Every finite system of $T_1$ uniformities comes from a single distance structure

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**Abstract.** Using the general notion of distance function introduced in an earlier paper, a construction of the finest distance structure which induces a given quasi-uniformity is given. Moreover, when the usual defining condition $x U_\varepsilon y :\iff d(y, x) \leq \varepsilon$ of the basic entourages is generalized to $nd(y, x) \leq n\varepsilon$ (for a fixed positive integer $n$), it turns out that if the value-monoid of the distance function is commutative, one gets a countably infinite family of quasi-uniformities on the underlying set. It is then shown that at least every finite system and every descending sequence of $T_1$ quasi-uniformities which fulfill a weak symmetry condition is included in such a family. This is only possible since, in contrast to real metric spaces, the distance function need not be symmetric.

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**1. Introduction**

Since Fréchet’s invention of real metric spaces in [2], many generalizations of this concept have been studied in the literature. Much research has been done on *generalized metric spaces*, in which the distance functions are replaced by certain set systems (cf. [9]). On the contrary, many authors independently suggested more general types of *distance functions*, the references [8], [12], [7], [5], [6], [11], [10], and [1] are only a small selection. In [3] and [4], a common framework for most if not all of these general concepts of distance functions has been developed to a certain extent.

In this paper, the induction of quasi-uniformities on a *distance space* $(X, d, M, P)$ will be studied. In such a structure, $d : X \times X \to M$ is a general *distance function* on $X$, that is, it fulfills the zero-distance condition $d(x, x) = 0$ and the triangle inequality $d(x, y) + d(y, z) \geq d(x, z)$, and takes its values in a quasi-ordered monoid (q.o.m.) $M = (M, +, 0, \leq)$. The set $P \subseteq M$ must be a *set of positives* (or idempotent zero-filter) for $M$, that is, a filter of $(M, \leq)$ with
Jobst Heitzig

infimum 0 such that, for every \( \varepsilon \in P \), there is \( \delta \in P \) with \( 2\delta \leq \varepsilon \). The triple \((d, M, \varepsilon)\) is called a distance structure on \( X \). For examples and categorical aspects of distance functions on various mathematical objects, see \([3, 4]\).

Using Kelley's metrization lemma, one can easily show that every quasi-uniformity is induced by a suitable multi-quasi-pseudo-metric, that is, a "quasi-pseudo-metric" taking values in a real vector space instead of the non-negative reals. There is no doubt that this fact must have been noticed early. In this article however, we will see that also every finite family of \( T_1 \) uniformities (and many families of \( T_1 \) quasi-uniformities) on a fixed set \( X \) comes from a single distance structure. In Theorem 8, this is proved by constructing the finest such structure. This construction is a combinatorially more complex variant of the construction of a finest distance structure for a given quasi-uniformity, which is given in Theorem 2. In contrast to multi-quasi-pseudo-metric spaces, the "topological" information in the resulting spaces will be mostly contained in the set of positives \( P \) rather than in the distance function \( d \) itself. For example, each \( T_1 \) quasi-uniformity on some set \( X \) can be induced using one and the same distance function.

2. Preliminaries

In generalization of the usual definition of entourages in a metric space, let

\[ U_n(\varepsilon) := \{(x, y) \in X \times X : nd(y, x) \leq n\varepsilon\} \]

for every \( \varepsilon \in P \) and every positive integer \( n \). As \( P \) is a filter, the set \( \mathcal{E}_n := \{U_n(\varepsilon) : \varepsilon \in P\} \) is a base for a filter \( \mathcal{U}_n \) of reflexive relations on \( X \) for each \( n \). Moreover, when \( M \) is commutative,

\[ nd(y, x) \leq \delta \geq nd(z, y) \implies nd(z, x) \leq n(d(z, y) + d(y, x)) \leq 2\delta, \]

so that, for every \( \varepsilon \in P \), there is \( \delta \in P \) with \( U_n(\delta)^2 \subseteq U_n(\varepsilon) \), that is, \( \mathcal{U}_n \) is a quasi-uniformity.

Of course, there are certain relationships between the \( \mathcal{U}_n \), and in many cases most of them coincide. Obviously,

\[ n = n_1 + \cdots + n_k \text{ implies } U_{n_1}(\varepsilon) \cap \cdots \cap U_{n_k}(\varepsilon) \subseteq U_n(\varepsilon). \]

Also, \( nd(x, y) \leq nmd(x, y) + (m - 1)nd(y, x) \), so that

\[ (2m - 1)n\delta \leq n\varepsilon \text{ implies } U_m(\delta) \cap U_n^{-1}(\delta) \subseteq U_n(\varepsilon). \]

For a positive \( d \) (that is, when \( d(x, y) \geq 0 \) for all \( x, y \)),

\[ n \leq m \text{ and } m\delta \leq n\varepsilon \text{ imply } U_m(\delta) \subseteq U_n(\varepsilon). \quad (\dagger) \]

On the other hand, a symmetric \( d \) (that is, one with \( d(x, y) = d(y, x) \)) fulfills \( 2d(x, y) = d(x, y) + d(y, x) \geq d(x, x) = 0 \), so that here the implication \((\dagger)\) holds at least when \( m - n \) is even. This proves the following
Lemma 2.1.

(a) \( n = n_1 + \cdots + n_k \) implies \( \mathcal{U}_n \subseteq \mathcal{U}_{n_1} \vee \cdots \vee \mathcal{U}_{n_k} \), in particular, the map \( n \mapsto \mathcal{U}_n \) is antitone with respect to divisibility.

