Applied General Topology
© Universidad Politécnica de Valencia
Volume 3, No. 2, 2002
pp. 225–233

A curious example involving ordered compactifications

Thomas A. Richmond

Abstract. For a certain product $X \times Y$ where X is a compact, connected, totally ordered space, we find that the semilattice $K_o(X \times Y)$ of ordered compactifications of $X \times Y$ is isomorphic to a collection of Galois connections and to a collection of functions $\mathcal{F}$ which determines a quasi-uniformity on an extended set $X \cup \{\pm\infty\}$, from which the topology and order on X is easily recovered. It is well-known that each ordered compactification of an ordered space $X \times Y$ corresponds to a totally bounded quasi-uniformity on $X \times Y$ compatible with the topology and order on $X \times Y$, and thus $K_o(X \times Y)$ may be viewed as a collection of quasi-uniformities on $X \times Y$. By the results here, these quasi-uniformities on $X \times Y$ determine a quasi-uniformity on the related space $X \cup \{\pm\infty\}$.

2000 AMS Classification: 54F05, 54D35, 06F30, 54E15
Keywords: ordered topological space, ordered compactification, Galois connection, quasi-uniformity, F-poset

1. Introduction.

An *ordered space* is a triple $(X, \tau, \leq)$ where (X, τ) is a topological space and $\leq$ is a partial order on X. All ordered spaces considered here will have a *convex topology* (τ has a base of $\leq$-convex sets) and will satisfy the T_2*-ordered property* (the graph of $\leq$ is closed in $(X, \tau)^2$). An *ordered compactification* of $(X, \tau, \leq)$ is a compact T_2-ordered space $(X', \tau', \leq')$ such that (X, τ) is (homeomorphic to) a dense subset of (X', τ') and $\leq'$ extends the order $\leq$ on X. An ordered space has an ordered compactification if and only if it is *completely regular ordered*, as defined in [11]. The collection $K_o(X)$ of all ordered compactifications of a completely regular ordered space X may be ordered by taking $X' \geq X''$ if and only if there exists a continuous increasing function $f : X' \to X''$ with $f(x) = x$ for all $x \in X$. $K_o(X)$ is a complete upper semilattice with largest element $\beta_o X$, the *Stone-Čech ordered-* or *Nachbin- compactification.*

A quasi-uniformity $\mathcal{U}$ is said to be *compatible* with an ordered space $(X, \tau, \leq)$ if $\bigcap \mathcal{U}$ is the graph of the partial order $\leq$ and the topology from the uniformity $\mathcal{U} \cup \mathcal{U}^{-1}$ is τ. There is a one-to-one correspondence (via completion) between the elements of the set $\mathcal{Q}(X)$ of compatible totally bounded quasi-uniformities on $(X, \tau, \leq)$ and the ordered compactifications of $(X, \tau, \leq)$. Details of this correspondence as well as other basic information on quasi-uniformities may be found in [4]. As posets, $(K_o(X), \leq) \approx (\mathcal{Q}(X), \subseteq)$.

For a particular example $X \times Y$ below, we will find that the poset $K_o(X \times Y) \approx \mathcal{Q}(X \times Y)$ is also isomorphic to a poset of Galois connections and to a collection $\mathcal{F}$ of functions on an extended space $X \cup \{\pm\infty\}$. Furthermore, the collection $\mathcal{F}$ is shown to be an "F-poset" on $X \cup \{\pm\infty\}$, thereby determining a quasi-uniformity on $X \cup \{\pm\infty\}$ which, after a simple quotient identifying the introduced points $\pm\infty$ with the extreme points of X, gives the original topology and order on X. This gives an example of a set of quasi-uniformities $\mathcal{Q}(X \times Y)$ on one set determining a quasi-uniformity (determined by the F-poset $\mathcal{F}$) on another set $X \cup \{\pm\infty\}$. This example was announced, without proofs, in [10].

