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On the structure of completely useful topologies

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Abstract. Let **X** be an arbitrary set. Then a topology t on **X** is *completely useful* if every upper semicontinuous linear preorder on **X** can be represented by an upper semicontinuous order-preserving real-valued function. In this paper we characterize in **ZFC** (Zermelo-Fraenkel + Axiom of Choice) and **ZFC+SH** (**ZFC** + Souslin Hypothesis) completely useful topologies on **X**. This means, in the terminology of mathematical utility theory, that we clarify the topological structure of any type of semicontinuous utility representation problem.

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1. Introduction

1.1. **The problem.** Let **X** be an arbitrary set. In Herden [18] a topology t on **X** is said to be *useful* if every continuous linear (total) preorder $\precsim$ on **X** can be represented by a continuous real-valued order preserving function, i.e. has a continuous utility representation. Continuity of $\precsim$ means that the order topology $t^{\precsim}$ induced by $\precsim$ is coarser than t. Sufficient conditions for a topology t on **X** to be useful are, for instance, given by the classical Eilenberg-Debreu theorems **EDT** and **DT** (Eilenberg [11], Debreu [8, 9]). Necessary and sufficient conditions for a topology t on **X** to be useful have been presented in 1995 by Estévez and Hervés in case that t is a metrizable topology on **X** and have been generalized recently by Herden and Pallack [20] to arbitrary topologies on **X**. Using the concept of a useful topology t on **X** the Eilenberg-Debreu theorems and the theorem of Estévez and Hervés (**EHT**) can be restated as follows:

EDT: *Every connected and separable topology t on* **X** *is useful.*

DT: *Every second countable topology t on* **X** *is useful.*

EHT: *A metrizable topology t on $\mathbf{X}$ is useful if and only if t is second countable.*

In utility theory besides continuous linear preorders also semicontinuous linear preorders are of interest. In contrast to useful topologies t on $\mathbf{X}$ *completely useful* topologies t on $\mathbf{X}$, i.e. topologies t on $\mathbf{X}$ for which every upper semicontinuous linear preorder admits an upper semicontinuous utility representation, have not systematically been studied in the literature. In this paper we, thus, focus our attention on these topologies. Of course, analogous considerations also work for lower semicontinuous order-preserving real-valued functions.

The reader may recall that a linear preorder $\precsim$ on $\mathbf{X}$ is said to be upper semicontinuous if for every point $x \in \mathbf{X}$ the set $\underset{\sim}{L}(x) = \{y \in \mathbf{X} \mid y \prec x\}$ is an open subset of $\mathbf{X}$. It will be shown in Proposition 4.4 that completely useful topologies t on $\mathbf{X}$ are useful, which justifies the concept of a completely useful topology t on $\mathbf{X}$ and implies, in particular, that the semicontinuous analogue of **EHT** exists (cf. Corollary 4.5). On the other hand, useful topologies t on $\mathbf{X}$ are not necessarily completely useful. Indeed, on every sufficiently large set the cardinality of the set of useful topologies t on $\mathbf{X}$ coincides with the cardinality of the power set of the set of all completely useful topologies t on $\mathbf{X}$ (cf. Theorem 5.1). This theorem shows in a very strong way up to which degree the concept of a completely useful topology t on $\mathbf{X}$ strengthens the concept of a useful topology t on $\mathbf{X}$.

Considering the theorems **EDT** and **DT** only the semicontinuous analogue of **DT**, which has been proved for the first time by Rader [30], is known. Using the concept of a completely useful topology t on $\mathbf{X}$ Rader's theorem (**RT**) can be restated as follows (cf. also Richter [31], Isler [22] and Mehta [27]):

RT: *Every second countable topology t on $\mathbf{X}$ is completely useful.*

EDT cannot be generalized to the semicontinuous case, which shows, in addition, that useful topologies t on $\mathbf{X}$ are not necessarily completely useful (cf. Theorem 5.1). Indeed, let $\aleph_1 := \Omega$ be the first uncountable ordinal and let t be the topology on $\mathbf{X} := [0, \aleph_1[= [0, \Omega[$ that is induced by the sets $[0, \alpha]$, where α runs through all countable ordinals. Let us denote by $\overline{A}$ the topological closure of any subset A of $\mathbf{X}$. Then $\overline{\{0\}} = \overline{[0, \alpha]} = \mathbf{X}$ for every countable ordinal α. Hence, t is a separable and connected topology on $\mathbf{X}$. On the other hand, the natural order $\leq$ on $\mathbf{X}$ is an upper semicontinuous linear preorder on $\mathbf{X}$ that, obviously, has no (upper semicontinuous) utility representation (cf. Example 4.6).

Let c be the cardinality of the real line $\mathbb{R}$ and let $[0,1]$ be the closed interval of all reals that are not smaller than 0 and not greater than 1. Then a more appealing example that shows that **EDT** cannot be generalized to the semicontinuous case is given by the topological product $(\mathbf{X}, t) := ([0,1]^c, t_{prod})$. Indeed, $([0,1]^c, t_{prod})$ is a compact, connected and separable space and, thus, satisfies the assumptions of **EDT**. On the other hand, we shall show in Corollary 4.13 that in case that κ is an ordinal number and t is a completely useful topology on $\mathbf{X}$ such that $| t |> 2$ the topological product $(\mathbf{X}^{\kappa}, t_{prod})$ is completely useful if and only if κ is countable, which, in particular, implies that $([0, 1]^{c}, t_{prod})$ is not completely useful.

1.2. **The relevance for mathematical utility theory.** In mathematical utility theory one often considers compact sets. In this case it suffices to assume upper (or lower) semicontinuity of a preference relation. Indeed, then a semicontinuous utility representation obtains its minimum (respectively maximum) (cf., for instance, Bridges and Mehta [5, Remark 3.2.8]).

