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Minimal T_{UD} spaces

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Abstract. A topological space is T_{UD} if the derived set of each point is the union of disjoint closed sets. We show that there is a minimal T_{UD} space which is not just the Alexandroff topology on a linear order. Indeed the structure of the underlying partial order of a minimal T_{UD} space can be quite complex. This contrasts sharply with the known results on minimality for weak separation axioms.

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1. Introduction

Definition 1.1. [**2**] A topological space is said to be T_{UD} if the derived set of each point is the (possibly empty) union of disjoint closed sets.

In this introduction, we provide a complete brief survey and bibliography of minimality. The family $LT(X)$ of all topologies definable for an infinite set X is a complete atomic and complemented lattice (under set inclusion). If $\mathcal{T}$ and $\mathcal{S}$ are two members of $LT(X)$ with $\mathcal{S} \subseteq \mathcal{T}$, then $\mathcal{S}$ is said to be *weaker* than $\mathcal{T}$. Given a topological invariant P, a member $\mathcal{T}$ of $LT(X)$ is said to be *minimal* P if and only if $\mathcal{T}$ possesses property P but no weaker member of $LT(X)$ possesses property P.

The concept of minimal topologies was first introduced in 1939 by Parhomenko [27] when he showed that compact Hausdorff spaces are minimal Hausdorff. Motivation for such an investigation is provided by realising that it is in seeking to identify those members of $LT(X)$ which minimally satisfy an invariant that we are, in a very real sense, examining the topological essence of the invariant.

Given a topological space $(X, \mathcal{T})$, $(X, \mathcal{T})$ is

- *minimal Hausdorff* if and only if it is Hausdorff and every open filterbase which has a unique adherent point is convergent to this point (see [5], [9], [10], [27], [31], [32], and [36])
- *minimal* T_1 if and only if $\mathcal{T}$ is the cofinite topology $\mathcal{C}$ on X