In all that follows, we assume that X and Y are totally ordered spaces, and that $X \times Y$ has the product topology and the product order $(a, b) \leq (c, d)$ if and only if $a \leq c$ and $b \leq d$. In general $\beta_o X \times \beta_o Y \leq \beta_o(X \times Y)$. In [5] it was shown that for totally ordered spaces X and Y, $\beta_o X \times \beta_o Y \neq \beta_o(X \times Y)$ if and only if $\beta_o X \setminus X$ contains a point which is the limit of a monotone sequence in X and Y contains a strictly monotone, oppositely directed sequence, or the dual condition (obtained by interchanging the roles of X and Y) holds.

In [9], the part of the semilattice $K_o(X \times Y)$ consisting of those ordered compactifications of $X \times Y$ below $\beta_o X \times \beta_o Y$ was described. In case $\beta_o X \times \beta_o Y = \beta_o(X \times Y)$, we have a description of the entire semilattice $K_o(X \times Y)$.

2. The Example via Galois Connections.

Let X be a compact, connected, totally ordered space. We will denote the least and greatest elements of X, respectively, by 0 and 1. Let $Y = [0, \omega_1) \cup \{\omega_1 + 1\}$ be the set of ordinals less than the first uncountable ordinal, together with an isolated top point $\omega_1 + 1$, and give Y the usual topology and order. From the results of [5], we have

$$\beta_o(X \times Y) = \beta_o X \times \beta_o Y = X \times [0, \omega_1] \cup \{\omega_1 + 1\}.$$

The results of [9] allow us to completely describe $K_o(X \times Y)$, and we shall do so here. The points of $X \times \{\omega_1 + 1\}$ prevent any identification of points of $\beta_o(X \times Y) \setminus (X \times Y)$, so all ordered compactifications of $X \times Y$ are topologically equivalent to $\beta_o(X \times Y)$. That is, all smaller ordered compactifications of $X \times Y$ are obtained from $\beta_o(X \times Y)$ by adding order to $\beta_o(X \times Y)$ in a way to get a closed order relation on $\beta_o(X \times Y)$ which introduces no new order on the original space $X \times Y$. The latter condition implies that any added order must be between points of the segment $X \times \{\omega_1\}$ and points of the segment $X \times \{\omega_1 + 1\}$. We may add order by making a point x of $X \times \{\omega_1\}$ greater than a point $f(x)$ of $X \times \{\omega_1 + 1\}$ (and by transitivity, x must also be greater than a decreasing segment $[\leftarrow, f(x)]$ of $X \times \{\omega_1 + 1\}$. Dually, order may be added by making a point a of $X \times \{\omega_1 + 1\}$ less than each point of an increasing segment $[g(a), \rightarrow]$ of $X \times \{\omega_1\}$. Figure 1 suggests the possible additional order.

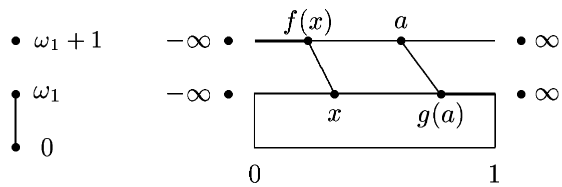

FIGURE 1. Additional order on $X \times [0, \omega_1] \cup \{\omega_1 + 1\}$.

Thus, any ordered compactification of $X \times Y$ determines a pair of functions f and g where, for $x \in X \times \{\omega_1\}$, $f(x)$ is the greatest element of $X \times \{\omega_1 + 1\}$ which is less than x, with $f(x) = -\infty$ if x is not greater than any points of $X \times \{\omega_1 + 1\}$; and for $x \in X \times \{\omega_1 + 1\}$, $g(x)$ is the least element of $X \times \{\omega_1\}$ which is greater than x, with $g(x) = \infty$ if x is not less than any elements of $X \times \{\omega_1\}$. Now f and g may be considered to be functions on $X \cup \{\pm\infty\}$, where $\pm\infty$ are topologically isolated fixed points of f and g, with $-\infty < x < \infty \ \forall x \in X$. One may show that f and g are increasing functions, f is continuous from the right, g continuous from the left, and f and g satisfy the inequality