Semicontinuous preference relations and their representability by a semicontinuous numerical function are frequently discussed in the literature (cf., for instance, Rader [30], Jaffray [23], Richter [31], Sondermann [32], Bridges and Mehta [5], Subiza and Peris [34], Alcantud and Gutiérrez [2], Alcantud [1], Droste [10] and many others). The particular relevance of semicontinuous preference relations is mainly based upon three aspects. In the first place, it often suffices to only assume semicontinuity (cf. Alcantud [1] where spaces are considered that satisfy a weaker property than compactness). Secondly, continuity often cannot be reached without adding additional (artificial) properties (cf. the negative result of Alcantud [1]). Finally, semicontinuity often appears in a natural way and, thus, can be applied to construct continuous utility representations (cf., for instance, the Arrow-Hahn approach [3] or, more generally, the Euclidean distance approach that is thoroughly discussed in Bridges and Mehta [5] and the approach of Sondermann [32] that generalizes Neuefeind's construction of utility representations (cf. Neuefeind [29]).

In opinion of the authors the main advantage of the following considerations is the clarification of the topological structure of the general semicontinuous utility representation problem. This means that our approach is within the main stream of results that clarify the general structure of the utility representation problem (cf. Eilenberg [11], Wold [36], Birkhoff [4], Debreu [8, 9], Fleischer [14, 15], Jaffray [23, 24], Mehta [25, 26], Herden [16, 17, 18], Estévez and Hervés [12], Candeal, Hervés and Induráin [6], Herden and Pallack [20], Herden and Mehta [19] and many others). In particular, we shall widely generalize the semicontinuous analogue of the afore-quoted result of Estévez and Hervés [12]. Indeed, in the following sections we shall discuss necessary and sufficient conditions for a topology to be completely useful. These conditions can be applied in any concrete situation as follows: In case that a given consumption set is endowed with some topological structure (in practice this is nearly always the case), then one has to check if the given structure satisfies any of the assumptions of the corresponding characterization theorems that will be proved in this paper in order to guarantee the semicontinuous and, thus, also continuous, representability of an arbitrary semicontinuous, respectively continuous, preference relation on this consumption set. If none of the assumptions of the these theorems are satisfied, then a given semicontinuous preference relation may be not semicontinuously representable and one has to look for additional conditions like countably boundedness in order to guarantee semicontinuous representability (cf. Monteiro [28] or Estévez, Hervés and Verdejo [13]). Another useful condition could be convexity (cf. Candeal, Induráin and Mehta [7, Theorem 3]).

2. Notation and preliminaries

Let for the remainder of this paper an arbitrary but fixed chosen non-empty set $\mathbf{X}$ be given. Then a preorder $\precsim$ on $\mathbf{X}$ is a reflexive and transitive binary relation on $\mathbf{X}$. The induced ordered set of equivalence classes of $\mathbf{X}$ will be denoted by $\mathbf{X}_{|\sim}$. The reader may recall that an antisymmetric preorder on $\mathbf{X}$ is said to be an *order*. A function $f : (\mathbf{X}, \precsim) \rightarrow (\mathbb{R}, \leq)$ is said to be a *utility function* if it is order-preserving, i.e. isotone and $f(x) < f(y)$ whenever $x \prec y$. For every point $x \in \mathbf{X}$ we set $L(x) := \{z \in \mathbf{X} \mid z \prec x\}$ and $K(x) := \{z \in \mathbf{X} \mid x \prec z\}$. A pair (x, y) of points of $\mathbf{X}$ is said to be a *jump* if $x \prec y$ and $]x, y[= \{u \in \mathbf{X} : x \prec u \prec y\}$ is empty. If $\preceq$ is a *linear* order on $\mathbf{X}$, then the related set $(\mathbf{X}, \preceq)$ is said to be a *chain*.

The *order topology* $t^{\precsim}$ for $\mathbf{X}$, which is induced by some preorder $\precsim$ on $\mathbf{X}$, is generated by the order intervals $L(x)$ and $K(x)$, where x runs through all points of $\mathbf{X}$. If t is a topology on $\mathbf{X}$, then a preorder $\precsim$ on $\mathbf{X}$ is said to be *continuous* if $L(x)$ and $K(x)$ are open subsets of $\mathbf{X}$ for every point $x \in \mathbf{X}$, or equivalently the order topology $t^{\precsim}$ is coarser than t. A topology t on $\mathbf{X}$ is said to be *useful* if for every continuous total preorder $\precsim$ on $\mathbf{X}$ there exists a continuous utility function $u : (\mathbf{X}, \precsim, t) \rightarrow (\mathbb{R}, \leq, t_{nat})$ (t_{nat} is the natural topology on $\mathbb{R}$).

The *upper order topology* $t_u^{\precsim}$ on $\mathbf{X}$, which is induced by some preorder $\precsim$ on $\mathbf{X}$, is generated by the order intervals $L(x)$, where x runs through $\mathbf{X}$. If t is a topology on $\mathbf{X}$, then a preorder $\precsim$ on $\mathbf{X}$ is said to be *upper semicontinuous* if $L(x)$ is an open subset of $\mathbf{X}$ for every point $x \in \mathbf{X}$, or equivalently the upper order topology $t_u^{\precsim}$ is coarser than t. A topology t on $\mathbf{X}$ is said to be *completely useful* if for every upper semicontinuous linear preorder $\precsim$ on $\mathbf{X}$ there exists an upper semicontinuous utility function $u : (\mathbf{X}, \precsim, t) \rightarrow (\mathbb{R}, \leq, t_{nat})$.

3. A general characterization theorem

In this section we shall show that completely useful topologies on $\mathbf{X}$ are closely related to second countable topologies on $\mathbf{X}$.

Let, therefore, t be a second countable topology on $\mathbf{X}$. Then it is easily seen that also every subtopology of t that is linearly ordered by set inclusion (briefly: linearly ordered subtopology) must be second countable. Indeed, let t^l be a linearly ordered subtopology of t. Then we choose a countable base b of t and consider the countable subset

$$\begin{aligned} b^l :=\ & \{O \in t^l \mid \exists B \in b\,(B \subset O \wedge \forall O' \in t^l\,(O' \subsetneqq O \Rightarrow B \not\subset O'))\} \\ \cup\ & \{O \in t^l \mid \exists B \in b\,(O = \bigcup_{B \not\subset O' \in t^l} O')\} \cup \{\varnothing, \mathbf{X}\} \end{aligned}$$

of t^l in order to immediately verify that b^l is a base of t^l.

