and  $E = \text{Base}_{\mathfrak{J}}$ ), I use the following conventions:

$$\text{Epi}_{\text{Base}_{\mathfrak{J}}}^T = \text{Epi}_{\text{Base}_{\mathfrak{J}}}^{\$} = A_{\mathfrak{J}},$$

<sup>15</sup>that is,  $f \times \text{Id}_Y$  is  $L$ -quotient for every  $Y \in \text{Ob}(\mathbf{L})$ .

<sup>16</sup>that is, the two point set  $\{0, 1\}$  in which  $\mathcal{V}(1) = \{0, 1\}$  and  $\mathcal{V}(0) = \{\{0\}, \{0, 1\}\}$ .

<sup>17</sup>The underlying set of  $\mathfrak{Y}$  is the three point set  $\{0, 1, 2\}$  endowed with the following pretopology:  $\mathcal{V}(0) = \{\mathfrak{Y}\}$ ,  $\mathcal{V}(1) = \{\mathfrak{Y}\}$ ,  $\mathcal{V}(2) = \{\{0, 1, 2\}, \{1, 2\}\}$ . See [7, II.2] for details.

and,

$$\text{Epi}^\S = A; \quad \text{Epi}_{\text{First}}^\S = A_\omega.$$

Indeed,  $A$  is the reflector on Antoine convergence spaces (see [7], [30]) and I call a convergence space  $X$   $\mathfrak{J}$ -Antoine whenever  $X = A_{\mathfrak{J}}X$ . If  $\mathfrak{J}$  is the class of countably based filters, I call such a convergence space *countably Antoine*.

By definition,

$$\text{ad}_X A = \bigcup_{a \in A} \lim_X(a), \quad \text{ad}_X^* A = \{x \in |X| : \lim_X(x) \cap A \neq \emptyset\}.$$

Let  $\mathcal{H}_{\text{ad}_{TX}}$  denote the filter generated by  $\{\text{ad}_{TX} H : H \in \mathcal{H}\}$ , and let  $\mathfrak{J}_{\text{ad}_{TX}}$  denote the class of  $\mathfrak{J}$ -filters  $\mathcal{H}$  for which  $\mathcal{H} = \mathcal{H}_{\text{ad}_{TX}}$ .

**Theorem 5.1.** [34, Theorem 2.2] *If  $\mathfrak{J}$  is a composable class of filters, then the reflector  $A_{\mathfrak{J}} : \mathbf{Conv} \rightarrow \mathbf{A}_{\mathfrak{J}}$  is given by*

$$\lim_{A_{\mathfrak{J}}X} \mathcal{F} = \bigcap_{\mathfrak{J}_{\text{ad}_{TX}} \ni \mathcal{H} \# \mathcal{F}} \text{cl}_X(\text{adh}_X \mathcal{H}).$$

By Corollary 4.6, the category of Antoine convergences is the cartesian closed hull of the category of topologies [7]. When  $\mathfrak{J}$  is the class of all filters, Theorem 5.1 provides a characterization of Antoine convergences. The characterization of Bourdaud [7] can be easily derived from this one (see [17]). In view of Theorem 4.8,  $A$ -quotient maps are exactly the maps introduced by Day and Kelly in [10] to characterize maps whose product with every identity is quotient. Moreover,  $A_{\mathfrak{J}}$ -quotient maps allowed to derive in [17] new variants of the theorem of Day and Kelly. The following is a combination of Theorems 11.7 and 11.3 of [17], but follows from Theorem 5.1 and Theorem 4.8. A filter  $\mathcal{L}$  is  $X$ - $\mathfrak{J}$ -compactoid in a family  $\mathcal{V}$  if  $\text{adh}_X \mathcal{H} \# \mathcal{V}$  whenever  $\mathcal{H}$  is a  $\mathfrak{J}$ -filter such that  $\mathcal{H} \# \mathcal{L}$ .

**Theorem 5.2.** *Let  $f : X \rightarrow Y$  be a continuous surjection. Then the following are equivalent:*

- (1)  $f$  is  $A_{\mathfrak{J}}$ -quotient;
- (2) If  $y \in \lim_Y \mathcal{F}$ , then  $\mathcal{F}$  is  $A_{\mathfrak{J}}(\overrightarrow{f}X)$ - $\mathfrak{J}$ -compactoid in  $\mathcal{N}_{\overrightarrow{f}X}(y)$ <sup>18</sup>;
- (3) If  $\mathcal{H} \in \mathfrak{J}_{\text{ad}_{T\overrightarrow{f}X}}$  and  $y \in \text{adh}_Y \mathcal{H}$ , then

$$f^-(y) \cap \text{cl}_{\overleftarrow{f}\overrightarrow{f}X}(\text{adh}_X f^-\mathcal{H}) \neq \emptyset;$$

- (4) If  $y \in \lim_Y \mathcal{F}$ ,  $V$  is a  $\overrightarrow{f}X$ -open set containing  $y$ , and  $\S$  is a  $X$ - $\mathfrak{J}$ -cover<sup>19</sup> of  $f^{-1}V$ , there exists a finite subfamily  $\P \subset \S$  such that the intersection of all  $\overrightarrow{f}X$ -open sets containing  $\bigcup_{P \in \P} f(P)$  is an element of  $\mathcal{F}$ ;

<sup>18</sup>If  $f$  is  $A_{\mathfrak{J}}$ -quotient, it is in particular quotient. In this case  $\mathcal{N}_{\overrightarrow{f}X}(y) = \mathcal{N}_Y(y)$  and  $A_{\mathfrak{J}}(\overrightarrow{f}X) = Y$  provided  $Y$  is a  $\mathfrak{J}$ -Antoine convergence space.

<sup>19</sup>Let  $\S_c = \{S^c : S \in \S\}$ . Recall that a family  $\S$  of subsets of  $X$  is a  $X$ - $\mathfrak{J}$ -cover of  $A \subset X$  whenever  $(\widehat{\S})_c$ , where  $\widehat{\S}$  stands for the ideal generated by  $\S$ , is a  $\mathfrak{J}$ -filter and  $\text{adh}_X(\widehat{\S})_c \cap A = \emptyset$  [12].

- (5)  $f \times \text{Id}_W$  is quotient for each  $\mathfrak{J}$ -based convergence space (equivalently for each convergence space in a concretely finally dense subclass of  $\mathfrak{J}$ -based convergence spaces<sup>20</sup>  $W$ ;
- (6)  $f \times g$  is  $A_{\mathfrak{J}}$ -quotient for every  $A$ -quotient map  $g$  with  $A\text{Base}_{\mathfrak{J}}$ -range.

Of course results deriving from Theorem 4.7 are just another facet of the same mechanism. The internal characterization of Theorem 5.1 leads in [34] to a resolution of a problem of Tanaka of characterizing topologies whose product with every first countable topology is sequential. Hence, the following [34, Theorem 3.1] follows from Theorem 4.7 (with  $\mathbf{L}=\mathbf{T}$  and  $E = \text{First}$ ) and Theorem 5.1.  $A_{\omega}$  First-convergences, called *strongly sequential*, are of particular interest in the study of product of sequential spaces (see [34] for details).

**Theorem 5.3.** *The following are equivalent:*

- (1)  $X$  is strongly sequential;
- (2)  $\text{adh}_X \mathcal{H} \subset \text{cl}_{\text{First } X}(\text{adh}_{\text{First } X} \mathcal{H})$  for each countably based  $\mathcal{H}$  such that  $\mathcal{H} = \mathcal{H}_{\text{ad}_{\mathbf{T}X}}$ ;
- (3)  $X \times Y$  is sequential for each first-countable convergence space  $Y$ ;
- (4)  $X \times Y$  is sequential for each metrizable atomic topological space  $Y$ ;
- (5)  $X \times Y$  is strongly sequential for each quasi-bisequential convergence space  $Y$ .