- *minimal regular* if and only if it is regular and every regular filter-base which has a unique adherent point is convergent ([4], [8])
- *minimal completely regular* if and only if it is compact and Hausdorff ([4], [5])
- *minimal normal* if and only if it is compact and Hausdorff ([5])
- *minimal Urysohn* if and only if it is Urysohn and every filter with a unique adherence point converges to this point ([11], [34])
- *minimal (locally compact and Hausdorff)* if and only if it is compact and Hausdorff ([5], [4])
- *minimal paracompact* if and only if it is compact and Hausdorff ([35])
- *minimal metric* only if it is compact and Hausdorff ([35])
- *minimal completely normal* only if it is compact and Hausdorff ([35])
- *minimal completely Hausdorff* only if it is compact and Hausdorff ([35])
- *minimal* T_0 if and only if it is T_0, nested and generated by the family $\{X \setminus \overline{\{x\}} : x \in X\} \cup \{\varnothing, X\}$ ([1], [12], [19], [22], [26])
- *minimal* T_D if and only if it is T_D and nested ([1], [12], [19], [22], [26])
- *minimal* T_δ if and only if it is T_δ and nested ([1], [22])
- *minimal* T_ξ if and only if it is T_ξ and nested ([1])
- *minimal* T_A if and only if it is T_A and partially nested ([22])
- *minimal* T_{ES} if and only if either $\mathcal{T} = \mathcal{C}$ or $\mathcal{T} = \mathcal{E}(X \setminus Y) \cup (\mathcal{C} \cap \mathcal{I}(Y))$ for some non-empty proper subset Y of X ([21])
- *minimal* T_{EF} if and only if $\mathcal{T} = \mathcal{C}$ or $\mathcal{T} = \mathcal{I}(x)$ or $\mathcal{T} = \mathcal{E}(x)$ for some $x \in X$ ([21])
- *minimal* T_{FF} if and only if there exists $x \in X$ such that either $\mathcal{T} = \mathcal{C} \cap \mathcal{I}(x)$ or $\mathcal{T} = \mathcal{C} \cap \mathcal{E}(x)$ ([15])
- *minimal* T_F if and only if either there exists $x \in X$ such that $\mathcal{T} = \mathcal{C} \cap \mathcal{I}(x)$ or there exists a non-empty proper non-singleton subset Y of X such that $\mathcal{T} = \mathcal{D}(Y)$ ([15])
- *minimal* T_{YS} if and only if $\mathcal{T} = \mathcal{W}(\mathcal{P}) \vee (\mathcal{C} \cap \mathcal{I}(K))$ for some subset K of X and some partition $\mathcal{P}$ of X such that $\mathcal{P}$ is simply associated with K and is associated with $X \setminus K$. ([16])
- *minimal* T_{DD} if and only if $\mathcal{T} = \mathcal{W}_K(\mathcal{P}) \vee (\mathcal{C} \cap \mathcal{I}(K))$ for some subset K of X and partition $\mathcal{P}$ of X such that $\mathcal{P}$ is simply associated with K and associated with $X \setminus K$ ([16])
- *minimal* T_{YY} if and only if $\mathcal{T} = \mathcal{M}_p(\mathcal{P}) \vee (\mathcal{C} \cap \mathcal{I}(K))$ for some $p \in X$, subset K of $X \setminus \{p\}$ such that $\mathcal{P}$ is simply associated with K ([23])
- *minimal* T_Y if and only if $\mathcal{T} = \mathcal{W}(\mathcal{F}) \vee (\mathcal{C} \cap \mathcal{I}(K))$ for some degenerate K-cover $\mathcal{F}$ of X ([23])
- *minimal* T_{SA} if and only if $\mathcal{T} = \mathcal{E}(X \setminus B) \vee \mathcal{S}_K(\mathcal{P}) \vee (\mathcal{C} \cap \mathcal{I}(K \cup B))$ for some disjoint subsets B and K of X such that $K \neq \varnothing$ and $K \cup B \neq X$, and partition $\mathcal{P}$ of $X \setminus B$ such that $\mathcal{P}$ is simply associated with $X \setminus (K \cup B)$ and associated with K. ([24])
- *minimal* T_{SD} if and only if $\mathcal{T} = \mathcal{S}_K(\mathcal{P}) \vee (\mathcal{C} \cap \mathcal{I}(K))$ for some non-empty proper subset K of X and partition $\mathcal{P}$ of X such that $\mathcal{P}$ is simply associated with $X \setminus K$ and associated with K ([24])

- *minimal* T_{FA} if and only if

either $(X, \mathcal{T})$ is minimal T_{ES} with at least one isolated point and at least two closed points

or $(X, \mathcal{T})$ is minimal T_{SD}

or $\mathcal{T} = \mathcal{E}(X \setminus B) \vee \mathcal{S}_K(\mathcal{P}) \vee \mathcal{D}(B \cup K))$ for some non-empty, disjoint subsets B, K of X such that $B \cup K$ is a proper, infinite subset of X with $|X \setminus (B \cup K)| > 1$, a subset G of $X \setminus (B \cup K)$ and a partition $\mathcal{P}$ of $X \setminus (B \cup G)$ such that $\mathcal{P}$ is simply associated with $X \setminus K$ and associated with K.

2. Constructing the Partial Order

We need an axiom for partial orders which implies the T_{UD} axiom for topological spaces.

Definition 2.1. A partial order $(X, \lhd)$ is said to be T^+_{UD} if there is a family $\{Y(x) : x \in X\}$ of subsets of X such that

- $(\forall y \in Y(x)) y \lhd x \wedge y \neq x$
- $(\forall z \lhd x) z \neq x \Rightarrow (\exists y \in Y(x)) z \lhd y$
- $(\forall y, y' \in Y(x))(\forall z \in X)(z \lhd y \wedge z \lhd y') \Rightarrow y = y'$

We shall write $Y_X(x)$ if the underlying partial order is ambiguous.

A few comments:

We conjecture that the minimal T_{UD} topologies must be the weak topologies on a minimal T^+_{UD} partial order.