(b) For all \( n, m \), \( \mathcal{U}_n \subseteq \mathcal{U}_m \).  

(c) For a positive \( d \), all \( \mathcal{U}_n \) coincide.

(d) For a symmetric \( d \) and all \( k \geq 1 \), \( \mathcal{U}_{2k} = \mathcal{U}_2 \subseteq \mathcal{U}_{k+1} = \mathcal{U}_{2k-1} \).

Note that there are indeed natural distance functions which are neither positive nor symmetric, the most important being perhaps the distance \( x^{-1}y \) on groups, introduced by Menger [8]:

Example 2.2. Let \( G := [0,2\pi) \) be the additive group of real numbers modulo \( 2\pi \), \( M := \langle \mathcal{P}(G), +, \{0\}, \subseteq \rangle \) the power set of \( G \) ordered by set inclusion and with the usual element-wise addition, \( P := \{(-\delta, \delta) : \delta \in (0,2\pi]\} \). Then \( d(x,y) := \{y-x\} \) defines a skew-symmetric distance function (that is, one with \( d(x,y) + d(y,x) = 0 \), and \( \mathcal{U}_1 \) is the usual “Euclidean” uniformity on \( G \), while \( \mathcal{U}_n \) is this uniformity “modulo \( \frac{2\pi}{n} \)” since 

\[ x \mathcal{U}_n(-\delta, \delta) y \iff x - y \in \bigcup_{k \in n}(\delta + \frac{2k\pi}{n}, \frac{2k\pi}{n} + \delta). \]

Likewise, for \( X := C \setminus \{0\} \), \( M' := M \otimes [0,\infty) \), \( P' := P \times (0,\infty) \), and \( d'(x,y) := (d'(|\arg x, \arg y|), |y|-|x|) \), the uniformity \( \mathcal{U}_n \) of \( (d', M', P') \) induces the Euclidean topology “modulo multiplication with \( n \)th roots of unity”.

3. Finest distance functions

Like for other topological structures on a set \( X \), we might compare two distance functions \( d, d' \) resp. distance structures \( D = (d, M, P) \) and \( D' = (d', M', P') \) on \( X \) with respect to their fineness. If the implication 

\[ d(x_1, y_1) + \cdots + d(x_n, y_n) \leq d(z_1, w_1) + \cdots + d(z_m, w_m) \]

\[ \implies d'(x_1, y_1) + \cdots + d'(x_n, y_n) \leq d'(z_1, w_1) + \cdots + d'(z_m, w_m) \]

holds for all \( x_i, y_i, z_i, w_i \in X \), we say that \( d \) is finer than \( d' \). If, additionally, for all \( \varepsilon' \in P' \), there is \( \varepsilon \in P \) such that 

\[ d(x_1, y_1) + \cdots + d(x_n, y_n) \leq \varepsilon \implies d'(x_1, y_1) + \cdots + d'(x_n, y_n) \leq \varepsilon' \]

for all \( x_i, y_i \in X \), we say that \( D \) is finer than \( D' \).

For a convenient notation, let me introduce the free monoid \( F \) of all words in \( X \) that have even length and define 

\[ d(x_1y_1 \cdots x_ny_n) := d(x_1, y_1) + \cdots + d(x_n, y_n), \]

\[ s \mathcal{R}_d t := d(s) \leq d(t) \quad (s, t \in F). \]

By definition, \( (F, \circ, 0, \mathcal{R}_d) \) is a q.o.m., where \( \circ \) is concatenation and \( 0 \) is the empty word. Given any quasi-order \( R \) on \( F \) which is compatible to \( \circ \) (that is, whenever \( (F, \circ, 0, R) \) is a q.o.m.), the following construction leads to a distance function \( d_R \) if and only if 

\[ xx R 0 R xx \text{ and } xz R xy yz \quad \text{for all } x, y, z \in X. \quad (*) \]
Let \((M_R, \subseteq) := \theta(F, R)\) be the lower set completion of \((F, R)\), that is, the system of all lower sets \(RA := \{s : sRt \text{ for some } t \in A\}\) of \((F, R)\) with set inclusion as partial order. Define an associative operation \(+_R\) on \(M_R\) and its neutral element \(0_R\) by
\[
RA +_R RB := R\{s \circ t : s \in A \text{ and } t \in B\} \quad \text{for all } A, B \subseteq F
\]
and \(0_R := R\{0\}\). Then let
\[
d_R : \begin{cases}
X \times X & \mapsto M_R = (M_R, +_R, 0_R, \subseteq) \\
(x, y) & \mapsto R\{xy\}.
\end{cases}
\]
It was shown in [3] that \(d_{R_L}\) is equivalent to \(d\), which motivates calling \(R_d\) the generating quasi-order of \(d\). Moreover, when \(R_L\) is the smallest quasi-order on \(F\) which fulfils \((*)\) and is compatible with \(\circ\) then \(d_L := d_{R_L}\) is a finest distance function on \(X\). In this relation, the step from \(s \in F\) to an upper neighbour w.r.t. \(R_L\) consists of inserting a pair \(yy\) at an arbitrary position in \(s\) or removing a pair \(yy\) after an even number of letters in \(s\), while the step to a lower neighbour is made by removing a pair \(yy\) at an arbitrary position or inserting a pair \(yy\) after an even number of letters.

4. Induction of a Single Quasi-Uni

We are now ready for the first main result of this paper:

**Theorem 4.1.** Every quasi-uniformity \(V\) admits a finest distance structure \((d_V, M_V, P_V)\) for which \(V = U_1\).