Of course, the converse does not hold. Indeed, in case that $\mathbf{X}$ is an infinite set, then there exist topologies t on $\mathbf{X}$ that are neither second nor first countable but for which every linearly ordered subtopology is second countable. Let, for instance, ϕ be any function from $\mathbf{X}$ onto the set $\mathbb{N}$ of natural numbers. Then each topology on $\mathbb{N}$ that is not second countable induces with respect to ϕ a topology on $\mathbf{X}$ that is neither second nor first countable but for which every linearly ordered subtopology is second countable.

We soon shall see that a topology t on $\mathbf{X}$ which has the property that all its linearly ordered subtopologies are second countable is completely useful, which generalizes Rader's theorem. Therefore, we choose some fixed given topology t on $\mathbf{X}$ and consider the family $\mathcal{O}$ of all sets $\mathbf{O}$ of open subsets of $\mathbf{X}$ which are linearly ordered by set inclusion.

Let some set $\mathbf{O} \in \mathcal{O}$ be arbitrarily chosen. A set $O \in \mathbf{O}$ is said to be *isolated* if $\bigcup_{\mathbf{O} \ni O'' \subsetneqq O} O'' \subsetneqq O \subsetneqq \bigcap_{O \subsetneqq O' \in \mathbf{O}} O'$. In some sense the following lemma is fundamental for our considerations.

Lemma 3.1. *In order for t to be completely useful it is necessary that every set $\mathbf{O} \in \mathcal{O}$ only contains countably many isolated sets.*

Proof. Let $\mathbf{O} \in \mathcal{O}$ be arbitrarily chosen. Then we set $\mathbf{O}^+ := \mathbf{O} \cup \{\mathbf{X}\}$ and define a linear preorder $\precsim_{\mathbf{O}}$ on $\mathbf{X}$ by setting for every pair of points $x, z \in \mathbf{X}$

$$x \precsim_{\mathbf{O}} z \Leftrightarrow \forall O \in \mathbf{O}^+ \, (z \in O \Rightarrow x \in O).$$

Since for every point $x \in \mathbf{X}$ the equality $L(x) := \{y \in \mathbf{X} \mid y \prec_{\mathbf{O}} x\} = \bigcup\{O \in \mathbf{O}^+ \mid x \notin O\}$ holds it follows that $\precsim_{\mathbf{O}}$ is upper semicontinuous. In order to, thus, prove the lemma it is because of the definition of $\mathbf{O}^+$ sufficient to show that there exists a bijective correspondence between the set $\mathbf{J}$ of jumps of $(\mathbf{X}_{|\sim}, \precsim_{\mathbf{O}|\sim})$ and the set $\mathbf{I}$ of isolated sets of $\mathbf{O}^+$. Indeed, since $\precsim_{\mathbf{O}}$ is representable $(\mathbf{X}_{|\sim}, \precsim_{\mathbf{O}|\sim})$ only has countably many jumps. Let $([x],[y]) \in \mathbf{J}$ be arbitrarily chosen. Then there exists some set $O_{]x,y[} \in \mathbf{O} \subset \mathbf{O}^+$ such that $x \in O_{]x,y[}$ and $y \notin O_{]x,y[}$. Let us abbreviate this property of $O_{]x,y[}$ by (*). Since $([x],[y])$ is a jump of $(\mathbf{X}_{|\sim}, \precsim_{\mathbf{O}|\sim})$ there exists no other set $O \in \mathbf{O}^+$ which also satisfies property (*). Therefore, we may set $\phi([x],[y]) := O_{]x,y[}$. In order to show that ϕ is a function from $\mathbf{J}$ into $\mathbf{I}$ we must verify that $O_{]x,y[} \in \mathbf{I}$. But this is easily to be seen since $O_{]x,y[}$ is the only set which satisfies property (*). Hence, no set $\mathbf{O}^+ \supset \mathbf{O} \ni O' \subsetneqq O_{]x,y[}$ can contain x and every set $O_{]x,y[} \subsetneqq O'' \in \mathbf{O}^+$ must contain y. Now we consider, on the other hand, an arbitrary set $O \in \mathbf{I}$. Then we may choose some point $x \in O \setminus \bigcup_{\mathbf{O} \ni O'' \subsetneqq O} O''$, and some point $y \in \bigcap_{O \subsetneqq O' \in \mathbf{O}^+} O' \setminus O$. Because of the definition of $\precsim_{\mathbf{O}}$ the pair $([x],[y])$ belongs to $\mathbf{J}$. In addition, since $[x] = O \setminus \bigcup_{\mathbf{O} \ni O'' \subsetneq O} O''$ and $[y] = \bigcap_{O \subsetneq O' \in \mathbf{O}^+} O' \setminus O$ the jump $([x],[y])$ is uniquely determined by O. Thus, we may set $\psi([x],[y]) := O$. Clearly, $\psi \circ \phi = id_{\mathbf{J}}$ and $\phi \circ \psi = id_{\mathbf{I}}$. So the proof is complete. □

For the remainder of this paper a topology t on $\mathbf{X}$ which satisfies the necessary condition of Lemma 3.1 is said to be *countably isolated*.

In order to prove the main result of this section we need some more notation:

Let $\mathbf{O} \in \mathcal{O}$ be arbitrarily chosen. Then a set $O \in \mathbf{O}$ is said to be *right isolated*, respectively *left isolated*, if $O \subsetneq \bigcap_{O \subsetneq O' \in \mathbf{O}} O'$, respectively $\bigcup_{\mathbf{O} \ni O'' \subsetneq O} O'' \subsetneq O$.

$\mathbf{O}$ is said to be *semi-separable* if there exists a countable subset $\mathbf{O}'$ of $\mathbf{O}$ such that for every right isolated set $O \in \mathbf{O}$ and every set $O'' \in \mathbf{O}$ such that $O'' \subsetneq O$ there exists some set $O' \in \mathbf{O}'$ such that $O'' \subset O' \subset O$.