A convergence space  $X$  is *quasi-bisequential* whenever  $X \geq A\text{First } X$ . Recall that a topological space  $X$  is *bisequential* if there exists a countably based filter  $\mathcal{H} \# \mathcal{F}$  such that  $x \in \lim_X \mathcal{H}$  whenever  $x \in \lim_X \mathcal{F}$  (see [33]). As indicated in section 2, this definition can be extended to convergences via  $X \geq S\text{First } X$ . By Theorem 5.1 (see also [7]),  $AX = SX$  for each Hausdorff convergence space  $X$ , so that quasi-bisequentiality and bisequentiality coincide for Hausdorff convergence spaces.

As examples, I rewrite Theorem 5.2 in case  $\mathfrak{J}$  is the class of all filters, and in case  $\mathfrak{J}$  is the class of countably based filters.

**Corollary 5.4.** *Let  $f : X \rightarrow Y$  be a continuous surjection. Then the following are equivalent:*

- (1)  $f$  is  $A$ -quotient;
- (2) If  $y \in \lim_Y \mathcal{F}$ , then  $\mathcal{F}$  is  $A(\vec{f}X)$ -compactoid in  $\mathcal{N}_{\vec{f}X}(y)$ ;
- (3) If  $y \in \lim_Y \mathcal{F}$ ,  $V$  is a  $\vec{f}X$ -open set containing  $y$ , and  $\mathfrak{s}$  is a  $X$ -cover of  $f^{-1}V$ , there exists a finite subfamily  $\mathfrak{q} \subset \mathfrak{s}$  such that the intersection of all  $\vec{f}X$ -open sets containing  $\bigcup_{P \in \mathfrak{q}} f(P)$  is an element of  $\mathcal{F}$ ;
- (4)  $f \times \text{Id}_W$  is quotient for each convergence space (equivalently each atomic Hausdorff topological space)  $W$ ;
- (5)  $f \times g$  is quotient for every  $A$ -quotient map  $g$ .

**Corollary 5.5.** *Let  $f : X \rightarrow Y$  be a continuous surjection. Then the following are equivalent:*

<sup>20</sup>For example for  $\mathfrak{J}$ -based atomic topologies.

- (1)  $f$  is  $A_\omega$ -quotient;
- (2) If  $y \in \lim_Y \mathcal{F}$ , then  $\mathcal{F}$  is countably  $A_\omega(\vec{f}X)$ -compactoid in  $\mathcal{N}_{\vec{f}X}(y)$ ;
- (3) If  $y \in \lim_Y \mathcal{F}$ ,  $V$  is a  $\vec{f}X$ -open set containing  $y$ , and  $\S$  is a countable  $X$ -cover of  $f^{-1}V$ , there exists a finite subfamily  $\P \subset \S$  such that the intersection of all  $\vec{f}X$ -open sets containing  $\bigcup_{P \in \P} f(P)$  is an element of  $\mathcal{F}$ ;
- (4)  $f \times \text{Id}_W$  is quotient for each first-countable convergence space (equivalently each metrizable atomic topological space)  $W$ ;
- (5)  $f \times g$  is quotient for every  $A_\omega$ -quotient map  $g$  with quasi-bisequential domain.

Corollary 5.4 recovers [10, Theorem 2] and could be deduced from [39, Theorem 3] and from an internal characterization of Antoine convergences (the one of G. Bourdaud [7] or from Theorem 5.1). Indeed, Antoine convergences form the cartesian closed hull of  $\mathbf{T}$  and a map is product-stable in a subcategory of  $\mathbf{Conv}$  if and only if it is quotient in its cartesian closed hull [39, Theorem 3]. In contrast, Corollary 5.5<sup>21</sup> seems to be of a new type and does not follow from a general result that involves classical categorical notions.

## 6. MIXED COMMUTATION OF THE TOPOLOGIZER AND OF ANOTHER REFLECTOR WITH PRODUCT

In the foregoing section I studied the case  $\mathbf{L}=\mathbf{T}$  and  $\mathbf{J}=\mathbf{Conv}$  in (3.24). In this section I investigate the case in which  $\mathbf{L}=\mathbf{T}$ ,  $E = \text{Base}_{\mathfrak{J}}$  and  $J = \text{Adh}_{\mathfrak{D}}$  for two composable classes of filters  $\mathfrak{J}$  and  $\mathfrak{D}$ . Therefore, I need a new concept of  $Q_{\mathfrak{D},\mathfrak{J}}$ -convergence that generalizes the notion of  $T$ -core-compactness used in [17] to characterize quasi-exponential convergence in  $\mathbf{T}$ . Let  $Q_{\mathfrak{D},\mathfrak{J}}$  be defined by  $x \in \lim_{Q_{\mathfrak{D},\mathfrak{J}}X} \mathcal{F}$  if and only if  $x \in \lim_X \mathcal{F}$  and for every  $V \in \mathcal{N}_X(x)$ , there exists a  $\mathfrak{D}$ -filter  $\mathcal{C}_V \subset \mathcal{F}$  which is  $X$ - $\mathfrak{J}$ -compactoid in  $V$ . The map  $Q_{\mathfrak{D},\mathfrak{J}}$  is isotone and expansive, but in general not idempotent.

**Proposition 6.1.**  *$Q_{\mathfrak{D},\mathfrak{J}}$  is a concrete endofunctor of  $\mathbf{Conv}$ . The subcategory  $\mathbf{E}_{\mathfrak{D},\mathfrak{J}}$  of  $\mathbf{Conv}$  of fixed objects for  $Q_{\mathfrak{D},\mathfrak{J}}$  is coreflective.*

*Proof.* By Proposition 2.1 it suffices to prove

$$\vec{f}(Q_{\mathfrak{D},\mathfrak{J}}X) \geq Q_{\mathfrak{D},\mathfrak{J}}(\vec{f}X).$$

Let  $y \in \lim_{\vec{f}(Q_{\mathfrak{D},\mathfrak{J}}X)} \mathcal{F}$  and let  $V \in \mathcal{N}_{\vec{f}X}(y)$ . Since

$$f^{-1}(\mathcal{N}_{\vec{f}X}(y)) = \mathcal{N}_{\vec{f}X}(f^{-1}y)$$

[17, Lemma 6.5],  $f^{-1}V \in \mathcal{N}_{\vec{f}X}(f^{-1}y) \subset \mathcal{N}_X(f^{-1}y) \subset \mathcal{N}_X(x)$ . Now, there exists a filter  $\mathcal{G}$  such that  $f(\mathcal{G}) = \mathcal{F}$  and  $x \in \lim_{Q_{\mathfrak{D},\mathfrak{J}}X} \mathcal{G}$  for some  $x \in f^{-1}y$ . Thus, for every  $U \in \mathcal{N}_X(x)$ , in particular for  $f^{-1}V$ , there exists a  $\mathfrak{D}$ -filter  $\mathcal{C}_U \leq \mathcal{G}$  which is  $X$ - $\mathfrak{J}$ -compactoid in  $U$ . By composability,  $f(\mathcal{C}_{f^{-1}V})$  is a  $\mathfrak{D}$ -filter.

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<sup>21</sup>Corollary 5.5 appeared at first in [17] as a combination of theorems 11.3 and 11.7.

Moreover it is  $fX$ - $\mathfrak{J}$ -compactoid in  $V$ . Indeed, If a  $\mathfrak{J}$ -filter  $\mathcal{H}\#f(\mathcal{C}_{f^{-1}V})$ , then  $f^{-1}\mathcal{H}\#\mathcal{C}_{f^{-1}V}$  so that  $\text{adh}_X f^{-1}\mathcal{H}\#f^{-1}V$ , equivalently  $f(\text{adh}_X f^{-1}\mathcal{H})\#V$ . Since  $f(\text{adh}_X f^{-1}\mathcal{H}) = \text{adh}_{\vec{f}X} \mathcal{H}$ , I conclude that  $\text{adh}_{\vec{f}X} \mathcal{H} \cap V \neq \emptyset$ .