$$f(x) < g(f(x)) \leq x \leq f(g(x)) < g(x) \quad \forall x \in X.$$

In particular, note that f is strictly below the diagonal on X; the function f can have no fixed points in X. Consider the copies of x^- and x^+ of x in $X \times \{\omega_1\}$ and $X \times \{\omega_1 + 1\}$, respectively. We already have $x^- \leq x^+$, and if x were a fixed point of f, this would imply $x^- \geq x^+$, and thus $x^- = x^+$, that is, x^- and x^+ should be identified in the ordered compactification. This is impossible, however, as $x^+ \in X \times Y$ and $x^- \in \beta_o(X \times Y) \setminus (X \times Y)$.

Now any element of $K_o(X \times Y)$ determines a pair of functions (f, g) as above, and conversely any such pair of functions determines an ordered compactification of $X \times Y$.

The definition and proposition below may be found in [3]. (A symmetric but contravariant form of the definition appears in the literature as well; we use the covariant form of [3].)

Definition 2.1. Suppose $(P, \leq)$ and $(Q, \leq')$ are partially ordered sets. If $f : P \to Q$ and $g : Q \to P$ are functions such that for all $p \in P$ and all $q \in Q$,

$$p \leq g(q) \iff f(p) \leq' q,$$

then the quadruple (P, f, g, Q) is called a *Galois connection.*

Proposition 2.2 (See [3]). *Let $(P, \leq)$ and $(Q, \leq')$ be partially ordered sets and $f : P \to Q$ and $g : Q \to P$ be functions. Then the following are equivalent:*

(1) *(P, f, g, Q) is a Galois connection.*
(2) *f is increasing, and $g(q) = \max\{z \in P : f(z) \leq' q\}$ for each $q \in Q$.*
(3) *f and g are increasing, $x \leq' f(g(x))$ for all $x \in Q$ and $g(f(x)) \leq x$ for all $x \in P$.*

With $P = Q = X \cup \{\pm\infty\}$, we see that each ordered compactification of $X \times Y$ corresponds to a Galois connection (P, f, g, Q), and, by (2) above, the second function g is in fact determined by the first function f. For our space $X \times Y$, it follows that $K_o(X \times Y)$ is isomorphic to the collection of functions

$$\begin{aligned}\mathcal{F} = \{f : X \cup \{\pm\infty\} \to X \cup \{\pm\infty\} \mid\ & f \text{ is increasing, continuous} \\ & \text{from the right, strictly below the diagonal on } X \text{, with } \pm\infty \\ & \text{as fixed points}\}.\end{aligned}$$

The order on $\mathcal{F}$ is the dual pointwise order on functions: $r \leq s$ if and only if $r(x) \geq s(x)\ \forall x$.

3. The Example via F-posets.

Given a poset $(D, \leq)$, certain families of functions on D may serve as the "lower edges" of entourages of a basis for a quasi-uniformity on D. Ralph Kummetz [7] has fruitfully investigated some such families. The definitions and results below are from [7].

Definition 3.1. If $(D, \leq)$ is a poset, a directed family $\mathcal{F}$ of functions on D is an *F-poset* on D if

(a) each $f \in \mathcal{F}$ is increasing,
(b) each $f \in \mathcal{F}$ is below the diagonal Δ_D, and
(c) $\forall f \in \mathcal{F}\ \exists g \in \mathcal{F}$ with $f \leq g \circ g$.

An F-poset $\mathcal{F}$ is *approximating* if $\sup \mathcal{F} = \Delta_D$.

Proposition 3.2. *If $\mathcal{F}$ is an F-poset on D and for $f \in \mathcal{F}$, $U_f = \{(x, y) \in D \times D : y \geq f(x)\}$, then $\{U_f : f \in \mathcal{F}\}$ is a basis for a quasi-uniformity $\mathcal{U}_{\mathcal{F}}$ on D.*

For our example $X \times Y$, we have seen that $K_o(X \times Y) \approx \mathcal{F}$ where $\mathcal{F}$ is as described at the end of the previous section. We will now show that $\mathcal{F}$ is an F-poset on $X \cup \{\pm\infty\}$.