**Proof.** Let \(V\) be some quasi-uniformity on \(X\) and \(V_0 := \bigcap V\). We will see that the essential information about \(V\) is contained in the set of positives \(P_V\) which we must construct, while the generating quasi-order \(R_{d_V}\) is fully determined by the very weak condition that \(xy \in R_{d_V} zz\) must hold for any triple \(x, y, z \in X\) which fulfils \(y V_0 x\) (otherwise \(d_V(x, y) \not\in \varepsilon\) for some \(\varepsilon \in P_V\), in contradiction to \(V_0 \subseteq U_1(\varepsilon)\)). Therefore, let \(R\) be the smallest quasi-order on \(F\) that is compatible with \(\circ\) and fulfils
\[
x'y' R 0Rx \text{ and } xz R xy yz \quad \text{for all } x, y, z, x', y' \in X \text{ with } y' V_0 x'. \quad (*)
\]
If we find a suitable s.o.p. \(P\) such that \((d_R, P)\) induces \(V\) then \(R\) must obviously be the smallest relation (and thus \(d_R\) a finest distance function) with this property.

Now observe that each of the resulting entourages \(U_1(\varepsilon)\) has to include some entourage \(V_1 \in \mathcal{V}\), hence every \(\varepsilon \in P\) must include some set \(\{xy \in F : y V_1 x\}\) with \(V_1 \in \mathcal{V}\). Since \(0_R = R\{xx\}\) is a neutral element, \(\varepsilon\) must even include the set
\[
\{xy \in F : y V_0 V_1 V_0 x\} \subseteq 0_R +_R \{xy \in F : y V_1 x\} +_R 0_R.
\]
The same must be true for any \(\delta \in P\) which fulfils \(\delta +_R \delta \subseteq \varepsilon\), so that \(\varepsilon\) must also include a set \(\{xyx'y' \in F : y V_0 V_2 V_0 x, y' V_0 V_2 V_0 x'\} \subseteq \delta +_R \delta\) for some \(V_2 \in \mathcal{V}\). This process of replacing some \(\varepsilon\) by some \(2\delta\) can be continued, and in
order to describe it formally, let us define $W$ to be the smallest set of tuples of positive integers that contains the 1-tuple $(1)$ and fulfils 

$$(n_1, \ldots, n_i-1, n_i + 1, n_i + 1, n_{i+1}, \ldots, n_k) \in W$$

whenever $(n_1, \ldots, n_k) \in W$ and $1 \leq i \leq k$. One can think of the elements of $W$ as coding exactly those terms of the form $'\varepsilon_{n_1} + \cdots + \varepsilon_{n_k}'$ that can be obtained when we start with the term $'\varepsilon_1'$ and then successively replace an arbitrary summand $'\varepsilon_n'$ by the term $'\varepsilon_{n+1} + \varepsilon_{n+1}'$. Accordingly, one shows by induction that for each element $\varepsilon_1$ of a set of positives $P$ there is a sequence $\varepsilon_2, \varepsilon_3, \ldots$ in $P$ such that

$$(n_1, \ldots, n_k) \in W \text{ implies } \varepsilon_{n_1} + \cdots + \varepsilon_{n_k} \leq \varepsilon_1.$$ 

In our situation, this observation implies that for each $\varepsilon \in P$ there must be a sequence $S = (V_1, V_2, \ldots)$ in $V$ with the property that $\varepsilon$ includes the set $A_S$ of all words $v_1w_1 \cdots v_kw_k \in F$ for which there is some $(n_1, \ldots, n_k) \in W$ such that $w_i V_1 V_2 \cdots V_l v_i$ for $i = 1, \ldots, k$. In particular, $\varepsilon_S := RA_S \subseteq R \varepsilon = \varepsilon$. It turns out that this is the only restraint on the set of positives $P_V$. More precisely, we will see that the system

$$B := \{ \varepsilon_S : S \text{ is a sequence in } V \}$$

of lower sets of $(F, R)$ is a base for a set of positives of $(M_R, +_R, 0_R, \subseteq)$, and that the distance structure $(d_R, P)$ induces the quasi-uniformity $\mathcal{V}$. It is then clear that $P$ is the largest set of positives with this property, so that $(d_V, P_V) := (d_R, P)$ is a finest distance structure inducing $\mathcal{V}$.

Since $V$ is a filter and the map $S \mapsto \varepsilon_S$ is isotone in every component of $S$, $B$ is a filter-base. In order to show that $P$ is a s.o.p., we first observe that $(n_1, \ldots, n_k), (m_1, \ldots, m_l) \in W$ implies

$$(n_1 + 1, \ldots, n_k + 1, m_1 + 1, \ldots, m_l + 1) \in W.$$ 

Indeed, after increasing each index by one, the replacements that produce $(n_1, \ldots, n_k)$ and $(m_1, \ldots, m_l)$ from the tuple $(1)$ can be combined to a sequence of replacements that produce $(n_1 + 1, \ldots, n_k + 1, m_1 + 1, \ldots, m_l + 1)$ from the tuple $(2, 2)$.