The iteration of  $Q_{\mathfrak{D},\mathfrak{J}}$  is defined for every ordinal  $\alpha$  by  $Q_{\mathfrak{D},\mathfrak{J}}^\alpha X = X$  if  $\alpha = 0$  and  $Q_{\mathfrak{D},\mathfrak{J}}^\alpha X = Q_{\mathfrak{D},\mathfrak{J}}(\bigvee_{\beta < \alpha} Q_{\mathfrak{D},\mathfrak{J}}^\beta X)$  otherwise. For every  $X$ , there exists the least ordinal  $\alpha$  such that  $Q_{\mathfrak{D},\mathfrak{J}}^\alpha X = Q_{\mathfrak{D},\mathfrak{J}}^{\alpha+1} X$ . I denote by  $E_{\mathfrak{D},\mathfrak{J}} X$  this convergence space. This is the coarsest convergence space fixed by  $Q_{\mathfrak{D},\mathfrak{J}}$  and finer than  $X$ . Moreover,  $\vec{f}(E_{\mathfrak{D},\mathfrak{J}} X) \geq E_{\mathfrak{D},\mathfrak{J}}(\vec{f}X)$ , because  $\vec{f}(Q_{\mathfrak{D},\mathfrak{J}} X) \geq Q_{\mathfrak{D},\mathfrak{J}}(\vec{f}X)$ . Thus  $E_{\mathfrak{D},\mathfrak{J}}$  is a coreflector.  $\square$

Let us review the more usual cases for  $Q_{\mathfrak{D},\mathfrak{J}}$ <sup>22</sup>.

$\mathfrak{D}$	$\mathfrak{J}$	property $Q_{\mathfrak{D},\mathfrak{J}}$	modifier
principal	countably based	$T$ -core-countably compact	$K_{\text{core}}^\omega$
	all filters	$T$ -core-compact	$K_{\text{core}}$
	$\wedge\omega$ -filters	$T$ -core-Lindelöf	$L_{\text{core}}$
countably based	countably based	$T$ -core-q	$Q_{\text{core}}$
	all filters	$T$ -core-pointwise countable type	$\text{First}_{K_{\text{core}}}$

For a regular topology the above notions are reduced to more usual ones:

$\mathfrak{D}$	$\mathfrak{J}$	property $Q_{\mathfrak{D},\mathfrak{J}}$	coreflector
principal	countably based	locally countably compact	$K_\omega$
	all filters	locally compact	$K$
	$\wedge\omega$ -filters	locally Lindelöf	$L$
countably based	countably based	q-topology	$\text{First}_{K_\omega}$
	all filters	pointwise countable type	$\text{First}_K$

**Theorem 6.2.** *Let  $\mathfrak{D}$  and  $\mathfrak{J}$  be two composable classes of filters.*

$$(6.40) \quad W \times \text{Adh}_{\mathfrak{D}} Y \geq T(X \times Y),$$

for every  $\mathfrak{J}$ -based convergence space  $Y$  if and only if

$$W \geq SQ_{\mathfrak{D},\mathfrak{J}} A_{\mathfrak{J}} X.$$

Notice that if  $X \geq W$  then

$$W \geq SQ_{\mathfrak{D},\mathfrak{J}} A_{\mathfrak{J}} X \iff W \geq SE_{\mathfrak{D},\mathfrak{J}} A_{\mathfrak{J}} X.$$

*Proof.* Assume that  $W = Q_{\mathfrak{D},\mathfrak{J}} A_{\mathfrak{J}} X$  and that  $(x, y) \in \lim_{W \times \text{Adh}_{\mathfrak{D}} Y} (\mathcal{F} \times \mathcal{G})$ . Let  $H$  be a  $(X \times Y)$ -closed set such that  $H\#(\mathcal{F} \times \mathcal{G})$ . I need to prove that  $(x, y) \in H$ , to the effect that  $W \times \text{Adh}_{\mathfrak{D}} Y \geq T(X \times Y)$ . Applying  $S$ , one gets the direct part of the theorem. By definition of  $W$ , there exists, for every  $V \in \mathcal{N}_X(x)$ , a  $\mathfrak{D}$ -filter  $\mathcal{C}_V \leq \mathcal{F}$  which is  $A_{\mathfrak{J}} X$ - $\mathfrak{J}$ -compactoid in  $V$ .

Since  $\mathfrak{D}$  is a composable class of filters,  $H\mathcal{C}_V$  is a  $\mathfrak{D}$ -filter and  $H\mathcal{C}_V\#\mathcal{G}$ . Hence,  $y \in \text{adh}_Y H\mathcal{C}_V$ , so that there exists a  $\mathfrak{J}$ -filter  $\mathcal{L}_V$  such that  $y \in \lim_Y \mathcal{L}_V$  and  $\mathcal{L}_V\#H\mathcal{C}_V$ . By compositability of  $\mathfrak{J}$ ,  $H^{-}\mathcal{L}_V$  is a  $\mathfrak{J}$ -filter such that  $H^{-}\mathcal{L}_V\#\mathcal{C}_V$ .

<sup>22</sup> $\wedge\omega$ -filters are filters closed for countable intersections.



By  $\mathfrak{J}$ -compactoidness of  $\mathcal{C}_V$ , there exists  $x_V \in \text{adh}_{A_3 X}(H^- \mathcal{L}_V) \cap V$ . Obviously,  $(x_V, y) \in \text{adh}_{A_3 X \times Y} H \subset H$ , because  $T(A_3 X \times Y) = T(X \times Y)$  for every  $Y = \text{Base}_3 Y$ . On the other hand,  $x \in \text{cl}_X(\{x_V : V \in \mathcal{N}_X(x)\})$  so that  $(x, y) \in \text{cl}_X(\{x_V : V \in \mathcal{N}_X(x)\}) \times \{y\} \subset \text{cl}_{X \times Y}(\{(x_V, y) : V \in \mathcal{N}_X(x)\}) \subset H$ .

Conversely, assume that  $W \not\geq SQ_{\mathfrak{D}, \mathfrak{J}} A_3 X$ . There exists an ultrafilter  $\mathcal{U}$  such that  $x_0 \in \lim_W \mathcal{U} \setminus \lim_{Q_{\mathfrak{D}, \mathfrak{J}} A_3 X} \mathcal{U}$ . Hence, there exists  $V_0 \in \mathcal{N}_X(x_0)$  such that for every  $\mathfrak{D}$ -filter  $\mathcal{H} \# \mathcal{U}$ , there exists a  $\mathfrak{J}$ -filter  $\mathcal{L}_{\mathcal{H}} \# \mathcal{H}$  such that  $\text{adh}_{A_3 X} \mathcal{L}_{\mathcal{H}} \cap V_0 = \emptyset$ . Let  $Y$  denote the atomic convergence space on  $|X|$  in which  $x_0 \in \lim_Y \mathcal{F}$  if and only if there exists a  $\mathfrak{D}$ -filter  $\mathcal{H} \# \mathcal{U}$  such that  $\mathcal{F} \geq \mathcal{L}_{\mathcal{H}} \wedge (x_0)$ . The convergence space  $Y$  is  $\mathfrak{J}$ -based. On the other hand,  $(x_0, x_0) \in \lim_{W \times \text{Adh}_{\mathfrak{D}} Y} (\mathcal{U} \times \mathcal{U})$  but  $(x_0, x_0) \notin \lim_{T(X \times Y)} (\mathcal{U} \times \mathcal{U})$ , because if