First observe that $\mathcal{F}$ is a directed family, for $f, g \in \mathcal{F} \Rightarrow f \vee g \in \mathcal{F}$. Indeed, as it is the dual pointwise order on $\mathcal{F}$ which makes it isomorphic to $K_o(X \times Y)$, this shows that the complete $\vee$-semilattice $K_o(X \times Y)$ is a lattice. However, $K_o(X \times Y)$ fails to be a complete lattice: Let $(z_\lambda)_{\lambda \in I}$ be an increasing net in X converging to the greatest element 1, and for each $\lambda \in I$, let K^λ be the ordered compactification of $X \times Y$ determined by the function f_λ defined by

$$f_\lambda(x) = \begin{cases} -\infty & \text{if } x < 1 \\ z_\lambda & \text{if } x = 1 \\ \infty & \text{if } x = \infty \end{cases}$$

Now $\bigvee\{f_\lambda : \lambda \in I\}$ has 1 as a fixed point, so $\bigvee\{f_\lambda : \lambda \in I\} \notin \mathcal{F}$. Consequently, the subset $\{K_\lambda\}_{\lambda \in I}$ of $K_o(X \times Y)$ has no infimum.

We have already noted that each $f \in \mathcal{F}$ is *strictly* below the diagonal on X, and therefore is below the diagonal on $X \cup \{\pm\infty\}$. To prove that $\mathcal{F}$ satisfies the third defining condition of an F-poset, we will need a definition and two lemmas.

Definition 3.3. A function f on a poset D is *finitely separated from the identity* if and only if there exists a finite subset M of D such that $\forall x \in D,\ \exists m_i \in M$ with $f(x) \le m_i \le x$.

Lemma 3.4. *With $\mathcal{F}$ as defined at the end of the previous section, each $f \in \mathcal{F}$ is finitely separated from the identity.*

Proof. As $\pm\infty$ are fixed points of $f \in \mathcal{F}$, the choice of m_i such that $f(\pm\infty) \le m_i \le \pm\infty$ is determined, so it suffices to show that $f \in \mathcal{F}$ is finitely separated from the identity on X. Suppose $f \in \mathcal{F}$ is given. Let m_1 be the least element 0 of X. Suppose m_i is defined. If $\{y \in X | f(y) \ge m_i\} = \varnothing$, then $\{m_1, \dots m_i\}$ finitely separates f from the identity. Otherwise, define $m_{i+1} = \inf\{y \in X | f(y) \ge m_i\}$ Since f is continuous from the right, $f(m_{i+1}) \ge m_i$. Since f is below the diagonal, $m_{i+1} > f(m_{i+1}) \ge m_i$. We will now show that this process must terminate after finitely many steps. Assume the procedure does not terminate. Then we get a strictly increasing sequence $\{m_i\}_{i=1}^{\infty}$ in a compact totally ordered space. This sequence must have a limit $m = \inf\{$upper bounds of $\{m_i\}_{i=1}^{\infty}\}$. Now $\forall i \in \mathbb{N}$, $m_{i+1} = \inf\{x | f(x) \ge m_i\} < m$ implies $\exists x = x(i) \in X$ such that $x < m$ and $f(x) \ge m_i$. For this x, we have $m_i \le f(x) < x < m$. This last inequality yields $f(x) \le f(m)$, and thus $m_i \le f(m) < m\ \forall i$. Now $f(m)$ is an upper bound of $\{m_i\}_{i=1}^{\infty}$ smaller than m, a contradiction. □

In the setting of totally ordered spaces, f finitely separated from the identity is equivalent to the existence of a step function with finite range between f and the identity. With the m_i's as defined in Lemma 3.4,

$$s(x) = \begin{cases} \max\{m_i | m_i \le x\} & \text{if } x \in X \\ x & \text{if } x = \pm\infty \end{cases}$$

is a step function with finite range, continuous from the right with $f(x) \le s(x) \le x$. Note that the last inequality may not be strict on X, so s itself may not be an element of $\mathcal{F}$. We will alter s to get a function $r \in \mathcal{F}$ with the properties of s.