We conjecture that the minimal T^+_{UD} partial orders are just the T^+_{UD} and suitable partial orders. This would provide a characterization which requires for each pair of elements an infinite set which satisfies a first order formula. Maybe those weak separation axioms which have simpler minimality characterizations do so because of logical considerations, i.e., must all first order weak separation axioms have minimalities which are weak topologies for partial orders either without infinite chains or without infinite antichains?

Next we describe a way in which two T^+_{UD} partial orders can be combined and yet preserve T^+_{UD}.

Definition 2.2. If $X_0 \subset X_1$ are partial orders, where X_0 has the order induced by X_1, then we say that X_0 is a *simple* subset of X_1 if there are distinct $x_0, x_1 \in X_0$ and $w \in X_1 - X_0$and $A \subset X_1 - X_0$ such that

- $x_1 \lhd w \notin A$
- x_0 and x_1 are incomparable in X_0
- A is the set of all elements of $X_1 - X_0$ strictly below x_0
- each element of A is minimal in X_1
- $(\forall x \in X_0)(\forall y \in X_1 - X_0)(x \lhd y \Rightarrow x \lhd x_1 \lhd y = w)$
- $(\forall x \in X_0)(\forall y \in X_1 - X_0)(y \lhd x \Rightarrow y \lhd x_0 \lhd x)$

Proposition 2.3. *If X_0 is a simple subset of X_1 and both X_0 and $X_1 - X_0$ are T^+_{UD}, then X_1 is also T^+_{UD}.*

Moreover, we can get $Y_{X_1}(x) \cap X_0 = Y_{X_0}(x)$ for each $x \in X_0$.

Proof. Let $x_0, x_1 \in X_0$ and $w \in X_1 - X_0$ and $A \subset X_1 - X_0$ be as in definition 2.2. Suppose $x \in X_1$. We must define $Y(x)$ as in definition 2.1. We do this by cases.

(1) If $x \in X_0$ and $x \neq x_0$, then we let $Y(x) = Y_{X_0}(x)$.
(2) If $x \in X_1 - X_0$ and $x \neq w$, then we let $Y(x) = Y_{X_1 - X_0}(x)$.
(3) If $x = x_0$, then we let $Y(x) = Y_{X_0}(x) \cup A$.
(4) If $x = w$, then we let $Y(x) = Y_{X_1 - X_0}(x) \cup \{x_1\}$.

It suffices to show that definition 2.1 is satisfied by $\{Y(x) : x \in X_1\}$.

Verifying the first condition requires us to use only the facts $(\forall a \in A) a \lhd x_0 \wedge a \neq x_0$ and $x_1 \lhd w \wedge x_1 \neq w$.

Verifying the second condition requires examination of the same four cases.

(1) If $y \lhd x$ and $y \in X_1 - X_0$ then simplicity says that $y \lhd x_0 \lhd x$. Now, since $x \neq x_0 \in X_0$, we know that $(\exists s \in Y_{X_0}(x)) x_0 \lhd s$ and thus that $y \lhd s$.
(2) If $y \lhd x$ and $y \in X_0$ then $x = w$ which is impossible.
(3) If $x = x_0$ and $y \lhd x$ and $y \in X_1 - X_0$ then $y \in A$ which suffices.
(4) If $x = w$ and $y \lhd x$ and $y \in X_0$ then $y \lhd x_1$ which suffices.

Verifying the third condition also requires the examination of these same four cases.

(1) Suppose y_0, y_1 are distinct elements of $Y(x)$ and $z \lhd y_0, y_1$. Then $z \in X_1 - X_0$ so that $x_0 \lhd y_0$ and $x_0 \lhd y_1$ by the sixth condition of simplicity—clearly a contradiction.
(2) Suppose y_0, y_1 are distinct elements of $Y(x)$ and $z \lhd y_0, y_1$. Then $z \in X_0$ so that, by the fifth condition of simplicity, $y_0 = w = y_1$!
(3) The first case for $Y_{X_0}(x)$ and the fact that A is a set of minimal points in X_1 suffices.
(4) The second case for $Y_{X_1 - X_0}(x)$ leaves the possibility that there is $z \lhd x_1$ and $z \lhd y \in Y_{X_1 - X_0}(x)$. If $z \in X_1 - X_0$, then $z \lhd x_0 \lhd x_1$ which is impossible. If $z \in X_0$, then $z \lhd x_1 \lhd y = w$—yet $y \in Y_{X_1 - X_0}(w)$!