$$A = \{(x, y) : y \neq x_0, x \in \lim_{A_3 X}(y)\},$$

then  $A \# (\mathcal{U} \times \mathcal{U})$  but  $(x_0, x_0) \notin \text{cl}_{X \times Y} A$ . Indeed, let  $\mathcal{M} \times \mathcal{G}$  be a filter on  $A$  that converges to  $(x, y)$  in  $X \times Y$ . If  $y \neq x_0$ , then  $\mathcal{G} = (y)$  because  $y$  is isolated in  $Y$ , so that  $\mathcal{M}$  is a filter on  $\lim_{A_3 X}(y)$ , because  $A \in \mathcal{M} \times (y)$ . Since  $A_3 X$  has closed limits, hence  $X$ -closed limits,  $(x, y) \in A$ . If  $y = x_0$ , then  $\mathcal{G} \supset \mathcal{L}_{\mathcal{H}}$  for some  $\mathfrak{D}$ -filter  $\mathcal{H} \# \mathcal{U}$ . In view of  $(\mathcal{M} \times \mathcal{L}_{\mathcal{H}}) \# A$ , I conclude  $\mathcal{M}_{\text{ad}_X^*} \# \mathcal{L}_{\mathcal{H}}$ . Thus  $\lim_{A_3 X} \mathcal{M}_{\text{ad}_X^*} \cap V_0 = \emptyset$ . Since  $A_3 X$  is  $\text{ad}_X^*$ -regular (see [17] or Theorem 5.1),  $\lim_{A_3 X} \mathcal{M}_{\text{ad}_X^*} = \lim_{A_3 X} \mathcal{M}$ , so that  $(x, x_0) \in V_0^c \times \{x_0\}$ . Thus  $(x_0, x_0) \notin \text{cl}_{X \times Y} A$  because  $V_0^c$  is  $X$ -closed.  $\square$

Theorem 6.2 combined with Theorem 3.1 leads to

**Theorem 6.3.** *Let  $\mathfrak{D}$  and  $\mathfrak{J}$  be two composable classes of filters. The following are equivalent:*

- (1) *For every convergence space  $Y \geq \text{Adh}_{\mathfrak{D}} \text{Base}_3 Y$*
- (6.41) 
$$W \times \text{Adh}_{\mathfrak{D}} Y \geq T(X \times Y);$$
- (2) *(6.41) holds for every  $\mathfrak{J}$ -based convergence space  $Y$ ;*
  - (3)  *$\text{Id}_{X, W} \times f$  is quotient for every  $\text{Adh}_{\mathfrak{D}}$ -quotient map  $f$  with  $\text{Adh}_{\mathfrak{D}} \text{Base}_3$ -domain;*
  - (4)  *$\text{Adh}_{\mathfrak{D}} \text{Base}_3([X, Z]) \geq [W, Z]$  for every topological space  $Z$ ;*
  - (5)  *$\text{Adh}_{\mathfrak{D}} \text{Base}_3[X, \$] \geq [W, \$]$ ;*
  - (6)  *$W \geq SQ_{\mathfrak{D}, \mathfrak{J}} A_3 X$ .*

Recall [11] that in case of a regular topology, the property  $SQ_{\mathfrak{D}, \mathfrak{J}}$  means

$\mathfrak{D}$	$\mathfrak{J}$	property $SQ_{\mathfrak{D}, \mathfrak{J}}$
principal	countably based	locally countably compact
	all filters	locally compact
	$\wedge\omega$ -filters	locally Lindelöf
countably based	countably based	bi-quasi- $k$
	all filters	bi- $k$

For a general convergence, if  $\mathfrak{D}$  is the class of countably based filters, I call a  $SQ_{\mathfrak{D}, \mathfrak{J}}$ -convergence *core-bi-quasi- $k$*  if  $\mathfrak{J} = \mathfrak{D}$  and *core-bi- $k$*  if  $\mathfrak{J}$  is the class of all filters.

**Theorem 6.4.** *Let  $f : X \rightarrow Y$  be a continuous surjection and let  $Y$  be a  $\mathfrak{J}$ -Antoine convergence space.  $f \times g$  is quotient for every  $\text{Adh}_{\mathfrak{D}}$ -quotient map  $g$  with  $\text{Adh}_{\mathfrak{D}}\text{Base}_{\mathfrak{J}}$ -domain if and only if  $f$  is  $A_{\mathfrak{J}}$ -quotient with a  $SQ_{\mathfrak{D},\mathfrak{J}}$ -range.*

*Proof.* By Theorem 6.2, Lemma 4.3 applies with  $\mathbf{L}=\mathbf{T}$ ,  $J = \text{Adh}_{\mathfrak{D}}$ ,  $E = \text{Base}_{\mathfrak{J}}$ ,  $R = S$  and  $C = Q_{\mathfrak{D},\mathfrak{J}}$ . In view of Theorem 5.1,  $\text{Epi}_E^L = A_{\mathfrak{J}}$  and the result follows.  $\square$

Theorem 6.4 applies with  $\mathfrak{D}$  the class of principal filters and  $\mathfrak{J}$  the class of all filters, to the effect that<sup>23</sup>

**Corollary 6.5.** [17, Theorem 11.1] *Let  $f : X \rightarrow Y$  be a continuous surjection and let  $Y$  be an Antoine convergence space.  $f \times g$  is quotient for every hereditarily quotient map  $g$  if and only if  $f$  is  $A$ -quotient with a  $T$ -core-compact range.*

In particular, when  $f = \text{Id}_X$ , Corollary 6.5 refines the classical Whitehead-Michael theorem [32, Theorem 2.1]. On the other hand, when  $\mathfrak{D}$  is again the class of principal filters but  $\mathfrak{J}$  is the class of countably based filters, then Theorem 6.4 applies to the effect that

**Corollary 6.6.** [17, Corollary 11.11] *Let  $f : X \rightarrow Y$  be a continuous surjection and let  $Y$  be a countably Antoine convergence space.  $f \times g$  is quotient for every hereditarily quotient map  $g$  with Fréchet domain if and only if  $f$  is  $A_{\omega}$ -quotient with a  $T$ -core-countably compact range.*

Once again, taking  $f = \text{Id}_X$ , one gets variants of the Whitehead-Michael theorem. More precisely, Theorem 6.3 applies with  $W = X$  to get specializations of Corollaries 6.5 and 6.6.

**Corollary 6.7.** *The following are equivalent:*

- (1)  $\text{Id}_X \times f$  is quotient for every hereditarily quotient map  $f$ ;
- (2)  $X$  is  $T$ -core-compact;
- (3)  $[X, Z]$  is pretopological for every topological space  $Z$ ;
- (4)  $[X, \$]$  is pretopological.

**Corollary 6.8.** *The following are equivalent:*

- (1)  $\text{Id}_X \times f$  is quotient for every hereditarily quotient map  $f$  with Fréchet domain;
- (2)  $X$  is  $T$ -core-countably compact;
- (3)  $P\text{First}[X, Z] \geq [X, Z]$  for every topological space  $Z$ ;
- (4)  $P\text{First}[X, \$] \geq [X, \$]$ .

When  $\mathfrak{D}$  is no longer the class of principal filters but the class of countably based filters, Theorem 6.4 leads to two new results (one in the case  $\mathfrak{J}$  is the class of all filters, the other in case  $\mathfrak{J}$  is the class of countably based filters) on product of quotient maps.

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<sup>23</sup>If  $Y$  is topological, this is also equivalent to  $f \times g$  is quotient for every quotient map  $g$ . See [17].