Lemma 3.5. *For each $f \in \mathcal{F}$, there exists a step function $r \in \mathcal{F}$ with a finite range R such that $r^{-1}(y)$ is not a singleton $\forall y \in R \setminus \{\infty\}$, $f(x) \le r(x) \le x\ \forall x \in X \cup \{\pm\infty\}$, and $r(x) < x\ \forall x \in X$.*

Proof. As a compact connected totally ordered space, X is *order dense*, that is, $\forall a, b \in X$ with $a < b$, there exists $c \in X$ with $a < c < b$. In particular, each $a \in X \setminus \{0\}$ is *accessible form the left* in the sense that there is a net in X of points below a which converges to a.

We will construct the required function r as a modification of s above. As before, we take $\pm\infty$ as fixed points of r and concentrate on the definition of r on X. Recall that $m_1 =$ the least element of X. Since $m_2 = \inf\{y|f(y) \geq m_1\} = \inf\{y|f(y) \neq -\infty\}$, continuity from the right implies $f(x) = -\infty$ for all $x < m_2$. Now $f(m_2) < m_2$ and order density implies that we may choose $k_2, l_2 \in X$ with

$$m_1 \leq f(m_2) < k_2 < l_2 < m_2.$$

Since f is continuous from the right and strictly below the diagonal on X, the definition of m_i implies $m_{i-1} \leq f(m_i) < m_i$. Since $f(m_2) < k_2$ and f is continuous from the right, $\exists n_2 \in X$ with $m_2 < n_2 < m_3$ and $f(n_2) < k_2$. (Otherwise, $f(n_2) \geq k_2 \ \forall n_2 \in (m_2, m_3) \Rightarrow f(m_2) \geq k_2$, a contradiction.)

$r(x)$ will be a piecewise defined function, defined inductively.

Define

$$r(x) = \begin{cases} -\infty & \text{if } x < l_2 \\ k_2 & \text{if } x \in [l_2, n_2) \end{cases}$$

Having defined $k_{i-1}, l_{i-1}, n_{i-1}$ with $k_{i-1} < l_{i-1} < m_{i-1} < n_{i-1} < m_i$, pick k_i, l_i, n_i with

$$f(m_i) \vee n_{i-1} < k_i < l_i < m_i < n_i < m_{i+1}$$

and with $f(n_i) \leq k_i$. [Since m_i is accessible from the left, such a k_i and l_i exist. If $f(n_i) \geq k_i \forall n_i \in (m_i, m_{i+1})$, then continuity of f from the right would imply $f(m_i) \geq k_i$, contrary to $f(m_i) < k_i$. Thus, such an n_i also exists.] Now define

$$r(x) = \begin{cases} m_{i-1} & \text{if } x \in [n_{i-1}, l_i) \\ k_i & \text{if } x \in [l_i, n_i) \end{cases} \qquad \text{for } i = 3, \ldots, z-1$$

and (with m_z being the last of the m_is) define

$$r(x) = \begin{cases} m_z & \text{if } x \in [n_{z-1}, 1] \\ \infty & \text{if } x = \infty. \end{cases}$$

We will verify that r satisfies the required conditions. The range of r is $R = \{-\infty, k_2, m_2, k_3, m_3, \ldots, k_{z-1}, m_z, \infty\}$, and $f^{-1}(y)$ is not a singleton for any $y \in R\{\infty\}$. Clearly r is continuous from the right. It remains to show $f(x) \leq r(x) < x$ for $x \in X$.

If $x \in (\leftarrow, l_2)$, then $f(x) = -\infty = r(x) < x$.

If $x \in [l_i, n_i)$, we have $r(x) = k_i$. Now $l_i \leq x < n_i$ implies

$$f(l_i) \leq f(x) \leq f(n_i) \leq k_i = r(x) < l_i \leq x,$$

and this shows the desired inequalities.