Hence also $v_1w_1 \cdots v_kw_k, v'_1w'_1 \cdots v'_{l'}w'_{l'} \in \varepsilon(V_2, V_3, V_4, \ldots)$ implies

$$v_1w_1 \cdots v_kw_kv'_1w'_1 \cdots v'_{l'}w'_{l'} \in \varepsilon(V_1, V_2, V_3, V_4, \ldots)$$

for each sequence $(V_1, V_2, \ldots)$ in $V$. Secondly, we must prove that $\bigcap B = 0_R$, which is the harder part. Let $s = x_1z_1 \cdots x_mz_m \in \bigcap B$ and $V_1 \in \mathcal{V}$. I will show that $z_j V_0 V_1 V_2 \cdots V_x j$ holds for all $j = 1, \ldots, m$. Choose a sequence $S = (V_1, V_2, \ldots)$ in $V$ such that $V_i V_0 V_{i+1} \subseteq V_i$ for all $i \geq 1$ (such a sequence always exists in a quasi-uniformity). Note that $(n_1, \ldots, n_k) \in W$ then implies $V_0 V_{n_1} V_0 V_{n_2} V_0 \cdots V_0 V_{n_k} V_0 \subseteq V_0 V_1 V_0$. Now $s \in RA_S$, that is, there exists a word $v_1w_1 \cdots v_kw_k$ and a $k$-tuple $(n_1, \ldots, n_k) \in W$ such that $w_i V_0 V_{n_i} V_0 v_i$ for $i = 1, \ldots, k$ and $s R v_1w_1 \cdots v_kw_k$. The latter means that, starting with $v_1w_1 \cdots v_kw_k$, one gets $x_1z_1 \cdots x_mz_m$ in finitely many steps in each of which
some pair of letters is inserted or removed corresponding to the condition \((\star')\). Now take the \(k\)-tuple
\[
\psi := (w_1 V_0 V_{n_1} V_0 v_1, \ldots, w_k V_0 V_{n_k} V_0 v_k)
\]
of formulae (which express true propositions about the word \(v_1 w_1 \cdots v_k w_k\)) and modify it, analogously to those finitely many steps, in the following way: (i) if (because of \(x z R x y y z\)) a pair \(yy\) is being removed after an odd number of letters, replace the two consecutive formulae \(\ldots V_0 y, y V_0 \cdots\) in \(\psi\) by one formula \(\ldots V_0 \cdots\) (that is, erase the symbols ‘\(y, y V_0\)’); (ii) if (because of \(0 R x x\)) a pair \(xx\) is being removed after an even number of letters, remove the corresponding formula \(x \ldots x\) from \(\psi\); (iii) if (because of \(x' y' R 0\)) a pair \(x' y'\) is inserted, insert the formula \(y' V_0 x'\) at the respective position in \(\psi\). By definition of \(R\), all these modifications preserve the truth of all formulae in the tuple, and each formula in the resulting tuple \((\psi_1, \ldots, \psi_k)\) expresses a true proposition of the form
\[
\psi_j = z_j V_0 V_{n_1} V_0 V_{n_{a+1}} V_0 \cdots V_0 V_{n_k} V_0 x_j,
\]
with \(1 \leq a, b \leq k\). Since all \(V_{n_i}\) are reflexive, \(\psi_j\) thus implies
\[
z_j V_0 V_{n_1} V_0 V_{n_2} V_0 \cdots V_0 V_{n_k} V_0 x_j,
\]
hence \(z_j V_0 V_1 V_0 x_j\). Because \(V_1\) was chosen arbitrarily, we conclude that \(z_j V_0 x_j\) for all \(j\), and therefore \(x_1 z_1 \cdots x_m z_m R 0\).

Finally, we have to show that \((d_R, P)\) induces the quasi-uniformity \(V\). For \(V \in V\), choose \(V_1 \in V\) such that \(V_0 V_1 V_0 \subseteq V\), then choose a sequence \(S\) as in the preceding paragraph. There we have shown that, in particular,
\[
d_R(x, z) \subseteq RA_S \text{ implies } (z, x) \in V_0 V_1 V_0 \subseteq V.
\]
On the other hand, for each \(\varepsilon \in P\) there is some sequence \(S = (V_1, \ldots)\) in \(V\) such that \(\varepsilon S \subseteq \varepsilon\), and
\[
(z, x) \in V_1 \subseteq V_0 V_1 V_0 \text{ implies } d_R(x, z) \subseteq \varepsilon S \subseteq \varepsilon.
\]

\(\square\)

A somewhat astonishing consequence of this construction is that one distance function is compatible to all \(T_1\) quasi-uniformities on \(X\):

**Corollary 4.2.** The distance function \(d_\perp\) is the finest distance function \(d\) on \(X\) such that for each \(T_1\) quasi-uniformity \(V\) on \(X\) there is a s.o.p. \(P\) such that \((d_\perp, P)\) induces \(V\) (namely \(P = P_V\)).

5. Induction of Systems of Quasi-Uniformities

I will now extend this result to certain systems of quasi-uniformities and show that, in particular, every finite system and every descending sequence of \(T_1\) uniformities is part of some system \((U_n)_{n \in \omega}\).

Some additional notation: Intervals of integers will be designated by \([a, b]\). A pair of letters \(xy \in F\) is a syllable of a word \(s \in F\) if and only if it occurs in \(s\) after an even number of letters. Let \(\hat{s} \in F\) be the word \(s\) after deletion of all syllables of the form \(xx\) (\(x \in X\)). The length of \(\hat{s}\) in letters is designated by
\[ \ell(s), \text{ and } s_a \text{ is the } a\text{th letter of } \tilde{s} \text{ for any position } a \in [1, \ell(s)]. \] The subword of \( \tilde{s} \) from position \( a \) to \( b \) is \( s_{a,b} \). Moreover, let \( \lambda(x,s) \) and \( \sigma(xy,s) \) denote the number of occurrences of the letter \( x \) resp. the syllable \( xy \) in \( \tilde{s} \). Finally, \((xy)^r = xy \cdots xy\) is a word consisting of \( r \) equal syllables.