The following theorem is a first characterization of completely useful topologies t on $\mathbf{X}$. It clarifies, in particular, the relations between completely useful topologies t on $\mathbf{X}$ and topologies t on $\mathbf{X}$ for which every linearly ordered subtopology t^l is second countable (cf. Corollary 3.4).

Theorem 3.2. *Let t be an arbitrary topology on $\mathbf{X}$. Then the following assertions are equivalent:*

(i) *t is completely useful.*
(ii) *Every linearly ordered subtopology t^l of t is semi-separable.*
(iii) *t is countably isolated and every linearly ordered subtopology t^l of t that only contains countably many left isolated sets is second countable.*
(iv) *Every linearly ordered subtopology t^l of t that has a base b that only contains countably many sets that are not right isolated is second countable.*

Proof. (i) $\Rightarrow$ (ii): Let t^l be a linearly ordered subtopology of t. As in the proof of Lemma 3.1 we consider the upper semicontinuous linear preorder $\precsim_l$ on $\mathbf{X}$ that is induced by t^l. Let $t^{\mathbf{L}}$ be the linearly ordered subtopology of t^l that is induced by the family $\mathbf{L} := \{L(x)\}_{x \in \mathbf{X}} = \{\{y \in \mathbf{X} \mid y \prec_l x\}\}_{x \in \mathbf{X}}$, i.e. t^L is the upper order topology on $\mathbf{X}$ with respect to $\precsim_l$. Since t is completely useful we may conclude with help of Proposition 1.6.11(v) of Bridges and Mehta [5] that the order topology $t^{\precsim_l}$ on $\mathbf{X}$ that is induced by $\precsim_l$ and, thus, $t^{\mathbf{L}}$ are second countable. Let, therefore, b be a countable base of $t^{\mathbf{L}}$. Then we choose some right isolated set $O \in t^l$ and some set $O'' \in t^l$ such that $O'' \subsetneq O$. Since O is right isolated it follows that $(\bigcap_{O \subsetneq O^+ \in t^l} O^+) \setminus O \neq \varnothing$ and that $O = L(x)$ for every point $x \in (\bigcap_{O \subsetneq O^+ \in t^l} O^+) \setminus O$. This means, in particular, that $O = \bigcup_{b \ni O^* \subset O} O^*$.

Hence, the linearity of t^l implies that there must exist some set $O' \in b$ such that $O'' \subset O' \subset O$, which proves assertion (ii).

(ii) $\Rightarrow$ (i): Let $\precsim$ be some upper semicontinuous total preorder on $\mathbf{X}$. Then we consider the linearly ordered subtopology $t^{\mathbf{L}}$ of t that is induced by the family $\mathbf{L} := \{L(x)\}_{x \in \mathbf{X}}$. The structure of $\mathbf{L}$ implies that the isolated sets of $t^{\mathbf{L}}$ are sets $L(y)$ for which there exists a unique set $L(x) \subsetneq L(y)$ such that for no set $L(z)$ the strict inclusions $L(x) \subsetneq L(z) \subsetneq L(y)$ hold. Hence, assertion (ii) implies that $t^{\mathbf{L}}$ only contains countably many isolated sets. This means because of the proof of Lemma 3.1 that $(\mathbf{X}_{|\sim}, \precsim_{|\sim})$ only has countably many jumps. In addition, assertion (ii) immediately implies that the order topology $t^{\precsim}$ which is induced by $\precsim$ is separable. Because of Proposition 1.6.11(iv) of Bridges and Mehta [5] it, thus, follows that $\precsim$ is representable. Now the famous Representation Lemma (**RL**) of Debreu [8, 9], which states that a continuous linear preorder $\precsim$ has a continuous utility representation if and only if $\precsim$ has a utility representation, implies that $\precsim$ has a utility representation that is continuous with respect to $t^{\precsim}$. Hence, we may conclude, in particular, that $\precsim$ has an upper semicontinuous utility representation, and assertion (i) follows.

(i) $\wedge$ (ii) $\Rightarrow$ (iii): Let t^l be a linearly ordered subtopology of t. Because of Lemma 3.1 it suffices to consider the case that t^l only contains countably many left isolated sets. We must verify that t^l is second countable. As in the proof of the implication "(i) $\Rightarrow$ (ii)" we, therefore, consider the upper semicontinuous linear preorder $\precsim_l$ on $\mathbf{X}$ that is induced by t^l. Assertion (ii) guarantees the existence of a countable subset $\mathbf{O}'$ of t^l such that for every right isolated set $O \in t^l$ and every set $O'' \in t^l$ such that $O'' \subsetneq O$ there exists some set $O' \in \mathbf{O}'$ such that $O'' \subset O' \subset O$. Let b be the union of $\mathbf{O}'$ with the set of all left isolated sets of t^l. Then b is countable and it suffices to show that b is a base of t^l. In order to verify that b is a base of t^l it remains to prove that every set $O \in t^l$ that is not left isolated is the union of sets $O' \in \mathbf{O}'$. Let, therefore, some set $O \in t^l$ that is not left isolated be arbitrarily chosen. We assume, in contrast, that $\bigcup_{\mathbf{O}' \ni O' \subset O} O' \subsetneq O$. Since O is not left isolated there exist sets $O^+, O^{++} \in t^l$ such that $\bigcup_{\mathbf{O}' \ni O' \subset O} O' \subsetneq O^+ \subsetneq O^{++} \subset O$. Now the definition of $\precsim_l$ implies the existence of some point $x \in O \setminus O^{++}$ such that $O^{++} \subset L(x) \subsetneq O$. Since $L(x)$ is right isolated there exists some set $O^* \in \mathbf{O}'$ such that $O^+ \subset O^* \subset L(x)$, which means that $O^+ \subset \bigcup_{\mathbf{O}' \ni O' \subset O} O'$. This contradiction implies the validity of assertion (iii).