If $x \in [n_{i-1}, l_i)$, then $r(x) = m_{i-1} < n_{i-1} \leq x$. To see that $f(x) \leq r(x) = m_{i-1}$, suppose not. Then $f(x) > m_{i-1}$, so $x \in \{y|f(y) \geq m_{i-1}\}$ so $m_i = \inf\{y|f(y) \geq m_{i-1}\} \leq x$, contrary to $x < l_i < m_i$. □

Now we are ready to verify that $\mathcal{F}$, the collection of functions isomorphic to $K_o(X \times Y)$, satisfies the final condition required of an F-poset.

Proposition 3.6. *For any $f \in \mathcal{F}$, there exists $g \in \mathcal{F}$ with $f \leq g \circ g$, and thus $\mathcal{F}$ is an F-poset.*

Proof. Without loss of generality, we may assume f is a step function with finite range, with the inverse image of any singleton in X never being a singleton. (For any $f \in \mathcal{F}$, we have seen there exists such a step function r with $f \leq r$. Now $r \leq g \circ g$ implies $f \leq g \circ g$). Suppose the elements of the range of f, listed in increasing order, are $m_0 = -\infty, m_1, \ldots, m_z, \infty$. Define a_i $(i = 0, 1, \ldots, z)$ by $f^{-1}(m_i) = [a_i, a_{i+1})$. In particular, note that $f(a_i) = m_i$. Furthermore, we may assume f is such that $a_i < m_{i+1}$ $\forall i = 0, 1, \ldots, z$ since for each index at which this fails, we have $m_i < m_{i+1} \leq a_i < a_{i+1}$, and we may replace m_{i+1} with a value m^*_{i+1} strictly between a_i and a_{i+1} (raising the height of that step). Now $m_1 = f(a_1) < a_1$, so there exist $y_1, w_1 \in X$ with $m_1 < y_1 < w_1 < a_1$. Define

$$g(x) = \begin{cases} m_0 = -\infty & \text{if } x \in (\leftarrow, y_1) \\ m_1 & \text{if } x \in [y_1, a_1) \end{cases}$$

Clearly $g(x) < x$ on this section of the domain of g. Observe that $f(x) \leq g \circ g(x)$:

$$x \in (\leftarrow, y_1) \Rightarrow g(g(x)) = g(m_0) = m_0 = -\infty = f(x)$$

$$x \in [y_1, a_1) \Rightarrow g(g(x)) = g(m_1) = m_0 = f(x).$$

Now $m_2 = f(a_2) < a_2$, so there exist $y_2, w_2 \in X$ with

$$a_1 \vee m_2 < y_2 < w_2 < a_2.$$

Define

$$g(x) = \begin{cases} w_1 & \text{if } x \in [a_1, y_2) \\ m_2 & \text{if } x \in [y_2, a_2) \end{cases}$$

Clearly $g(x) < x$.

$$x \in [a_1, y_2) \Rightarrow g(g(x)) = g(z_1) = m_1 = f(x)$$

$$x \in [y_2, a_2) \Rightarrow g(g(x)) = g(m_2) = z_1 \geq m_1 = f(x).$$

Now suppose we have defined y_i, w_i with

$$a_{i-1} \vee m_i < y_i < w_i < a_i,$$

and have defined g for $x \in (\leftarrow, a_i)$. Suppose $i < z$. Since $m_{i+1} = f(a_{i+1}) < a_{i+1}$, $\exists y_{i+1}, w_{i+1} \in X$ with

$$m_i < a_i \vee m_{i+1} < y_{i+1} < w_{i+1} < a_{i+1}.$$

Define

$$g(x) = \begin{cases} w_i & \text{if } x \in [a_i, y_{i+1}) \\ m_{i+1} & \text{if } x \in [y_{i+1}, a_{i+1})) \end{cases}$$

As above, we may show $f(x) \leq g \circ g(x) < x$. Define

$$g(x) = \begin{cases} w_z & \text{if } x \in [a_z, 1] \\ \infty & \text{if } x = \infty. \end{cases}$$

For $x \in (a_z, 1]$, clearly $g(x) < x$, and $g(g(x)) = g(w_z) = m_z = f(x)$. With g as defined, $g \in \mathcal{F}$ and $f \leq g \circ g$. $\square$