The proof is complete. □

Next, we describe when two incomparable elements of a T^+_{UD} partial order cannot be made comparable in a given "direction" without destroying T^+_{UD}. Moreover, since a T_{UD}-topology *may* induce an order which is *not* T^+_{UD}, we stipulate a condition to ensure that the resulting order has no compatible T_{UD}-topology.

Definition 2.4. If $X_0 \subset X_1$ are partial orders, where X_0 has the order induced by X_1, and $x_0, x_1 \in X_0$ are incomparable, then we say that X_0 is *a suitable subset in* X_1 *with respect to* (x_0, x_1), if there are, in X_1, elements w, $\{y_i : i \in \omega\}$ and $\{z_i : i \in \omega\}$ all distinct from each other and from x_0 and x_1 such that

- $(\forall i \in \omega) z_i \lhd y_i \lhd w$
- $(\forall i \in \omega) z_i \lhd x_0$
- $x_1 \lhd w$

- $(\forall F \in [X_1]^{<\omega})(((\forall f \in F) w \not\lhd f) \Rightarrow ((\exists i \in \omega)(\forall f \in F) y_i \not\lhd f))$

Note that this definition applies also when $X_0 = X_1$.

Indeed, we can "make" a T_{UD}^+ partial order suitable for two incomparable elements in a "simple" way.

Proposition 2.5. *If X is any T_{UD}^+ partial order and $x_0, x_1 \in X$ are incomparable, then there is a partial order $Y \supset X$ such that*

- *X is a simple subset of Y*
- *X is a suitable subset of Y with respect to (x_0, x_1)*
- *$Y - X$ is countable and T_{UD}^+*

Proof. We let $Y = X \cup \{y_i, z_i : i \in \omega\} \cup \{w\}$ where all these elements are distinct and not in X. We declare

- $(\forall i \in \omega) z_i \lhd x_0$
- $(\forall i \in \omega) z_i \lhd y_i \lhd w$
- $x_1 \lhd w$

and close off under transitivity.

To check that X is a simple subset of Y, define $A = \{z_i : i \in \omega\}$. Since nothing is defined to be below any z_i, we know that each z_i is minimal in Y. Thus we have conditions 1, 2 and 4 in definition 2.2. Further, clearly w cannot be below x_0 nor can any y_i be below x_0, so that condition 3 is satisfied.

If $x \in X$, $y \in Y - X$ and $x \lhd y$, then $x_1 \lhd w$ must be a step in the calculation. Since nothing is defined to be above w, w is maximal in Y and so $w = y$ as required. Thus condition 5 is satisfied.

If $x \in X$, $y \in Y - X$ and $y \lhd x$, then $z_i \lhd x_0$ must be a step in the calculation as required. Thus condition 6 is satisfied.

To check that X is a suitable subset of Y with respect to (x_0, x_1), suppose that there exists finite $F \subset Y$ such that $(\forall f \in F) w \neq f$ and $(\forall i \in \omega)(\exists f \in F) y_i \lhd f$. If $y_i \lhd f$ and $y_i \neq f$, then some step in the calculation must be $y_i \lhd w$. Since w is maximal in Y, we must have $f = w$ which is impossible. Thus we know that $(\forall i \in \omega)(\exists f \in F) y_i = f$ and thus $F \supset \{y_i : i \in \omega\}$!

To check that $Y - X$ is T_{UD}^+, let $Y(w) = \{y_i : i \in \omega\}$, $Y(y_i) = \{z_i\}$ and $Y(z_i) = \varnothing$. □

Definition 2.6. A partial order $(X, \lhd)$ is said to be *suitable* if, for each $x_0, x_1 \in X$ which are incomparable, X is suitable in itself with respect to (x_0, x_1).