The next constructions mainly rely on four lemmata. For the moment, let us fix some words \( s,t \in F \) with \( s \Rrightarrow t \), where
\[
\tilde{t} = (v_1w_1)^r_1 \cdots (v_\delta w_\delta)^r_\delta, \quad v_i \neq w_i, \quad \text{and all } r_i \text{ are even.}
\]

Then \( \tilde{s} \) can be derived from \( \tilde{t} \) by a finite number of successive deletions of pairs of identical letters which are neighbours at the time of deletion. A guiding example: for \( s = yyxyzzxyuzuz \), the deletion steps could be this: in \( \tilde{t} = xyxyzuuzuz \), first delete \( uu \), giving \( xyxyzzuzuz \), then delete \( zz \), giving \( xyxyuuzuz \) and thus \( \tilde{s} \).

We now also fix such a sequence of deletions and let \( D \subseteq [1,\ell(t)] \) be the set of positions in \( \tilde{t} \) whose corresponding letters are deleted in one of these steps (in the example: \( D = [5,8] \)). For \( a \in D \), let \( \pi(a) \in [1,\ell(t)] \) be that position in \( \tilde{t} \) such that \( t_a \) and \( t_{\pi(a)} \) build a deleted pair (in the example: \( \pi(5) = 8 \) and \( \pi(6) = 7 \)). Finally, we write \( a \bowtie b \) if and only if \( a \) and \( b - 1 \) are even numbers in \( D \) such that \( a < \pi(a) = b - 1 \) (in the example: \( 6 \bowtie 8 \)). Note that because \( t_c \) and \( t_{\pi(c)} \) must first become neighbours before they can be deleted, \( a \bowtie \cdots \bowtie b \) implies that (i) \([a,b-1] \subseteq D\), (ii) \( \pi(c) \in [a,b-1] \) for all \( c \in [a,b-1] \), and thus (iii) \( \lambda(x,t_{a,b-1}) \) is even for all \( x \in X \).

**Lemma 5.1.** Assume \( a \bowtie \cdots \bowtie b \bowtie \cdots \bowtie c \), \( t_a = t_{b-1} \), and \( t_b = t_{c-1} \). Then

(a) \( t_{a-1} = t_b \) or \( t_{b-1} = t_c \).

(b) If \( t_{a-1} \neq t_b \) then \( \lambda(t_a,t_c,\ell(t)) \) is odd.

(c) If \( t_{b-1} \neq t_c \) then \( \lambda(t_b,t_{1,a-1}) \) is odd.

**Proof.** Let \( e, f, e', f', e'', f'' \in [1,\ell(t)] \) with \( e < a \leq f < e' < b \leq f' < e'' < c \leq f'' \) such that \( t_{e,f}, t_{e',f'}, \) and \( t_{e'',f''} \) are three of the defining subwords \((v_iw_i)^r_i\) of \( \tilde{t} \). Moreover, let \( x := t_{a-1}, y := t_a = t_{b-1}, z := t_b = t_{c-1}, \) and \( w := t_c \), and assume \( x \neq z \). The situation and the parity arguments that will follow are sketched in Figure 1.

Because of \( x \neq z \), we have \( \lambda(x,t_{e',b-1}) = 0 \). Moreover, \( \lambda(x,t_{f+1,e'-1}) \) is even (since all \( r_i \) are even), and \( \lambda(x,t_{a,b-1}) \) is even because of (iii), so that also \( \lambda(x,t_{a,f}) \) is even and \( \lambda(y,t_{a,f}) \) is odd (since \(|[a,f]| \) is odd). As before, \( \lambda(y,t_{f+1,e'-1}) \) and \( \lambda(y,t_{a,b-1}) \) are even, thus \( \lambda(y,t_{e',b-1}) \) is odd. Because all \( r_i \) are even, \( \lambda(y,t_{f',e''}) \) is also odd. Again, \( \lambda(y,t_{f'+1,e''-1}) \) and \( \lambda(y,t_{b,c-1}) \) are even, hence \( \lambda(y,t_{e'',c-1}) \) is odd. In particular, \( y \in \{z,w\} \), that is, \( y = w \) (as \( yz \) is a syllable of \( \tilde{t} \)) and \( \lambda(y,t_{e'',f''}) \) is also odd. Finally, \( \lambda(y,t_{c,\ell(t)}) \) is odd because \( \lambda(y,t_{f'',\ell(t)}) \) is even. This proves (a) and (b), whereas (c) is strictly analogous to (b). \( \square \)
Lemma 5.2.

(a) Assume that $a_0 \cap b_0 \cap a_1 \cap b_1 \cdots a_k \cap b_k \cap c$ with $t_{a_0} = \cdots = t_{a_k} = y$, and $t_{b_0} = \cdots = t_{b_k} = z$. Then $t_{a_0-1} = z$ or $y = t_c$.

(b) Assume that $a \cap \cdots \cap b$ with $t_a = t_{b-1}$, and $t_{a-1} \neq t_b$. Then both $\lambda(t_a, t_{1,a-1})$ and $\lambda(t_a, t_{b,\ell(t)})$ are odd.