Having shown that $\mathcal{F} \approx K_o(X \times Y)$ is an F-poset on $X \cup \{\pm\infty\}$, $\mathcal{F}$ is a basis for a quasi-uniformity $\mathcal{U}_\mathcal{F}$ on $X \cup \{\pm\infty\}$. We will now investigate the associated order $\bigcap \mathcal{U}_\mathcal{F}$ and topology $\tau(\mathcal{U}_\mathcal{F} \cup \mathcal{U}_\mathcal{F}^{-1})$ on $X \cup \{\pm\infty\}$. We note again that the topology in question is the topology from the associated uniformity. For brevity, we will denote this topology by $\tau_\mathcal{F}$.

If $\mathcal{F}$ were an approximating F-poset on $X \cup \{\pm\infty\}$, then $\bigcap \mathcal{U}_\mathcal{F}$ would consist of the diagonal of $X \cup \{\pm\infty\}$ and everything above it; that is, $\bigcap \mathcal{U}_\mathcal{F}$ would be the graph of the order on $X \cup \{\pm\infty\}$. However, $\mathcal{F}$ fails to be approximating at exactly one point, namely the smallest element 0 of X. If $a \in X \setminus \{0\}$, then a is accessible from below by a net $(x_\lambda)_{\lambda \in I}$ in X. Now for any $\lambda \in I$, define

$$f_\lambda(x) = \begin{cases} x_\lambda & \text{if } x \geq a \\ -\infty & \text{if } x < a \end{cases}$$

Now $f_\lambda \in \mathcal{F}\ \ \forall \lambda \in I$ and $\sup\{f_\lambda(a)\} = \sup\{x_\lambda\} = a$. It follows that $\sup\{f(a) : f \in \mathcal{F}\} = id(a)\ \ \forall a \in X \setminus \{0\}$. The equality holds for $a = \pm\infty$ as well. Thus, if $\sup \mathcal{F}$ is not the identity on $X \cup \{\pm\infty\}$, equality can only fail at $a = 0$. As each $f \in \mathcal{F}$ is strictly below the diagonal on X, we have $f(0) = -\infty\ \ \forall f \in \mathcal{F}$, so $\sup\{f(0) : f \in \mathcal{F}\} = -\infty \neq id(0)$. Thus, $\bigcap \mathcal{U}_\mathcal{F}$, when restricted to X, gives the graph of the order on X except at the least element 0 of X. Instead of eliminating the introduced points $\pm\infty$ by considering the restriction of $\bigcap \mathcal{U}_\mathcal{F}$ to X, if we eliminate the introduced points $\pm\infty$ by identifying $-\infty$ with 0 and identifying ∞ with 1, the natural ordered quotient (see [8]) would have the identified point $\{-\infty, 0\}$ as least element and $\{1, \infty\}$ as greatest element. Thus, the order introduced by the quasi-uniformity $\mathcal{U}_\mathcal{F}$ gives, after this ordered quotient identifying the extreme points of X with the newly introduced extreme points $-\infty$ and ∞, the original order on X.

Turning our attention to the topology $\tau_\mathcal{F}$, we will find a similar situation. We note briefly that Kummetz has shown (2.9 of [7]) that if $\mathcal{F}$ is an F-poset with each $f \in \mathcal{F}$ finitely separated from the diagonal—as our $\mathcal{F}$ is by Lemma 3.4—then $\tau_\mathcal{F}$ is totally bounded. The topology of a compact T_2 space arises from a unique uniformity consisting of the neighborhoods of the diagonal. The neighborhoods of the diagonal of the compact totally ordered space X must touch the diagonal at the maximum and minimum points, yet the functions of $\mathcal{F}$ are all strictly below the diagonal at 0 and 1. As the functions of $\mathcal{F}$ serve as the "lower edges" of the basic entourages of $\mathcal{U}_\mathcal{F}$, it follows that restriction of the topology $\tau_\mathcal{F}$ on $X \cup \{\pm\infty\}$ to X does not agree with the original topology τ on X. However, on any compact subset $[x_\lambda, y_\lambda]$ of X where $0 < x_\lambda < y_\lambda < 1$, each neighborhood V of the diagonal does contain the restriction $f|_{[x_\lambda, y_\lambda]}$ of some $f \in \mathcal{F}$. (To see this, find a finite collection $\{N_i \times N_i : i = 1, \ldots, m\}$ of open squares whose union is contained in V, and construct a step function below the diagonal and just above the bottom edges of the squares.) Thus, the restriction of $\tau_\mathcal{F}$ to any subset W of $X \setminus \{0, 1\}$ agrees with the restriction of the original topology τ to W. The problem at the endpoints 0 and 1 shows that the restriction of $\tau_\mathcal{F}$ to X is not the appropriate topology on X. However, the quotient identifying $\{-\infty, 0\}$ and $\{1, \infty\}$ gives the correct topology τ on