Suitability can be obtained in a "simple" increasing sequence if suitability with respect to each incomparable pair is accomplished along the way.

Proposition 2.7. *If $\{X_i : i \in \omega\}$ is an increasing sequence of (partially ordered) subsets of a partial order X such that*

- *each X_i is a simple subset of X_{i+1}*
- $\bigcup\{X_i : i \in \omega\} = X$
- *$(\forall$ incomparable $x_0, x_1 \in X)(\exists i \in \omega) X_i$ is a suitable subset of X_{i+1} with respect to (x_0, x_1)*

then X is suitable.

Proof. Given any incomparable elements x_0, x_1 in X, we must check that X is suitable in itself with respect to (x_0, x_1). Now suppose that X_i is a suitable subset of X_{i+1} with respect to (x_0, x_1). This gives us distinct w, $\{y_i : i \in \omega\}$ and $\{z_i : i \in \omega\}$ as in definition 2.4. Thus the first three conditions of definition 2.4 are satisfied. We need to check the fourth condition.

Suppose $F \in [X]^{<\omega}$ and $(\forall f \in F) w \not\lhd f$ and $(\forall i \in \omega)(\exists f \in F) y_i \lhd f$. We shall argue that no such F can exist by mathematical induction. Find such an F with $j^* = \text{minimum}\{\max[j \in \omega : F \cap (X_{j+1} - X_j) \neq \varnothing]\}$ and furthermore such that $F \cap (X_{j^*+1} - X_{j^*})$ has minimum cardinality. Let $F' = \{f \in F : (\exists i \in \omega) y_i \lhd f\}$. We shall prove that $j^* \leq i$. Suppose $j^* > i$ and choose $f \in F' \cap (X_{j^*+1} - X_{j^*})$—such a choice is possible because of the minimum nature of j^*. We know that for certain $i \in \omega$, $y_i \lhd f$. Each such y_i is an element of $X_{i+1} \subset X_{j^*}$. Let the fact that X_{j^*} is a simple subset of X_{j^*+1} be witnessed by x_0^*, x_1^*, w^*. The fifth condition of simplicity gives us that $y_i \lhd x_1^* \lhd f = w^*$ so that $F' \cap (X_{j^*+1} - X_{j^*}) = \{w^*\} = \{f\}$. Thus f can be replaced by $x_1^* \in X_{j^*}$, giving a subset $F^* = (F - \{f\}) \cup \{x_1^*\}$ of F which again contradicts the minimum nature of j^*. Thus $j^* \leq i$. It follows that $F \subset X_{i+1}$, contradicting the suitability of X_i in X_{i+1} with respect to (x_0, x_1). □

Finally we can accomplish our aim of making a T_{UD}^+ partial order suitable without destroying T_{UD}^+.

Proposition 2.8. *Any countable T_{UD}^+ partial order can be embedded in a suitable T_{UD}^+ partial order.*

Proof. First, we define a partition $\{P_i : i \in \omega\}$ of ω. Given $P_i \subset \omega$ for $i < n$, define P_n to be any infinite, co-infinite subset of $\omega - \bigcup_{i<n} P_i$ such that $n \in P_n$ if $n \notin \bigcup_{i<n} P_i$ It is routine to verify that the family $\{P_i : i \in \omega\}$ is a partition of ω satisfying

(i) $\min P_i \geq i \ \forall i \in \omega$
(ii) $|P_i| = \omega \ \forall i \in \omega$.

Next, let X_0 be any countable T_{UD}^+ partial order with at least two incomparable elements. Write $\{(x, y) : x, y \in X_0,\ x \neq y\} = \{(x_i, y_i) : i \in P_0\}$. Consult (x_0, y_0). If x_0 and y_0 are comparable, then write $X_1 = X_0$; otherwise, extend X_0 to X_1 in the manner of proposition 2.5. We continue in this way, applying proposition 2.5 ω-many times to form an increasing sequence $\{X_i : i \in \omega\}$ and a sequence $\{(x_i, y_i) : i \in \omega\}$ such that