**Corollary 6.9.** *Let  $f : X \rightarrow Y$  be a continuous surjection and let  $Y$  be a Antoine convergence space.  $f \times g$  is quotient for every countably biquotient map  $g$  if and only if  $f$  is  $A$ -quotient with a core-bi- $k$  range.*

**Corollary 6.10.** *Let  $f : X \rightarrow Y$  be a continuous surjection and let  $Y$  be a countably Antoine convergence space.  $f \times g$  is quotient for every countably biquotient map  $g$  with strongly Fréchet domain if and only if  $f$  is  $A_\omega$ -quotient with a core-bi-quasi- $k$  range.*

Once again, both Corollaries 6.9 and 6.10 can be specialized with  $f = \text{Id}_X$  to get characterizations of core-bi- $k$ -ness and core-bi-quasi- $k$ -ness in a “Whitehead-like” formulation. Moreover, Theorem 6.3 applies with  $W = X$  to the effect that:

**Corollary 6.11.** *The following are equivalent:*

- (1)  $\text{Id}_X \times f$  is quotient for every countably biquotient map  $f$ ;
- (2)  $X$  is core-bi- $k$ ;
- (3)  $[X, Z]$  is paratopological for every topological space  $Z$ ;
- (4)  $[X, \$]$  is paratopological.

**Corollary 6.12.** *The following are equivalent:*

- (1)  $\text{Id}_X \times f$  is quotient for every countably biquotient map  $f$  with strongly Fréchet domain;
- (2)  $X$  is core-bi-quasi- $k$ ;
- (3)  $P_\omega \text{First}[X, Z] \geq [X, Z]$  for every topological space  $Z$ ;
- (4)  $P_\omega \text{First}[X, \$] \geq [X, \$]$ .

In particular, Corollaries 6.7, 6.8, 6.11 and 6.12 provide new results concerning the relationships between a convergence space  $X$  and the upper Kuratowski convergence  $[X, \$]$  on its closed subsets (homeomorphically Scott convergence on the lattice of its open subsets).

$[X, \$]$ verifies	iff $X$ is
$[X, \$] = P[X, \$]$	$T$ -core-compact
$[X, \$] = T[X, \$]$	(topological) core-compact
$[X, \$] = P_\omega[X, \$]$	core-bi- $k$
$P \text{First}[X, \$] \geq [X, \$]$	$T$ -core-countably compact
$P_\omega \text{First}[X, \$] \geq [X, \$]$	core-bi-quasi- $k$

Of course, Theorem 6.2 applies also to preservations of  $TE$ -properties, like sequentiality and leads to the following new result.

**Corollary 6.13.** *If  $X$  is strongly sequential and core-bi-quasi- $k$  then  $X \times Y$  is sequential for every strongly Fréchet convergence space  $Y$ .*

*Proof.* If  $X$  is strongly sequential and core-bi-quasi- $k$ , then  $X \geq A_\omega \text{First} X$  and  $X \geq SQ_{\text{core}} X$ , so that  $X \geq SQ_{\text{core} A_\omega} \text{First} X$ . Therefore Theorem 6.2 applies with  $\mathfrak{J} = \mathfrak{D}$  the class of countably based filters,  $W = X$ ,  $X = \text{First} X$  and  $Y = \text{First} Y$ , to the effect that  $X \times P_\omega \text{First} Y \geq T(\text{First} X \times \text{First} Y) = T \text{First}(X \times Y)$ . Hence  $X \times Y$  is sequential whenever  $Y$  is strongly Fréchet.  $\square$

Notice that Theorem 6.2 does not apply to the converse of Corollary 6.13. Indeed, if  $X \times Y$  is sequential for every strongly Fréchet convergence space  $Y$ , then

$$X \times P_\omega Y \geq T(\text{First } X \times \text{First } P_\omega Y),$$

for every first-countable  $Y$ , but this is not enough to conclude that

$$X \times P_\omega Y \geq T(\text{First } X \times Y)$$

for every first-countable  $Y$ .

## 7. THE REFLECTORS $\text{Adh}_{\mathfrak{J}}$ ARE DETERMINED BY COREFLECTIVELY MODIFIED BIDUALS

**7.1. Structural results.** Recall that a composable class of filter  $\mathfrak{J}$  contains principal filters, so that  $\text{Adh}_{\mathfrak{J}} \geq P$ .

**Theorem 7.1.** *Let  $\mathfrak{J}$  be a composable class of filters. Then  $\text{Epi}_{\mathfrak{J}}^{\forall} = \text{Adh}_{\mathfrak{J}}$ .*

*Proof.* By [17, Theorem 10.1],  $\text{Adh}_{\mathfrak{J}} X \times Y \geq P(X \times Y)$  for every  $\mathfrak{J}$ -based  $Y$ . On the other hand, if  $W \not\geq \text{Adh}_{\mathfrak{J}} X$ , there is a filter  $\mathcal{F}$  with  $x_0 \in \lim_W \mathcal{F}$  and a  $\mathfrak{J}$ -filter  $\mathcal{H}$  such that  $\mathcal{H} \# \mathcal{F}$  and  $x_0 \notin \text{adh}_X \mathcal{H}$ . Let  $Y$  be the atomic  $\mathfrak{J}$ -based topological space on  $|X|$  defined by  $\mathcal{N}_Y(x_0) = \mathcal{H} \wedge (x_0)$ . Then  $W \times Y \not\geq P(X \times Y)$ . Indeed,  $(x_0, x_0) \in \lim_{W \times Y} (\mathcal{F} \vee \mathcal{H} \times \mathcal{F} \vee \mathcal{H})$  but  $(x_0, x_0) \notin \text{adh}_{X \times Y} \{(x, x) : x \neq x_0\}$ . Indeed  $\mathcal{G} \geq \mathcal{H}$  whenever  $(x_0, x_0) \in \lim_{X \times Y} \mathcal{G} \times \mathcal{G}$ ; a contradiction to  $x_0 \notin \text{adh}_X \mathcal{H}$ .

Consequently, in view of Theorem 3.1 with  $\mathbf{L}=\mathbf{P}$ ,  $\mathbf{E} = \mathbf{Base}_{\mathfrak{J}}$  and  $\mathbf{J}=\mathbf{Conv}$ ,  $\text{Epi}_{\mathfrak{J}}^{\forall} = \text{Adh}_{\mathfrak{J}}$ .  $\square$

On the other hand, in the particular case of coprojectors  $E$  and  $B$  on convergence spaces based in composable class of filters (which are finitely productive by Lemma 2.2), Theorem 4.5 leads to

**Corollary 7.2.** *Let  $\mathfrak{D}$  and  $\mathfrak{J}$  be two composable classes of filters such that  $\mathfrak{D} \subset \mathfrak{J}$ . Then*

$$(7.42) \quad \text{SBase}_{\mathfrak{D}} \text{Adh}_{\mathfrak{J}} X \times \text{Adh}_{\mathfrak{D}} Y \geq \text{Adh}_{\mathfrak{D}} (X \times Y),$$

for every  $\mathfrak{J}$ -based convergence space  $Y$ .

Moreover, in this particular case, the converse is true.

**Proposition 7.3.** *Let  $\mathfrak{D}$  and  $\mathfrak{J}$  be two composable classes of filters. If*

$$(7.43) \quad W \times \text{Adh}_{\mathfrak{D}} Y \geq P(X \times Y)$$

for every  $\mathfrak{J}$ -based convergence space  $Y$ , then

$$W \geq \text{SBase}_{\mathfrak{D}} \text{Adh}_{\mathfrak{J}} X.$$

*Proof.* Assume that  $W \not\geq \text{SBase}_{\mathfrak{D}} \text{Adh}_{\mathfrak{J}} X$ . Thus, there exists an ultrafilter  $\mathcal{U}$  such that  $x_0 \in \lim_W \mathcal{U} \setminus \lim_{\text{Base}_{\mathfrak{D}} \text{Adh}_{\mathfrak{J}} X} \mathcal{U}$ . Hence, for every  $\mathfrak{D}$ -filter  $\mathcal{H}$  that meshes with  $\mathcal{U}$ , there exists a  $\mathfrak{J}$ -filter  $\mathcal{L}_{\mathcal{H}}$ , such that  $\mathcal{L}_{\mathcal{H}} \# \mathcal{H}$  but  $x_0 \notin \text{adh}_X \mathcal{L}_{\mathcal{H}}$ . Let  $Y$  denote the atomic convergence space on  $|X|$  in which  $x_0 \in \lim_Y \mathcal{F}$  if and only if there exists a  $\mathfrak{D}$ -filter  $\mathcal{H} \# \mathcal{U}$  such that  $\mathcal{F} \geq \mathcal{L}_{\mathcal{H}} \wedge (x_0)$ . The convergence