X. Essentially, the problem that each $f \in \mathcal{F}$ was strictly below the diagonal at 0 and 1 is solved by identifying these endpoints, respectively, with the fixed-points $-\infty$ and ∞, allowing the associated function on the quotient to touch the diagonal at the extreme points $\{-\infty, 0\}$ and $\{1, \infty\}$ of the quotient space.

For our example $X \times Y$, we have seen that $((K_o(X \times Y), \leq) \approx (\mathcal{F}, \geq) \approx (\mathcal{Q}, \subseteq)$, where $\mathcal{Q}$ is the collection of compatible totally bounded quasi-uniformities on $X \times Y$. Since $\mathcal{F}$ determined a quasi-uniformity on $X \cup \{\pm\infty\}$, we have an example of a collection $\mathcal{Q}$ of quasi-uniformities on one set determining a quasi-uniformity on another set.

Acknowledgements. The author is grateful to Ralph Kummetz and Hans-Peter Künzi for several helpful suggestions during the preparation of this paper.

References

[1] J. Blatter, *Order compactifications of totally ordered topological spaces*, J. Approx. Theory **13** (1975), 56–65.

[2] R. E. Chandler, *Hausdorff Compactifications*, Lecture Notes in Pure and Applied Mathematics **23**, (Marcel Dekker, New York, 1976).

[3] M. Erné, J. Koslowski, A. Melton and G. E. Strecker, *A primer on Galois connections*, in Papers on General Topology and Applications, Annals of the New York Academy of Sciences, **704**, S. Andima, R. Kopperman, P. R. Misra, M. E. Rudin and A. R. Todd (1993), 103–125.

[4] P. Fletcher and W. Lindgren, *Quasi-Uniform Spaces*, (Marcel Dekker, New York, 1982).

[5] D. Kent, D. Liu and T. Richmond, *On the Nachbin compactification of products of totally ordered spaces*, Int. J. Math. Math. Sci. **18** (1995), 665–676.

[6] D. Kent and T. Richmond, *Ordered compactifications of totally ordered spaces*, Int. J. Math. Math. Sci., **11** (1988), 683–694.

[7] R. Kummetz, *Continuous domains with approximating mappings and their uniformity*, Topology Proc. **24** (Summer 1999), 267–294.

[8] D. Mooney and T. Richmond, *Ordered quotients and the semilattice of ordered compactifications*, Proceedings of the Tennessee Topology Conference, P. R. Misra and M. Rajagopalan, editors, (World Scientific Inc, 1997), 141–155.

[9] D. Mooney and T. Richmond, *Ordered compactifications of products of two totally ordered spaces*, Order **16** (1999), 113–131.

[10] T. Richmond, *Ordered compactifications, Galois connections, and quasi-uniformities*, Unisa Topology Workshop Proceedings, vol 1. To appear, August 2002.

[11] L. Nachbin. *Topology and Order*, (Van Nostrand Math. Studies **4**, Princeton, N.J., 1965).

Received February 2002
Revised June 2002

Thomas A. Richmond
Department of Mathematics
Western Kentucky University
Bowling Green, KY 42101
USA

E-mail address: Tom.Richmond@wku.edu