(i) X_i is a simple subset of X_{i+1}, when $X_i \neq X_{i+1}$
(ii) $x_i, y_i \in X_i$
(iii) if x_i, y_i are incomparable elements of X_i then X_i is a suitable subset of X_{i+1} with respect to (x_i, y_i) and $\exists j \in \omega$ such that X_j is a suitable subset of X_{j+1} with respect to (y_i, x_i)
(iv) each non-empty $X_{i+1} - X_i$ is T_{UD}^+ and countable
(v) $(\forall \text{ distinct } x, y \in \bigcup\{X_i : i \in \omega\})(\exists i \in \omega)(x, y) = (x_i, y_i)$

Conditions (i), (iii) and (iv) follow from proposition 2.5, while (ii) and (v) follow from the properties of the index sets P_i. Let $X = \bigcup\{X_i : i \in \omega\}$ have the smallest partial order $\lhd$ which gives each X_i the subset order. By proposition 2.7, X is suitable. Applying proposition 2.3 iteratively, we deduce that each X_i is T^+_{UD} and that furthermore the family $\{Y_{X_i}(x) : i \in \omega,\, x \in X_i\}$ satisfies $(\forall i < j) x \in X_i \Rightarrow Y_{X_j}(x) \cap X_i = Y_{X_i}(x)$.

Since X is the increasing union of the X_i's, it is routine to verify that X is also T^+_{UD}. □

Indeed any T^+_{UD} partial order whatsoever can be embedded in a suitable T^+_{UD} partial order.

Corollary 2.9. *There is a suitable non-linear T^+_{UD} partial order.*

Proof. By proposition 2.8, the empty partial order on two elements (which is T^+_{UD}) can be embedded in a suitable T^+_{UD} partial order. Any partial order which embeds the empty partial order on two elements must be non-linear. □

3. The Relation between the Partial Order and the Topology

Lemma 3.1. *If a partial order is T^+_{UD}, then the associated weak topology is T_{UD}.*

Proof. Let $(X, \lhd)$ be a T^+_{UD} partial order and let $\mathcal{W}$ be the associated weak topology for $(X, \lhd)$. Given $x \in X$, the derived set of $\{x\}$ is $\overline{\{x\}}^{\mathcal{W}} - \{x\} = \{y \in X : y \lhd x \wedge y \neq x\}$. With $Y(x)$ as in definition 2.1, for each $y \in Y(x)$, $\overline{\{y\}}^{\mathcal{W}} = \{z \in X : z \lhd y\}$ so that it follows (by definition 2.1) that $\overline{\{x\}}^{\mathcal{W}} - \{x\} = \bigcup\{\overline{\{y\}}^{\mathcal{W}} : y \in Y(x)\}$, a union of disjoint closed sets. That is, $\mathcal{W}$ is a T_{UD} topology for $(X, \lhd)$. □

Proposition 3.2. *If the weak topology $\mathcal{W}$ on a suitable partial order $(X, \lhd)$ is T_{UD}, then it is minimally T_{UD}.*

Proof. Suppose not and that there exists a T_{UD}-topology $\mathcal{T}$ for X which is a proper subset of $\mathcal{W}$. Let $\preceq$ be the specialization preorder induced on X by $\mathcal{T}$. We note that $\preceq$ is, in fact, a partial order since T_{UD} implies T_0. Since $\mathcal{W}$ is the smallest topology on X to induce $\lhd$, it follows that $\preceq$ must strictly contain $\lhd$ as a relation. Thus, there must exist elements x_0, x_1 in X which are incomparable in $\lhd$ and yet $x_0 \preceq x_1$. Since $\lhd$ is suitable, then in particular it is suitable in itself with respect to (x_0, x_1) so that there exist w, $\{y_i : i \in \omega\}$ and $\{z_i : i \in \omega\}$ as in definition 2.4. It follows that

(i) $(\forall i \in \omega) z_i \preceq y_i \preceq w$
(ii) $(\forall i \in \omega) z_i \preceq x_0 \preceq x_1 \preceq w$