Proof. (a) Define $e''$, $f''$ as above. Similarly, for each $i \in [0, k]$, find positions $e_i, f_i, e'_i, f'_i \in [1, \ell(t)]$ with $e_i < a_i \leq f_i < e'_i < b_i \leq f'_i$ such that $t_{e_i, f_i}$ and $t_{e'_i, f'_i}$ are two of the defining subwords of $\tilde{t}$. Assuming $t_{a_0-1} = x \neq z$, one proves that $\lambda(y, t_{b_0, f_0})$ is odd exactly as before. Since, for $i \in [1, k]$, all of $\lambda(y, t_{b_{i-1}, a_i-1})$, $\lambda(y, t_{a_i, b_i-1})$, $\lambda(y, t_{f'_i, e'_i-1}), \lambda(y, t_{f_i, 1, e_i-1}), \lambda(y, t_{f_i+1, e_i-1}), \lambda(y, t_{e'_i, f'_i})$ are even, and since also $\lambda(y, t_{b_k, c-1})$ and $\lambda(y, t_{e''_a, c-1})$ are even, we conclude that $\lambda(y, t_{e'', c-1})$ is odd, hence $y = t_c$.

(b) Again as in the previous lemma, one proves that, for $y := t_a$, the number $\lambda(y, t_{b, f''})$ is odd, so that the first claim follows because $\lambda(y, t_{f'', \ell(t)})$ is even. The second claim is just the dual. \hfill \Box

Lemma 5.3. Assume that $s_{e-1} = xz$ is the syllable of $\tilde{s}$ that remains after all the deletions in a subword $t_{a-1,b}$ of $\tilde{t}$, with $a < b$, $t_{a-1} = x$, and $t_b = z$. Then there is $y \in X$ such that $\lambda(y, s) > 0$, $\sigma(xy, t_{a-1,b}) > 0$, and $\sigma(yz, t_{a-1,b}) > 0$.

Proof. Although $t_a$ and $t_{b-1}$ may be different, we find $k \geq 2, c_1, \ldots, c_k \in [1, \ell(t)]$, and $y_0, y_1, \ldots, y_k \in X$ such that

$$a = c_1 \cap \cdots \cap c_2 \cap \cdots \cap c_3 \cdots c_{k-1} \cap \cdots \cap c_k \leq b.$$
Proof. If also \( \sigma \) and \( \rho \) are letters, and \( Q \) for all \( j \neq j' \) (Start with \( a := c_1' \cap c_2' \cap \cdots \cap c_j' := b \) and \( y_j' := t_{c_j'} \). As long as there are indices \( j > i > 1 \) with \( y_i' = y_j' \), remove all the indices \( i + 1, \ldots, j \), so that finally all remaining \( y_i' \) are different. Since \( y_1' = t_0 \neq z = y_1' \), at least \( k \geq 2 \) of the original indices are not removed, including the index 1, and the corresponding \( c_i' \) build the required positions \( c_1, \ldots, c_k \).

Then \( k = 2 \) since otherwise Lemma 5.1 (a) would imply that either \( y_0 = y_2 \) or \( y_1 = y_3 \). With \( y_1 \) for \( y \) and \( c_2 \) for \( b \), Lemma 5.2 (b) implies that \( \lambda(y, t_{1, a-1}) \) is odd. Now, also \( \lambda(y, s_{1, c-1}) \) is odd, because \( c \in [1, a - 1] \cap D \) implies \( \pi(c) \in [1, a - 1] \) (since the letter \( x \) at position \( a - 1 \) is not deleted). In particular, \( \lambda(y, s_{1, c-1}) > 0 \). \( \square \)

**Lemma 5.4.** Assume that \( k \geq 2 \), \( c_0 \cap c_1 \cdot \cdot \cdot c_{k-1} \cap c_k \subseteq D \), and \( \pi(c_k) = \pi(c_0) - 1 \), representing a number of deletions of the form

\[
\begin{array}{cccccccc}
t_{0} & t_{1} & \cdots & \cdots & \cdots & \cdots & t_{k-1} & t_{k} \\
\end{array}
\]

Let \( t' := t_{c_0} - t_{c_1} t_{c_1} - t_{c_2} \cdot \cdot \cdot t_{c_k} - 1 t_{c_k} \) be the word consisting only of the “boundary letters”, and \( i \in [0, k] \). Then \( \sigma(t_{c_i} - 1 t_{c_i}, t') = \sigma(t_{c_i}, t_{c_i}, t') \).

Proof. Put \( c_{i-1} := c_k \). Obviously, \( t_{c_{i-1}} = t_{c_{i-1}} \) for all \( i \in [1, k] \), and \( t_{c_k} = t_{c_0} - 1 \). If also \( t_{c_i} - 1 = t_{c_i} \) for all \( i \in [0, k] \) then \( k \) must be odd (since \( t_{c_k} \neq t_{c_0} \)), and \( \sigma(t_{c_i} - 1 t_{c_i}, t') = \sigma(t_{c_i}, t_{c_i} - 1, t') = k/2 \). Otherwise, there are \( r \geq 1 \) positions \( i(1) < \cdots < i(r) \) in \([0, k]\) with \( t_{c_i(r)} - 1 = t_{c_i(r)} \). Then \( i(i + 1) - i(j) \) is even for all \( j \) (otherwise, put \( a_0 := c_{i(j)-1}, b_0 := c_{i(j)}, \ldots, c := c_{i(j+1)-1} \) and apply Lemma 5.2 (a)). In case that all \( i(j) \) are even, we have

\[
t_{c_{i-1}} \neq t_{c_k} = t_{c_0} - 1 = t_{c_1} = t_{c_i}
\]

for all odd \( i \), so that \( k \) must be odd. On the other hand, if all \( i(j) \) are odd, we have

\[
t_{c_k} = t_{c_0} - 1 \neq t_{c_0} = t_{c_i}
\]

for all even \( i \), so that again \( k \) must be odd. This shows that \( t' \) is of one of the following two forms:

\[
t' = (yxyxy)^{m_0}(yz_1z_1y)^{m_1} \cdots (yz_{r-1}z_{r-1}y)^{m_r-1}(yxyxy)^{m_r}
\]

or

\[
t' = xy(yxyxy)^{m_0}(yz_1z_1y)^{m_1} \cdots (yz_{r-1}z_{r-1}y)^{m_r-1}(yxyxy)^{m_r}yx,
\]

from which the claim follows immediately. \( \square \)