(iii) $\Rightarrow$ (iv): Let $b \in \mathcal{O}$ be a base of some linearly ordered subtopology t^l of t such that only countably many sets $O \in b$ are not right isolated. In case that b contains uncountably many left isolated sets it, thus, follows that b contains uncountably many isolated sets. Hence, assertion (iii) implies that b only contains countably many left isolated sets. With help of the definition of a left isolated set we, therefore, may conclude that also t^l only contains countably many left isolated sets. Assertion (iii), thus, implies that t^l is second countable.

(iv) $\Rightarrow$ (i): Let $\precsim$ be some upper semicontinuous linear preorder on $\mathbf{X}$. Then we consider the linearly ordered subtopology $t^{\mathbf{L}}$ of t that is induced by the family $\mathbf{L} := \{L(x)\}_{x \in \mathbf{X}}$. Since $\mathbf{L} \cup \{\varnothing, \mathbf{X}\}$ is a base of t^l and since every set $L(x) \in \mathbf{L}$ is right isolated assertion (iv) implies that $t^{\mathbf{L}}$ is second countable. Now the arguments that have been applied in the last part of the proof of the implication "(ii) $\Rightarrow$ (i)" imply that $\precsim$ has an upper semicontinuous utility representation, which proves assertion (i).

□

Remark 3.3. Theorem 3.2 implies, in particular, that a topology t on $\mathbf{X}$ for which every linearly ordered subtopology t^l is second countable is completely useful. Hence, **RT** is a consequence of Theorem 3.2.

In addition to Remark 3.3 the following observation is a consequence of Lemma 3.1, the equivalence of the assertions (i) and (iii) of Theorem 3.2 and some straightforward additional consideration which, for the sake of brevity, is left to the reader.

Corollary 3.4. *Let t be an arbitrary topology on $\mathbf{X}$. Then the following assertions are equivalent:*

(i) *The condition that t is completely useful and the conditon that every linearly ordered subtopology of t is second countable are equivalent conditions for t.*
(ii) *Every linearly ordered subtopology t^l of t contains only countably many left isolated sets.*

Example 3.5. Example 3.5 Because of Corollary 3.4 the following example is characteristic for a topology t on $\mathbf{X}$ that is completely useful but contains linearly ordered subtopologies that are not second countable:

Let $\mathbf{X}$ be the real line. Then the topology t on $\mathbf{X}$ which is induced by the sets $]-\infty, r]$ where r runs through all reals satisfies the equivalent assertions of Theorem 3.2 and contains uncountably many left isolated sets. Hence, t is completely useful but contains linearly ordered subtopologies that are not second countable.

In case that $\mathbf{X}$ is the real interval $[0,1]$ the afore-described topology can be extended to a compact (Hausdorff-)topology t on $\mathbf{X}$ that is completely useful but contains linearly ordered subtopologies t^l that are not second countable. Indeed, one only has to consider the topology t on $\mathbf{X}$ that is generated by the sets $[0,r]$ and $]s,1]$ where r runs through all reals that are greater than 0 but not greater that 1 and s runs through all reals that are smaller than 1 but not smaller than 0.

4. The structure of completely useful topologies ZFC+SH and ZFC

A topology t on $\mathbf{X}$ is said to be *short* if there exists no uncountable ordinal α that can be order-embedded into $(t, \subsetneq)$ or $(t, \supsetneq)$.

The following lemma is an immediate consequence of Lemma 3.1.

Lemma 4.1. *In order for t to be completely useful it is necessary that t is short.*

The reader may recall that a topology t on $\mathbf{X}$ is said to be a *hereditarily Lindelöf-topology* if for every subset A of $\mathbf{X}$ and every open covering $\mathbf{C}$ of A there exists a countable covering $\mathbf{C}' \subset \mathbf{C}$ of A. In case that for every open covering $\mathbf{C}$ of $\mathbf{X}$ there exists a countable covering $\mathbf{C}' \subset \mathbf{C}$ of $\mathbf{X}$ the topology t on $\mathbf{X}$ is said to be a *Lindelöf-topology.* In addition, the reader may recall that t is said to be *hereditarily separable* if every subspace $(A, t_{|A})$ of $(\mathbf{X}, t)$ is separable. With help of this notation short topologies easily can be characterized.

Proposition 4.2. *Let t be a topology on $\mathbf{X}$. Then the following assertions are equivalent:*

(i) *t is short.*
(ii) *t is hereditarily Lindelöf and hereditarily separable.*

Proof. (i) $\Rightarrow$ (ii): At first we show that t is a hereditarily Lindelöf-topology. Let, therefore, some subset A of $\mathbf{X}$ and some open covering $\mathbf{C}$ of A be arbitrarily chosen. Assume that there exists no countable covering $\mathbf{C}' \subset \mathbf{C}$ of A. Then we consider some well-ordering $\{O_\alpha\}_{\alpha<|\mathbf{C}|}$ of $\mathbf{C}$ and define by transfinite induction a well ordered chain of open subsets of $\mathbf{X}$ that is increasing by set inclusion.

$\alpha = 0$: We set $O'_0 := O_0$.

$\alpha > 0$ is not a limit ordinal: In this case we set $O'_\alpha := O'_{\alpha-1} \cup O_\alpha$.

$\alpha > 0$ is a limit ordinal: Now we set $O'_\alpha := \bigcup_{\beta<\alpha} O'_\beta$.

Finally we set $\mathbf{O} := \{O'_\alpha\}_{\alpha<|\mathbf{C}|}$. Since there exists no countable subset $\mathbf{C}'$ of $\mathbf{C}$ which also covers A we may assume without loss of generality that $O'_\beta \subsetneq O'_\gamma$ for all ordinals $\beta < \gamma <| \mathbf{C} |$, which contradicts the shortness of t.

It remains to prove that t is hereditarily separable. Let, therefore, A be some arbitrarily chosen subset of $\mathbf{X}$. We have to show that A contains some countable subset B such that $A \subset \overline{B}$. Without loss of generality we may assume that $A \neq \varnothing$. Then we define by transfinite induction a well ordered chain of open subsets of $\mathbf{X}$ that is decreasing by set inclusion.

$\alpha = 0$: We set $B_0 := \{a_0\}$ for some arbitrarily chosen point $a_0 \in A$.