space  $Y$  is  $\mathfrak{J}$ -based. On the other hand,  $(x_0, x_0) \in \lim_{W \times \text{Adh}_{\mathfrak{D}} Y} (\mathcal{U} \times \mathcal{U})$  but  $(x_0, x_0) \notin \lim_{P(X \times Y)} (\mathcal{U} \times \mathcal{U})$ . Indeed, if  $(x_0, x_0) \in \lim_{X \times Y} (\mathcal{G} \times \mathcal{G})$  for  $\mathcal{G} \neq (x_0)$ , then there exists a  $\mathfrak{D}$ -filter  $\mathcal{H} \# \mathcal{U}$  such that  $\mathcal{G} \geq \mathcal{L}_{\mathcal{H}}$ . Then  $x_0 \notin \lim_X \mathcal{G}$  because  $x_0 \notin \text{adh}_X \mathcal{L}_{\mathcal{H}}$ . Hence  $W \times \text{Adh}_{\mathfrak{D}} Y \not\geq P(X \times Y)$ .  $\square$

If  $\mathfrak{D}$  is the class of principal filters and  $\mathfrak{J}$  the class of countably based filters, Proposition 7.3 can be refined as follows.

**Proposition 7.4.** *If  $W \times PY \geq P(X \times Y)$ , for every atomic  $Y = \text{First } PY$ , then  $W \geq S\text{Fin } P_{\omega} X$ .*

*Proof.* Consider  $\mathcal{U}$  as in the proof of Proposition 7.3. for every  $U \in \mathcal{U}$ , consider the atomic topological space  $Y_U$  on  $|X|$  defined by  $\mathcal{N}_{Y_U}(x_0) = \mathcal{L}_U \cap (x_0)$  and let  $Z$  be the convergence space obtained from the disjoint sum of every  $Y_U$  by identifying all points  $x_0$  to a single point  $\infty$ . If  $Z$  is endowed with the corresponding final convergence, then  $Z$  is a first countable atomic convergence space such that  $PZ = TZ = P \text{First } PZ$ . Notice that  $\mathcal{V}_Z(x_0) = \bigwedge_{U \in \mathcal{U}} \mathcal{L}_U \wedge (x_0)$ . Let  $Y_0 = \text{First } PZ$ . Then  $W \times PY_0 \not\geq P(X \times Y_0)$ . Indeed, consider  $A = \{(x, x) \in |X \times Y_0| : x \neq x_0\}$ . The point  $(x_0, x_0)$  belongs to  $\text{adh}_{W \times PY_0} A$  because  $\mathcal{U} \times \mathcal{V}_{Y_0}(x_0) \# A$ . However,  $(x_0, x_0) \notin \text{adh}_{X \times Y_0} A$ . If  $\mathcal{G}$  is a countably based filter such that  $\mathcal{G} \geq \mathcal{V}_{\alpha}(x_0)$ , define a free sequence  $(x_n)_n \geq \mathcal{G}$  by  $x_n \in G_n \setminus G_{n+1}$ , where  $(G_n)_n$  denotes a decreasing base of  $\mathcal{G}$ . By standard arguments, there exists a subsequence  $(x_{n_k})_k$  and  $U_0 \in \mathcal{U}$  such that  $x_{n_k} \in |Y_{U_0}|$  for every  $k$ . Hence,  $\lim_X \mathcal{G} \subset \lim_{(x_{n_k})_k} \subset \text{adh}_X \mathcal{L}_{U_0}$ . Thus  $x_0 \notin \lim_X \mathcal{G}$ .  $\square$

In view of Corollary 7.2, Proposition 7.3 and Theorem 3.1

**Theorem 7.5.** *Let  $\mathfrak{D}$  and  $\mathfrak{J}$  be two composable classes of filters such that  $\mathfrak{D} \subset \mathfrak{J}$ . The following are equivalent*

- (1)  $W \times \text{Adh}_{\mathfrak{D}} Y \geq \text{Adh}_{\mathfrak{D}}(X \times Y)$ , for every  $\text{Adh}_{\mathfrak{D}} \text{Base}_3$ -convergence space  $Y$ ;
- (2)  $W \times \text{Adh}_{\mathfrak{D}} Y \geq P(X \times Y)$ , for every  $\mathfrak{J}$ -based convergence space  $Y$ ;
- (3)  $\text{Id}_{X, W} \times f$  is  $\text{Adh}_{\mathfrak{D}}$ -quotient for every  $\text{Adh}_{\mathfrak{D}}$ -quotient map  $f$  with  $\text{Adh}_{\mathfrak{D}} \text{Base}_3$ -domain;
- (4)  $\text{Id}_{X, W} \times f$  is hereditarily quotient for every  $\text{Adh}_{\mathfrak{D}}$ -quotient map  $f$  with  $\mathfrak{J}$ -based domain;
- (5)  $\text{Adh}_{\mathfrak{D}} \text{Base}_3[X, Z] \geq [W, Z]$  for every  $\text{Adh}_{\mathfrak{D}}$ -object  $Z$ ;
- (6)  $\text{Adh}_{\mathfrak{D}} \text{Base}_3[X, \mathbb{Y}] \geq [W, \mathbb{Y}]$ ;
- (7)  $W \geq S\text{Base}_{\mathfrak{D}} \text{Adh}_3 X$ .

The converse of Theorem 4.5 is true in the particular case of  $Z_0 = \mathbb{Y}$  and  $E = \text{Base}_{\mathfrak{D}}, B = \text{Base}_3$  with  $\mathfrak{D} \subset \mathfrak{J}$  two composable classes of filters, but this is not true in general. Indeed, if  $Z_0 = \$$ ,  $\mathbf{E}$  is the (finitely productive) coreflective subcategory of discrete convergence spaces and  $\mathbf{B} = \mathbf{Conv}$ , then  $\text{Epi}_E^{Z_0}$  is the topologizer and Theorem 4.5 reads as follows:

$$\text{ADis} AX \times TY \geq T(X \times Y),$$

for every  $Y$ . But exponential objects in  $\mathbf{T}$  are the core-compact topologies (see [17] for details on  $T$ -exponential convergences or for  $\mathbf{B} \neq \mathbf{Conv}$ ). In particular, Hausdorff core-compact topologies are exactly the locally compact ones. But Hausdorff topologies verifying  $X \geq A\text{Dis}AX$  are discrete.

**7.2. Categorical comments.** By Theorem 7.1 with  $\mathfrak{J}$  the class of all filters,  $\text{Epi}^{\mathfrak{Y}} = S$ , the pseudotopologizer. Hence, by Corollary 4.6, the category of pseudotopologies is the cartesian closed hull of the category of pretopologies [7, Théorème II.4.1] (and of course also of the category of paratopologies). On the other hand, since  $W \times Y \geq P(X \times Y)$  for every first-countable  $Y$  if and only if  $W \geq P_{\omega}X$ , the category of paratopologies is the cartesian closed hull of the category of pretopologies relatively to First.

A pretopology  $X$  is an exponential object in the category of pretopologies if and only if  $X \times PY \geq P(X \times Y)$  for every convergence space  $Y$ . In view of Theorem 7.5 applied with  $\mathfrak{D}$  the class of principal filters, the exponential objects in the category of pretopologies are the pretopological spaces  $X$  verifying  $X \geq S\text{Fin}SX = S\text{Fin}X$ . It is easy to see that, as  $X$  is pretopological,  $S\text{Fin}X = \text{Fin}X$  is also pretopological. Each point of  $|X|$  has a smallest neighborhood, that is  $X$  is *finitely generated* [29].