Now we are ready for the construction. Let \( p_i \) be the \( i \)th odd prime number, and \( S(A) := \{a_1 + \cdots + a_k : k \geq 1, a_i \in A\} \) for any set \( A \) of integers. In the next theorem, we need the following sets of even numbers: for any positive integer \( u \), let \( q_{uj} = \frac{2}{p_j} \prod_{i=1}^{u} p_i \) for all \( j \in [1, u], Q_u := \{q_{u1}, \ldots, q_{uu}\} \), and \( Q_{uj} := Q_u \setminus \{q_{uj}\} \). It is easy to see that then, for each \( j \in [1, u] \) and \( k \in S(Q_{uj}) \), \( k - q_{uj} \notin S(Q_{uj}) \) (since \( p_j \) divides \( k \) but not \( q_{uj} \)).
Theorem 5.5.

(a) Let $V_1, \ldots, V_u$ be a finite system of $T_1$ quasi-uniformities such that, for all $i, j \in [1, u]$, $V_j \subseteq V_j^{-1} \forall V_i$. Then there is a finest s.o.p. $P$ such that, for $j \in [1, u]$, $V_j = \cup_{q_{uj}}$.

(b) Let $V_1 \supseteq V_2 \ldots$ be a descending sequence of $T_1$ quasi-uniformities such that, for all $j$ and all $U \in V_j$, there are $V_1 \in V_1, V_2 \in V_2, \ldots$ with $V_j^{-1} \cap \bigcup_{i \neq j} V_i \subseteq U$. Then there is a finest s.o.p. $P$ such that $V_j = \cup_{q_{uj}}$ for all $j$.

Proof. For part (a), let $I := [1, u]$, while for part (b), let $I$ be the set of natural numbers. In both cases, $P$ is defined quite analogously to the proof of Theorem 4.1: its filter-base is now the system

$$B := \{\varepsilon_S : S \text{ is a sequence in } V\}$$

of lower sets $\varepsilon_S = R_\perp A_S$ of $R_\perp$, where $V := \bigcap_i V_i$, and the definition of $A_S$ changes to this: for

$$S = ((V_{11}, V_{12}, \ldots), (V_{21}, V_{22}, \ldots), \ldots),$$

$A_S$ is now the set of all words $(v_1 w_1)^{r_1} (v_2 w_2)^{r_2} \cdots (v_p w_p)^{r_p} \in F$ for which there is some $(n_1, \ldots, n_u) \in W$ and some tuple of indices $(i_1, \ldots, i_u)$ such that, for all $a \in [1, \rho]$, $w_a V_{n_a i_a} v_a$ and either $r_a = q_{u_a}$ (for the proof of (a)) or $r_a = 2^{u_a}$ (for the proof of (b)).

As before, $P$ turns out to be a s.o.p., where the only major change is the proof of $\bigcap B = \emptyset$: Let $s \in \bigcap B$, $\sigma(xz, s) > 0$, and $V = (V_{11}, V_{12}, \ldots) \in V$. Choose $\delta$ so that $V_{k+1,i} V_{k+1,i} \subseteq V_{ki}$ for all $i \in I$ and all $k$, and some $t \in A_S$ with $s R_\perp t$. Assume that $t = (v_1 w_1)^{r_1} (v_2 w_2)^{r_2} \cdots (v_p w_p)^{r_p}$. If $\sigma(xz, t) > 0$, put $y_{V'} := x$, otherwise choose some $y_{V'} \in X$ with $\lambda(y_{V'}, s) > 0$, $\sigma(xy_{V'}, t) > 0$, and $\sigma(y_{V'} z, t) > 0$, according to Lemma 5.3. Since $\ell(s)$ is finite and $V$ is filtered, there is some $y$ such that, for all $V \in V$, there is $V' \in V$ with $V' \subseteq V$ and $y_{V'} = y$, where $\subseteq$ denotes component-wise set inclusion. Consequently, $x U_{V'} y U_{V'} z$ for all $V \in V$, where $U_{V'} = \bigcup_i V_i$. This implies that $x, y, z \in \bigcap_i V_i$, and hence $x = y = z$. Since this is a contradiction to $x \neq z$, we have shown that $\delta$ is the empty word, that is, $s \in 0_R$.

Finally, let us show that $V_j = \cup_{q_{uj}}$ resp. $V_j = \cup_{q_{uj}}$ for each $j \in I$. Fix some $j \in I$ and let $V_{0j} \in V_j$. Because of the premises, the following choices can now be made. For part (a), choose for all $i \in I \setminus \{j\}$ some $V_{0i} \in V_j$ and $V_{1i} \in V_i$ such that $(V_{0i})^{-1} \cap V_{1i} \subseteq V_{0j}$. Then choose $V_{1j} \in V_j$ such that $V_{1j} \subseteq 0_{V_i}$ for all of the finitely many $i \in I \setminus \{j\}$. For part (b), choose instead some $(V_{11}, V_{12}, \ldots) \in V$ with $V_{1h} = V_{1j} \subseteq 0_{V_j}$ for all $h \leq j$ and $(V_{1j})^{-1} \cap \bigcup_{i \neq j} V_{1i} \subseteq V_{0j}$.