$\alpha > 0$ is not a limit ordinal: In this case we set $B_\alpha := B_{\alpha-1}$ if $A \setminus \overline{B}_{\alpha-1} = \varnothing$ and $B_\alpha := B_{\alpha-1} \cup \{a_\alpha\}$ for some arbitrarily chosen point $a_\alpha \in A \setminus \overline{B}_{\alpha-1}$ if $A \setminus \overline{B}_{\alpha-1} \neq \varnothing$.

$\alpha > 0$ is a limit ordinal: Now we set $B_\alpha := \bigcup_{\beta<\alpha} B_\beta$.

Let us now assume, in contrast, that there exists no countable subset B of A

such that $A \subset \overline{B}$. Then there exists, in particular, no countable ordinal α such that $A \setminus \overline{B}_\alpha = \varnothing$. This means that the well ordered chain $(\mathbf{O}, \underset{\neq}{\supset}) := (\{\mathbf{X} \setminus \overline{B}_\alpha \mid \alpha \textit{ is an ordinal}\}, \underset{\neq}{\supset})$ of open subsets of $\mathbf{X}$ is uncountable, a contradiction. Hence, t is hereditarily separable.

(ii) $\Rightarrow$ (i): The proof that no uncountable ordinal α can be order-embedded into $(t, \underset{\neq}{\subset})$ is based upon the assumption that t is a hereditarily Lindelöf-topology. The proof is similar to the proof that no uncountable ordinal α can be order-embedded into $(t, \underset{\neq}{\supset})$. For the sake of brevity we, thus, concentrate on the proof that no uncountable ordinal α can be order-embedded into $(t, \underset{\neq}{\supset})$. Let us assume, in contrast, that there exists an uncountable ordinal α that can be order-embedded into $(t, \underset{\neq}{\supset})$. We may assume without loss of generality that $cf(\alpha) > \omega$. As usual $cf(\alpha)$ denotes the cofinality of α and ω is the first infinite ordinal (cardinal). Then there exists a well ordered chain $B_0 \underset{\neq}{\subset} B_1 \underset{\neq}{\subset} ... \underset{\neq}{\subset} B_\beta \underset{\neq}{\subset} ... \underset{\neq}{\subset} ...$ of length (order-type) α of closed subsets B_β of $\mathbf{X}$. We, thus, set $B := \bigcup_{\beta < \alpha} B_\beta$. Since t is hereditarily separable there exists some countable subset C of B such that $B \subset \overline{C}$. The inequality $cf(\alpha) > \omega$ implies because of the countability of C that there exists some ordinal $\gamma < \alpha$ such that $C \subset B_\gamma$. Hence, $\overline{C} \subset B_\gamma \underset{\neq}{\subset} B$. This contradiction completes the proof of assertion (i). □

At first we note that Proposition 4.2 allows us to generalize assertion (ii) of Theorem 3.2.

Corollary 4.3. *Let t be an arbitrary topology on $\mathbf{X}$. Then the following assertions are equivalent:*

(i) *t is completely useful.*
(ii) *Every set $\mathbf{O} \in \mathcal{O}$ is semi-separable.*

Proof. Because of assertion (ii) of Theorem 3.2 it suffices to show that the implication "(i) $\Rightarrow$ (ii)" holds. Let, therefore, some set $\mathbf{O} \in \mathcal{O}$ be arbitrarily chosen. Then we consider the upper semicontinuous linear preorder $\precsim_{\mathbf{O}}$ on $\mathbf{X}$ that is induced by $\mathbf{O}$ and the corresponding subtopology $t^{\mathbf{L}}$ of t that is generated by the family $\{L(x)\}_{x \in \mathbf{X}}$. Because of the proof of the implication "(i) $\Rightarrow$ (ii)" of Theorem 3.2 there exists a countable subset b of $\mathbf{L}$ that is a base of $t^{\mathbf{L}}$. In addition, Proposition 4.2 implies that every set $L(x) \in b$ is the union of countably many sets $O \in \mathbf{O}$. Let, thus, for every set $L(x) \in b$ some countable subset $\mathbf{O}(x) \subset \mathbf{O}$ such that $L(x) = \bigcup_{O \in \mathbf{O}(x)} O$ be chosen. Then we set $\mathbf{O}' := \bigcup_{L(x) \in b} \mathbf{O}(x)$. Obviously, $\mathbf{O}'$ is a countable set. Furthermore, one verifies immediately that for any pair of sets $O'', O \in \mathbf{O}$ such that $O'' \underset{\neq}{\subset} O$ and O is right isolated there exists some set $O' \in \mathbf{O}'$ such that $O'' \subset O' \subset O$, which was to be shown. □

A comparison of assertion (ii) of Proposition 5.1 in Herden and Pallack [20] with assertion (ii) of Corollary 4.3 immediately implies the following proposition that, in particular, shows that **DT** is a consequence of **RT**. In addition, it guarantees that any characterization of completely useful topologies t on $\mathbf{X}$ also provides sufficient conditions for a topology t on $\mathbf{X}$ to be useful.

Proposition 4.4. *Every completely useful topology on* $\mathbf{X}$ *is useful.*

Since for a metrizable topology t on $\mathbf{X}$ the concepts of a Lindelöf-topology, separability and second countability are equivalent the following corollary is an immediate consequence of Proposition 4.2 and **RT**. Of course, the corollary also is a consequence of Proposition 4.4 and **EHT**.

Corollary 4.5. *Let t be a metrizable topology on* $\mathbf{X}$. *Then the following assertions are equivalent:*

(i) *t is completely useful.*
(ii) *t is second countable.*

Example 4.6. Let $1 < \alpha$ be an ordinal number. Then the topology t^α on α that is induced by the sets $[0, \beta[$ $(\beta < \alpha)$ is always a hereditarily separable topology on α. On the other hand, it is a (hereditarily) Lindelöf-topology on α if and only if α is countable. Conversely, the topology t_α on α that is induced by the sets $]\beta, \alpha[$ $(\beta < \alpha)$ is always a hereditarily Lindelöf-topology on α, but, on the other hand, a (hereditarily) separable topology on α if and only if α is countable. Let, furthermore, $\leq_L$ be the lexicographic (linear) order on the plane $\mathbb{R}^2$. Then the topology $t^{\mathbf{L}}$ on $\mathbb{R}^2$ that is induced by the family $\{L(r,s)\}_{(r,s)\in\mathbb{R}^2}$ is a short topology on $\mathbb{R}^2$ that is not completely useful.