**Corollary 7.6.** [29] *Exponential objects in the category  $\mathbf{P}$  of pretopological spaces are the finitely generated pretopological spaces.*

More generally, I call a convergence space *finitely generated* if  $X \geq S\text{Fin}X$ . Within pretopologies, there is no difference between exponentiality and exponentiality relatively to First.

On the other hand, in view of Theorem 7.5 applied with  $\mathfrak{D}$  the class of countably based filters,

**Theorem 7.7.** *Exponential objects in the category  $\mathbf{P}_{\omega}$  of paratopological spaces are bisequential paratopological spaces.*

Once again, exponentiality and exponentiality relatively to First coincide in paratopologies.

Now we are in position to gather some of the reflectors that can be characterized as a  $\text{Epi}_E^{Z_0}$ -reflector, for a particular coreflector  $E$  and a particular  $Z_0$ . Recall that the usual topology  $\mathbb{R}$  of the real line is initially dense in the category of completely regular topologies:

$$(7.44) \quad OX = \bigvee_{f \in C(X, \mathbb{R})} \overleftarrow{f} \mathbb{R}.$$

The reflector  $\text{Epi}^{\mathbb{R}}$  is the reflector on the  $c$ -spaces of E. Binz [6]<sup>24</sup>. They form the cartesian closed hull of the category of completely regular topologies [8]. I denote  $c$  this reflector, and  $c_{\omega}$  the reflector  $\text{Epi}_{\text{First}}^{\mathbb{R}}$ .

<sup>24</sup>Actually, the  $c$ -spaces in the sense of E. Binz are the Hausdorff  $\mathbf{Epi}^{\mathbb{R}}$ -object. See [8] for details.

	Dis	Fin	First	$I$
$\mathbb{R}$	$O$	$O$	$c_\omega$	$c$
$\$$	$T$	$T$	$A_\omega$	$A$
$\mathbb{Y}$	$P$	$P$	$P_\omega$	$S$

Each reflector is smaller than the reflectors below in the same column and than the reflectors on its righthandside in the same row. The second column follows from (7.44), (4.38) and (4.39). For the third, notice that  $\$, \mathbb{Y}$  and  $\mathbb{R}$  verify the following separation axiom:

**Condition 7.8.** *If  $\mathcal{F} \rightarrow y$  and  $(y) \rightarrow x$  then  $\mathcal{F} \rightarrow x$*

Therefore, the third column follows from

**Lemma 7.9.** *If  $Z_0$  is a pretopological space that verifies Condition 7.8 and is initially dense in a reflective subcategory  $\mathbf{L}$  of  $\mathbf{Conv}$  then,*

$$\text{Epi}_{\text{Fin}}^{Z_0} = L.$$

*Proof.* By definition  $\text{Epi}_{\text{Fin}}^{Z_0} \geq L$ . On the other hand,  $LX \times Y \geq L(X \times Y)$  for every  $X$  and for every  $Y = \text{Fin } Y$ . so that  $L \geq \text{Epi}_{\text{Fin}}^{Z_0}$ . Indeed, if  $(x_0, y_0) \in \lim_{LX \times Y} \mathcal{F} \times G$ , then I need to show that  $f(x_0, y_0) \in \lim_{Z_0} f(\mathcal{F} \times G)$  for every  $f \in C(X \times Y, Z_0)$ . For every  $y \in Y$ , the map  $f_y : X \rightarrow Z_0$  defined by  $f_y(x) = f(x, y)$  is continuous, so that  $f_y(x_0) \in \lim_{Z_0} f_y(\mathcal{F})$ . On the other hand, the map  $f_{x_0} : Y \rightarrow Z_0$  defined by  $f_{x_0}(y) = f(x_0, y)$  is continuous, so that  $f(x_0, y_0) \in \lim_{Z_0} f_{x_0}(G)$ . Hence,  $f(x_0, y_0) \in \lim_{Z_0} f_{x_0}(y)$ , for every  $y \in G$ . By Condition 7.8,  $f(x_0, y_0) \in \lim_{Z_0} f_y(\mathcal{F})$  for every  $y \in G$ . Since  $f(\mathcal{F} \times G) = \bigwedge_{y \in G} f_y(\mathcal{F})$  and  $Z_0$  is a pretopology,  $f(x_0, y_0) \in \lim_{Z_0} f(\mathcal{F} \times G)$ .  $\square$

Hence, the most classically used concrete reflectors of convergence theory can be handled simultaneously, as  $\text{Epi}_E^{Z_0}$ -reflectors.

**7.3. Product of sequential spaces.** If in Theorem 7.1,  $\mathfrak{J}$  is the class of countably based filters, then  $\text{Epi}_{\mathfrak{J}}^{\mathbb{Y}} = P_\omega$ . Recall that the usual topological notion of strong Fréchetness can be extended to convergences via  $X \geq P_\omega \text{First } X$ . In this particular context, Theorem 4.7 leads to Theorem 7.10, just like it led to Theorem 5.3 in the foregoing section.

**Theorem 7.10.** *The following are equivalent:*

- (1)  $X$  is strongly Fréchet;
- (2)  $\text{adh}_X \mathcal{H} \subset \text{adh}_{\text{First } X} \mathcal{H}$  for each countably based  $\mathcal{H}$ ;
- (3)  $X \times Y$  is Fréchet for each first-countable convergence space  $Y$ ;
- (4)  $X \times Y$  is Fréchet for each metrizable atomic topological space  $Y$ ;
- (5)  $X \times Y$  is strongly Fréchet for each bisequential convergence space  $Y$ .

This last theorem is just an extension to convergences of a combination of well-known results of E. Michael: [33, Proposition 4.D.4] and [33, Proposition 4.D.5]<sup>25</sup>, but the interesting point is that both the new Theorem 5.3 and the classical Theorem 7.10 are facets of the same result Theorem 4.7.

<sup>25</sup>In [33], E. Michael uses the term *countably bisequential* for strongly Fréchet.

In particular, *the product of a bisequential convergence space with a strongly Fréchet convergence space is strongly Fréchet*. As Theorem 6.2 for Corollary 6.13, Proposition 7.3 does not apply to the converse. Indeed, by [3, Proposition 6.27, Corollary 6.28], there exists a non bisequential topological space which product with every strongly Fréchet topological space is strongly Fréchet.

On the other hand, Theorem 7.5 applies with  $\mathfrak{D}$  the class of principal filters and  $\mathfrak{J}$  the class of countably based filters to the effect that

**Corollary 7.11.** *Let  $X$  be a paratopological space. The following are equivalent:*

- (1)  $\text{Id}_X \times f$  is hereditarily quotient for every hereditarily quotient map  $f$  (equivalently with Fréchet domain);
- (2)  $X \times Y$  is Fréchet for every Fréchet convergence space (equivalently Fréchet atomic topological space)  $Y$ ;
- (3)  $X$  is finitely generated;
- (4)  $[X, Z]$  is pretopological for every pretopological space  $Z$ ;
- (5)  $P \text{First}[X, \mathfrak{F}] \geq [X, \mathfrak{F}]$ .

Indeed,  $S \text{Fin } X = S \text{Fin } P_\omega X = S \text{Fin } P_\omega \text{First } X$  whenever  $X = P_\omega X$ . Hence, if  $X$  is paratopological,  $X \times PY \geq P(X \times Y)$  for every  $Y$  is equivalent with  $X \times PY \geq P(X \times Y)$  for every first-countable  $Y$ , equivalently for every  $Y = \text{First } PY$ , in view of Proposition 7.4. Therefore, the result follows from Theorem 7.5.