After that, choose the remaining components of a sequence

$$S = ((V_{11}, V_{12}, \ldots), (V_{21}, V_{22}, \ldots), \ldots)$$

in $V$ so that $V_{k+1,i} V_{k+1,i} \subseteq V_{ki}$ for all $i \in I$ and all $k$, and assume that $rd_{R_i}(x, y) \in \varepsilon_S$, that is, $s := (xz)^{r_1} R_\perp t \in A_S$ with (a) $r = q_{uj}$ resp. (b) $r = 2^j$. We have to show that $z V_{0j} x$. 

By definition of $A_S$, we have $\tilde{t} = (v_1 w_1)^r_1 (v_2 w_2)^r_2 \cdots (v_p w_p)^r_p$, and there is some corresponding tuple $(i_1, \ldots, i_p)$. Since the only letters in $\tilde{s}$ are $x$ and $z$, there are exactly $r$ occurrences of the syllable $xz$ in $\tilde{t}$ which are not deleted (because otherwise Lemma 5.3 would imply the existence of a third letter $y$ in $\tilde{s}$). All other occurrences of $xz$ in $\tilde{t}$ are deleted as part of some set of deletions of the form represented in Lemma 5.4, that is, there are $c_0, \ldots, c_k$ with properties as in Lemma 5.4 and with $t_{c_i-1} t_{c_i} = xz$ for some $i \in \{0, k\}$. Then the lemma implies that $\sigma(xz, t) = r + \sigma(xz, t) =: k$.

For (a): If $(v_a w_a)^r_a = (xz)^q_{a_i}$ for some $a \in [1, \rho]$, then $i_a = j$ and

$$(z, x) \in V_{n_a, i_a} \subseteq V_{1j} \subseteq V_{0j}.$$ 

Otherwise, we know that $k \in S(Q_w)$, that is, $\sigma(xz, t) = k - q_{uj} \in S(Q_u) \setminus S(Q_w)$, so that $(v_a w_a)^r_a = (xz)^q_{a_i}$ and $i_a = j$ for some $a \in [1, \rho]$. Also, $(v_b w_b)^r_b = (xz)^q_{b_i}$ and $i_b = i$ for some $b \in [1, \rho]$ and some $i \in I \setminus \{j\}$, so that $(z, x) \in (V_{1j})^{-1} \cap V_{ii} \subseteq V_{0j}$.

For (b) instead: If $(v_a w_a)^r_a = (xz)^{2^j}$ for some $a \in [1, \rho]$ and $i \leq j$, then $i_a = i$ and $(z, x) \in V_{n_a, i_a} \subseteq V_{ii} \subseteq V_{0j}$. Otherwise, $k$ is a multiple of $2^{j+1}$ so that $\sigma(xz, t) = k - 2^j$ is not such a multiple. Therefore, $(v_a w_a)^r_a = (xz)^{2^j}$ and $i_a \neq j$ for some $a \in [1, \rho]$. Also, $(v_b w_b)^r_b = (xz)^{2^{j+1}}$ and $i_b \neq j$ for some $b \in [1, \rho]$, so that again $(z, x) \in (V_{1i_a})^{-1} \cap V_{i_b} \subseteq V_{0j}$. \hfill \Box

Unfortunately, this proof highly depends on the fact that $M_{R,+}$ is not commutative, so that conjecture that there is also a suitable distance structure with a commutative value monoid is yet unproved.

The most familiar example for a descending sequence of uniformities is perhaps the following. Let $X := C_b[0, 1]$ be the (infinite-dimensional) vector space of bounded, continuous, and real-valued functions on the unit interval $[0, 1]$, and, for positive integers $p$, let $V_p$ be the uniformity on $X$ induced by the usual $p$-norm.

For a second example, take $u$ different primes $p_1, \ldots, p_u$ and let $V_i$ be the $p_i$-adic uniformity on the rationals. As these are transfinite uniformities with countable bases, we may use a slightly simpler construction. More precisely, a base for $V_i$ is the set of equivalence relations $U_{i, m} := \{(x, y) : p_i^m \text{ divides } \nu(|x - y|)\}$, where $m$ is a positive integer, and $\nu(z/n) := z$ whenever $z, n$ have no common divisor (that is, $\nu(q)$ is the nominator of $q$). Therefore, it suffices to use only those $\varepsilon_S$ where all tuples in $S$ are equal, that is, $V_{h+1, i} = V_{hi}$ for all $i, h$. In this case, the resulting s. o. p. $P$ has a countable base $B = \{\varepsilon_m : m \text{ a positive integer}\}$, where

$$\varepsilon_m := \bigcup_{n=0}^{\infty} \left( n \cdot \bigcup_{j \in [1, a], (x, y) \in U_{j, m}} q_{uj} d_1(x, y) \right).$$
As a concluding remark, I note that with similar methods, one can show that, for each pair of comparable $T_1$ uniformities $V_2 \subseteq V_1$, there is some symmetric distance structure $(d, P)$ such that $U_i = V_i$, which gives a complete characterization of the symmetric $T_1$ case.

References


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