In order to proceed we now return to the concept of a countably isolated topology t on $\mathbf{X}$.

We need the following notation:

Let $\mathbf{O} \in \mathcal{O}$ be some arbitrarily chosen set. Then a *gap* of $\mathbf{O}$ is a pair (O, B) of subsets of $\mathbf{X}$ that satisfies the following conditions:

G1: $O = \bigcup_{\mathbf{O} \ni O' \subset O} O'$ and $B = \bigcap_{O \subsetneqq O'' \in \mathbf{O}} O''$.

G2: There exists an open set $O^+ \in t \setminus \mathbf{O}$ such that $O \subsetneqq O^+ \subsetneqq B$.

$\mathbf{O} \in \mathcal{O}$ is *gap free* if it has no gaps.

The reader may compare our concept of a gap with the concept of a *Debreu gap* (Debreu [8, 9]). Indeed, both concepts are closely related.

t is said to be *thin* if there exists no uncountable chain $(Z, <)$ such that the lexicographic product $(Z \times \{0,1\}, <_L)$ can be order embedded into $(t, \subsetneqq)$.

t is said to be *small* if for every uncountable chain $(Z, <_L)$ and every order-embedding $\phi : (Z \times \{0,1\}, <_L) \to (t, \subsetneqq)$ there exists some point $z \in Z$ such that $\phi(z,1) \setminus \overline{\phi(z,0)} \neq \varnothing$.

An open subset O of $\mathbf{X}$ is said to be *thin bounded* if it satisfies the following conditions:

TB1: There exists no uncountable chain $(Z,<)$ for which there exists some order embedding $\phi : (Z \times \{0,1\}, <_L) \to (t, \subsetneq)$ such that $O \subset \phi(z,0) \subset \phi(z,1)$ and $\overline{O} = \overline{\phi(z,0)} = \overline{\phi(z,1)}$ for every point $z \in Z$.

TB2: For each set $\mathbf{O} \in \mathcal{O}$ such that $O \subset O'$ and $\overline{O} \subsetneq \overline{O'}$ for all set $O' \in \mathbf{O}$ there exists a set $O'' \in \mathbf{O}$ such that for every uncountable chain $(Z,<)$ for which there exists some order-embedding $\phi : (Z \times \{0,1\}, <_L) \to ([O, O''], \subsetneq)$ there exists some point $z \in Z$ such that $\phi(z,1) \setminus \overline{\phi(z,0)} \neq \varnothing$.

In addition, the reader may recall that t satisfies **ccc** (countable chain condition) if every family of pairwise disjoint open subsets of $\mathbf{X}$ is countable. Of course, any separable topology t on $\mathbf{X}$ satisfies **ccc**.

Now the following lemma holds.

Lemma 4.7. *Let t be a topology on $\mathbf{X}$. Then the following assertions are equivalent:*

(i) *t is countably isolated.*
(ii) *Every set $\mathbf{O} \in \mathcal{O}$ has at most countably many gaps.*
(iii) *t is thin.*
(iv) *t is small and satisfies* **ccc**.
(v) *t is short and every non-empty open subset O of $\mathbf{X}$ is thin bounded.*

Proof. (i) $\Rightarrow$ (ii): Let, in contrast, $\mathbf{O} \in \mathcal{O}$ be some set that has uncountably many gaps. Then we interpose in every gap (O, B) of $\mathbf{O}$ some open subset O' of $\mathbf{X}$ such that $O \subsetneq O' \subsetneq B$. It follows that the union $\mathbf{O}'$ of $\mathbf{O}$ with these sets O' contains uncountably many isolated sets. This contradiction proves assertion (ii).

(ii) $\Rightarrow$ (i): Let $\mathbf{O} \in \mathcal{O}$ be arbitrarily chosen. Then $\mathbf{O}' := \mathbf{O} \setminus \{O \in \mathbf{O} \mid O \textit{ is an isolated set}\}$ has at most countably many gaps. Hence, $\{O \in \mathbf{O} \mid O \textit{ is an isolated set}\}$ is at most countable and assertion (i) follows.

(i) $\Rightarrow$ (iii): Let us assume, in contrast, that t is not thin. Then $\mathcal{O}$ contains some set $\mathbf{O}$ such that the induced upper semicontinuous linear preorder $\precsim_{\mathbf{O}}$ on $\mathbf{X}$ (cf. the proof of Lemma 3.1) has uncountably many jumps. Because of the proof of Lemma 3.1 the topology t on $\mathbf{X}$, thus, cannot be countably isolated. This contradiction proves assertion (i).

(iii) $\Rightarrow$ (i): In order to verify this implication we assume, in contrast, that t is not countably isolated. Then there exists some set $\mathbf{O} \in \mathcal{O}$ that contains uncountably many isolated sets. Hence, $(Z,<) := (\{ \bigcup_{\mathbf{O} \ni O' \subsetneq O} O' \mid O \textit{ is an isolated set of } \mathbf{O}\}, \subsetneq)$ is an uncountable chain such that $(Z \times \{0,1\}, <_L)$ can be order-embedded into $(t, \subsetneq)$, a contradiction. Indeed, for every point $z = \bigcup_{\mathbf{O} \ni O' \subsetneq O} O' \in Z$ the pair $(z, 0)$ corresponds to $\bigcup_{\mathbf{O} \ni O' \subsetneq O} O'$ and the pair $(z,1)$ corresponds to O. This indirect argument implies the validity of assertion (i).

(i) $\wedge$ (iii) $\Rightarrow$ (iv): This implication follows immediately with help of Lemma 4.1, Proposition 4.2 and the definition of a small topology t on $\mathbf{X}$.