#### 7.4. Product of $\text{Adh}_{\mathfrak{D}}$ -quotient maps.

**Theorem 7.12.** *Let  $\mathfrak{D} \subset \mathfrak{J}$  be two composable classes of filters. Let  $f : X \rightarrow Y$  be a continuous surjection and let  $Y$  be a  $\text{Adh}_{\mathfrak{J}}$ -object.  $f \times g$  is  $\text{Adh}_{\mathfrak{D}}$ -quotient for every  $\text{Adh}_{\mathfrak{D}}$ -quotient map with  $\text{Adh}_{\mathfrak{D}} \text{Base}_{\mathfrak{J}}$ -domain if and only if  $f$  is  $\text{Adh}_{\mathfrak{J}}$ -quotient with  $S \text{Base}_{\mathfrak{D}}$ -range.*

*Proof.* By Theorem 7.5, Lemma 4.3 applies with  $L = J = \text{Adh}_{\mathfrak{D}}$ ,  $E = \text{Base}_{\mathfrak{J}}$ ,  $R = S$  and  $C = \text{Base}_{\mathfrak{D}}$ . In view of Theorem 7.1,  $\text{Epi}_E^L = \text{Adh}_{\mathfrak{J}}$  and the result follows.  $\square$

In particular, if  $\mathfrak{D}$  is the class of principal filters and  $\mathfrak{J}$  is the class of all (respectively countably based) filters, then

**Corollary 7.13.** *Let  $f : X \rightarrow Y$  be a continuous surjection and let  $Y$  be a pseudotopological (resp. paratopological) space.  $f \times g$  is hereditarily quotient for every hereditarily quotient map  $g$  (resp. with Fréchet domain) if and only if  $f$  is biquotient (resp. countably biquotient) with finitely generated range.*

Dually, Theorem 4.8 applies with  $E = \text{Fin}$  to the effect that

**Corollary 7.14.** *Let  $f$  be a continuous surjection. The following are equivalent*

- (1)  $f$  is a hereditarily quotient map;



- (2) For every  $F \subset |Y|$ ,
 
$$y \in \text{adh}_Y F \implies f^-(y) \bigcap \text{adh}_X f^-(F) \neq \emptyset;$$
- (3) For every  $y$ , if  $\{Q\}$  is a cover of  $f^{-1}y$  then  $\{f(Q)\}$  is a cover of  $y$ ;
- (4)  $f \times \text{Id}_Y$  is hereditarily quotient for every finitely generated convergence  $Y$ ;
- (5)  $f \times \text{Id}_Y$  is hereditarily quotient for every finitely generated topological space  $Y$ ;
- (6)  $f \times g$  is hereditarily quotient for every biquotient map  $g$  with finitely generated range.

The equivalence between 1, 2 and 3 is observed in [11].

In case  $\mathbf{E}=\mathbf{Conv}$  and  $Z_0 = \mathfrak{Y}$ , Theorem 4.8 applies to the effect that

**Theorem 7.15.** *Let  $f : X \rightarrow Y$  be a continuous surjection. The following is equivalent:*

- (1)  $f$  is biquotient;
- (2)  $y \in \text{adh}_Y \mathcal{F} \implies f^-(y) \bigcap \text{adh}_X f^-(\mathcal{F}) \neq \emptyset$ ;
- (3) For every  $X$ -cover  $\mathcal{F}$  of  $f^{-1}(y)$  there exists a finite subfamily  $\mathcal{R} \subset \mathcal{F}$  such that  $\bigcup_{R \in \mathcal{R}} f(R)$  is a  $Y$ -cover of  $y$ ;
- (4)  $f \times \text{Id}_Y$  is hereditary quotient for every convergence space  $Y$ ;
- (5)  $f \times \text{Id}_Y$  is hereditary quotient for every compact Hausdorff topological space  $Y$ ;
- (6)  $f \times g$  is biquotient for every biquotient map  $g$ .

*Proof.* The characterization 2 of biquotient maps can be found in [11], while 3 is the original definition given by E. Michael in [31].  $1 \iff 4$  follows from Theorem 4.8.  $4 \iff 5$  follows from [9, Theorem 6.5]. Obviously 6 implies 4, while 1 implies 6, because  $S$  commutes with the product<sup>26</sup>.  $\square$

Once again, the equivalence between 1 and 4 can be deduced from [39, Theorem 3], since the category of pseudotopologies is the cartesian closed hull of  $\mathbf{P}$ , with the observation that biquotient maps are the quotient maps in the category of pseudotopologies (see for example [39, Theorem 2]). In contrast, the counter-part of Theorem 4.8 in case  $\mathfrak{J}$  is the class of countably based filters recovers [33, Propositions 4.3 and 4.4] but does not follow from a categorical result.

**Theorem 7.16.** *Let  $f$  be a continuous surjection. The following are equivalent*

- (1)  $f$  is a countably biquotient map;
- (2) For every countably based filter  $\mathcal{F}$ ,

$$y \in \text{adh}_Y \mathcal{F} \implies f^-(y) \bigcap \text{adh}_X f^-(\mathcal{F}) \neq \emptyset;$$

---

<sup>26</sup>In fact  $S$  commutes with arbitrary product [22], [13], so that every product of biquotient maps is biquotient [31].

- (3) For every countable  $X$ -cover  $\xi$  of  $f^{-1}(y)$  there exists a finite subfamily  $\mathcal{R} \subset \xi$  such that  $\bigcup_{R \in \mathcal{R}} f(R)$  is a  $Y$ -cover of  $y$ ;
- (4)  $f \times \text{Id}_Y$  is hereditary quotient for every bisequential convergence space  $Y$ ;
- (5)  $f \times \text{Id}_Y$  is hereditary quotient for every metrizable atomic topological space  $Y$ ;
- (6)  $f \times g$  is countably biquotient for every biquotient map  $g$  with bisequential range.

The characterization 2 of countably biquotient maps can be found in [11] while 3 is the definition given by E. Michael in [31].

Dually, Theorem 7.12 applied with  $\mathfrak{D}$  the class of countably based filters and  $\mathfrak{J}$  the class of all (respectively countably based filters) leads to Corollary 7.17 (resp. Corollary 7.18).

**Corollary 7.17.** *Let  $f : X \rightarrow Y$  be a continuous surjection and let  $Y$  be a pseudotopological space.  $f \times g$  is countably biquotient for every countably biquotient map  $g$  if and only if  $f$  is biquotient with bisequential range.*

**Corollary 7.18.** *Let  $f : X \rightarrow Y$  be a continuous surjection and let  $Y$  be a paratopological space.  $f \times g$  is countably biquotient for every countably biquotient map  $g$  with strongly Fréchet domain if and only if  $f$  is countably biquotient with bisequential range.*

Moreover if Theorem 7.5 is specialized with  $W = X$  we obtain a new “Whitehead-like” result. If  $X$  is a paratopological space then  $S \text{First } X = S \text{First } P_\omega X = S \text{First } P_\omega \text{First } X$ . Hence,  $X \times P_\omega Y \geq P_\omega(X \times Y)$  for every  $Y$  is equivalent with  $X \times P_\omega Y \geq P(X \times Y)$  for every first-countable  $Y$ , so that the case where  $\mathfrak{D} = \mathfrak{J}$  is the class of countably based filters and the case where  $\mathfrak{D}$  is the class of countably based filters while  $\mathfrak{J}$  is the class of all filters can be gathered to the effect that:

**Corollary 7.19.** *Let  $X$  be a paratopological space. The following are equivalent:*

- (1)  $\text{Id}_X \times f$  is countably biquotient for every countably biquotient map  $f$ ;
- (2)  $\text{Id}_X \times f$  is hereditarily quotient for every countably biquotient map  $f$  with strongly Fréchet domain;
- (3)  $[X, Z]$  is paratopological for every paratopological space  $Z$ ;
- (4)  $[X, \mathbb{Y}]$  is paratopological;
- (5)  $P_\omega \text{First}[X, \mathbb{Y}] \geq [X, \mathbb{Y}]$ ;
- (6)  $X$  is bisequential.

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