Now, $\mathcal{T}$ is T_{UD} and so there exist disjoint $\mathcal{T}$-closed sets $\{A_\alpha : \alpha \in \kappa\}$ whose union is the derived set of $\{w\}$. Suppose without loss of generality that $x_1 \in A_0$. Then, by (ii), $\{z_i : i \in \omega\} \subset A_0$. Fix $i \in \omega$; since y_i is an element of the derived set of $\{w\}$, then there exists $\beta \in \kappa$ such that $y_i \in A_\beta$. But, by (i), $z_i \preceq y_i$ so that $z_i \in A_\beta$, whence $\beta = 0$. Thus $\{y_i : i \in \omega\} \subset A_0$, whence $w \notin \overline{\{y_i : i \in \omega\}}^{\mathcal{T}}$. It follows that $w \notin \overline{\{y_i : i \in \omega\}}^{\mathcal{W}}$ (since $\mathcal{T} \subset \mathcal{W}$) and so there exists a finite set $F \subset X$ such that

- $(\forall i \in \omega)(\exists f \in F) y_i \lhd f$
- $(\forall f \in F) w \ntrianglelefteq f$

This contradicts the fifth condition of suitability. □

Corollary 3.3. *There is a minimal T_{UD} topology which is not nested.*

Proof. Apply corollary 2.9 to get a suitable non-linear T^+_{UD} partial order and then take the weak topology $\mathcal{W}$. Observe that the weak topology on any non-linear partial order is not nested. Lemma 3.1 says that $\mathcal{W}$ is T_{UD}. Proposition 3.2 says that $\mathcal{W}$ is minimal T_{UD}. □

References

[1] S.J. Andima and W.J. Thron, *Order-induced topological properties*, Pacific J. Math. (2) **75** (1978).

[2] C.E. Aull and W.J. Thron, *Separation axioms between T_0 and T_1*, Indag. Math. **24** (1962), 26–37.

[3] V.K. Balachandran, *Minimal bicompact spaces*, J. Ind. Math. Soc. **12** (1948), 47–48.

[4] B. Banaschewski, *Über zwei Extramaleigenschaften topologischer Räume*, Math. Nachr. **13** (1955), 141–150.

[5] M.P. Berri, *Minimal topological spaces*, Trans. Amer. Math. Soc. **108** (1963), 97–105.

[6] M.P. Berri, *Categories of certain minimal topological spaces*, J. Austral. Math. Soc. **3** (1964), 78–82.

[7] M.P. Berri, J.R. Porter and R.M. Stephenson Jr., *A survey of minimal topological spaces*, General Topology and its relation to modern analysis and algebra, Proc. Conference topology, Academic Press, New York (1970).

[8] M.P. Berri and R.H. Sorgenfrey, *Minimal regular spaces*, Proc. Amer. Math. Soc. **14** (1963), 454–458.

[9] N. Bourbaki, *Espaces minimaux et espaces complètement séparés*, C. R. Acad. Sci. Paris **212** (1941), 215–218.

[10] D.E. Cameron, *Maximal and minimal topologies*, Trans. Amer. Math. Soc. **160** (1971), 229–248.

[11] H. Herrlich, *T_ν-Abgeschlossenheit und T_ν-Minimalität*, Math. Z. **88** (1965), 285–294.

[12] R-E. Hoffmann, *Essentially complete T_0-spaces*, Manuscripta Math. **27** (1979), 401-432.

[13] S. Ikenaga, *Product of minimal topological spaces*, Proc. Japan Acad. **40** (1964), 329–331.

[14] B. Johnston and S.D. McCartan, *Minimal T_F-spaces and minimal T_{FF}-spaces*, Proc. R. Ir. Acad. **80A** (1980), 93–96.

[15] B. Johnston and S.D. McCartan, *Minimal T_{YS}-spaces and minimal T_{DD}-spaces*, Proc. R. Ir. Acad. **88A** (1988), 23-28.

[16] H. Kawashima, *On the topological product of minimal Hausdorff spaces*, TRU Math. **1** (1965), 62–64.

[17] D.C. Kent and G.D. Richardson, *Minimal convergence spaces*, Trans. Amer. Math. Soc. **160** (1971), 487–500.

[18] S.M. Kim, *Quasi-compact spaces and topological products of minimal Hausdorff spaces*, Kyungpook Math. J. **6** (1965), 49–52.

[19] R.E. Larson, *Minimal T_0-spaces and minimal T_D-spaces*, Pac. J. Math. **31** (1969), 451-458.

[20] R.E. Larson, *Minimum and maximum topological spaces*, Notices Amer. Math. Soc. **16** (1969), 347. Bull. Polish Acad. Sci. **18** (1970), 707–710.
[21] S.D. McCartan, *Minimal T_{ES}-spaces and minimal T_{EF}-spaces*, Proc. R. Ir. Acad. **79A** (1979), 11-13.
[22] A.E. McCluskey and S.D. McCartan, *The minimal structures for T_A*, Ann. New York Acad. Sci. **659** (1992), 138-155.
[23] A.E. McCluskey and S.D. McCartan, *Minimality with respect to Youngs' axiom*, Houston J. Math. **21** (1995), 413-428.
[24] A.E. McCluskey and S.D. McCartan, *Minimality with respect to T_{SA} and T_{SD}*, Top. with Applications, Szekszard (Hungary), (1993), 83-97.
[25] A.E. McCluskey and S.D. McCartan, *Minimality structures for T_{FA}*, *Rend. dell' Istituto di Matematica dell' Università di Trieste*, **27** (1995), 11–24.
[26] Ki-Hyun Pahk, *Note on the characterizations of minimal T_0 and T_D spaces*, Kyungpook Math. J. **8** (1968), 5-10.
[27] A.S. Parhomenko, *Über eineindeutige stetige Abbildungen*, Mat. Sb. 5 **47** (1939), 197–210.
[28] J.R. Porter, *Minimal first countable spaces*, Bull. Austral. Math. Soc. **3** (1970), 55–64.
[29] J.R. Porter and J.P. Thomas, *On H-closed and minimal-Hausdorff spaces*, Trans. Amer. Math. Soc. **138** (1969), 159–170.
[30] T.G. Raghavan and I.L. Reilly, *On minimal Hausdorff first countable spaces*, Ind. J. Pure and App. Math. 14 **2** (1983), 244–252.
[31] A. Ramanathan, *A characterization of maximal-Hausdorff spaces*, J. Indian Math. Soc. **11** (1947), 73–80.
[32] A. Ramanathan, *Maximal Hausdorff spaces*, Proc. Indian Acad. Sci. Sec. A **26** (1947), 31–42.
[33] A. Ramanathan, *Minimal-bicompact spaces*, J. Indian Math. Soc. **12** (1948), 40–46.
[34] C.T. Scarborough, *Minimal Urysohn spaces*, Pac. J. Math. **27** (1968), 611–618.
[35] C.T. Scarborough and R.M. Stephenson Jr., *Minimal topologies*, Colloq. Math. **19** (1968), 215–219.
[36] N. Smythe and C.A. Wilkins, *Minimal Hausdorff and maximal compact spaces*, J. Austral. Math. Soc. **3** (1963), 167–171.
[37] R.M. Stephenson Jr., *Minimal first countable topologies*, Trans. Amer. Math. Soc. **138** (1969), 115–128.
[38] R.M. Stephenson Jr., *Minimal first countable Hausdorff spaces*, Pac. J. Math. **36** (1971), 821–828.
[39] W.J.Thron and S.J. Zimmerman, *A characterization of order-topologies by means of minimal T_0-topologies*, Proc. Amer. Math. Soc. **27** (1971), 161–167.
[40] H. Tong, *Minimal bicompact spaces*, Bull. Amer. Math. Soc. **54** (1948), 478–479.
[41] J. Vermeer, *Embeddings in minimal Hausdorff spaces*, Proc. Amer. Math. Soc. **87** (1983), 533–535.
[42] G. Vigilino, *A co-topological application to minimal spaces*, Pac. J. Math. **27** (1969), 197–200.

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