(iv) $\Rightarrow$ (i): Now we assume, in contrast, that there exists some set $\mathbf{O} \in \mathcal{O}$ that contains uncountably many isolated sets. Let $O_0 := O'_0 := \varnothing$ and $\mathbf{O}'_0 := \{O'_0\}$. Then we choose the family $\mathbf{O_I}$ of all isolated sets of $\mathbf{O}$. Assertion (iv) implies the existence of some set $O_1 \in \mathbf{O_I}$ such that $O_1 \setminus \overline{\bigcup_{\mathbf{O} \ni O' \subsetneq O_1} O'} \neq \varnothing$.

Therefore, we set $O'_1 := O_1 \setminus \overline{\bigcup_{\mathbf{O} \ni O' \subsetneq O_1} O'}$ and consider the set $\mathbf{O}'_1 := \{O'_0, O'_1\}$.

Let Ω be the first uncountable ordinal (cardinal). We proceed by transfinite induction on all countable ordinals α, i.e. all ordinals $\alpha < \Omega$.

$1 < \alpha < \Omega$ is not a limit ordinal: In this situation the family of all sets $O \in \mathbf{O_I}$ such that $O_\beta \subsetneq O$ for every ordinal $0 \leq \beta \leq \alpha - 1$ is uncountable or there exist ordinals $0 \leq \beta \neq \gamma \leq \alpha - 1$ such that the family of all sets $O \in \mathbf{O_I}$ such that $O_\beta \subsetneq O \subsetneq O_\gamma$ is uncountable. In both situations there exists a set $O_\alpha \in \mathbf{O_I}$ that is different from all sets O_τ, where τ runs through all ordinals that are strictly smaller that α, such that $O_\alpha \setminus \overline{\bigcup_{\mathbf{O} \ni O' \subsetneq O_\alpha} O'} \neq \varnothing$. Hence, we set

$O'_\alpha := O_\alpha \setminus \overline{\bigcup_{\mathbf{O} \ni O' \subsetneq O_\alpha} O'}$ and consider the set $\mathbf{O}'_\alpha := \mathbf{O}'_{\alpha-1} \cup \{O'_\alpha\}$.

$1 < \alpha < \Omega$ is a limit ordinal: Now we set $\mathbf{O}'_\alpha := \{O'_\beta \mid \beta < \alpha\}$.

Finally we consider the set $\mathbf{O}' := \bigcup_{\alpha < \Omega} \mathbf{O}'_\alpha$. It follows that $\mathbf{O}'$ is an uncountable family of pairwise disjoint open subsets of $\mathbf{X}$, which contradicts **ccc**. Thus, assertion (i) is proved.

(i) $\wedge$ (iv) $\Rightarrow$ (v): With help of Lemma 4.1 this implication easily can be verified.

(v) $\Rightarrow$ (i): Let $\precsim$ be some upper semicontinuous linear preorder on $\mathbf{X}$. Because of the proof of Lemma 3.1 it suffices to show that $\precsim$ only has countably many jumps. Let $\mathbf{L} := \{L(x) \mid x \in \mathbf{X}\}$. Then the jumps of $\precsim$ correspond bijectively to the sets $L(y) \in \mathbf{L}$ for which there exists a unique set $L(x) \subsetneq L(y)$ such that for no set $L(z) \in \mathbf{L}$ the strict inclusions $L(x) \subsetneq L(z) \subsetneq L(y)$ hold (cf. the proof of the implication "(ii) $\Rightarrow$ (i)" of Theorem 3.2). Let $J(\mathbf{L})$ be the family of these uniquely determined sets $L(x) \subsetneq L(y)$. Then we assume, in contrast, that $J(\mathbf{L})$ is not countable. Since t is short we may conclude that there exists some isolated set $L(y) \in J(\mathbf{L})$ such that the set $\mathbf{O}_y$ of all sets $L(u) \in J(\mathbf{L})$ that properly contain $L(y)$ is uncountable. Indeed, otherwise a (meanwhile) routine transfinite induction argument implies the existence of some uncountable ordinal α that can be order-embedded into $(t, \supsetneq)$. Let $C(L(y)) := \{L(v) \in J(\mathbf{L}) \mid L(y) \subsetneq L(v) \wedge \overline{L(y)} = \overline{L(v)}\}$. Then we distinguish between the following two cases:

Case 1: $C(L(y)) = \varnothing$. In this case the application of condition **TB2** in each step of the transfinite induction argument in the proof of the implication "(iv) $\Rightarrow$ (i)" allows us to use this transfinite induction argument in order to conclude that there exists some set $L(p) \in \mathbf{O}_y$ such that $[L(y), L(p)]$ is a countable interval of $(J(\mathbf{L}), \subsetneq)$.

Case 2: $C(L(y)) \neq \varnothing$. Now condition **TB1** guarantees the existence of some set $L(q) \in C(L(y))$ such that $[L(y), L(q)]$ is a countable interval of $(J(\mathbf{L}), \subsetneq)$. Since both intervals $[L(y), L(p)]$, respectively $[L(y), L(q)]$ are countable we may construct by transfinite induction an uncountable well ordered increasing subchain $(Z, \subsetneq)$ of $(t, \subsetneq)$ (cf. the proof of the first part of the implication "(i) $\Rightarrow$ (ii)" of Proposition 4.2). The reader may notice that at limit steps the same transfinite induction argument can be applied that has been used in order to guarantee the uncountability of $\mathbf{O}_y$. On the other hand, at limit steps also the arguments that have been used in the afore-discussed two cases may be applied. The existence of $(Z, \subsetneq)$ contradicts the shortness of t, which completes the proof of assertion (i).

□

The reader may recall that a chain $(Z, <)$ satisfies **ccc** (countable chain condition) if every family of pairwise disjoint open intervals of $(Z, <)$ is countable, or, equivalently, if the order topology $t^<$ that is induced by $<$ satisfies **ccc**. The Souslin Hypothesis (**SH**) states that every order-dense and (almost) complete unbordered chain that satisfies **ccc** is order-isomorphic to the real line. **SH** was posed by M. Souslin (1894-1919) in the only paper that he published during his life. Since the late sixties it is known that **SH** is independent of **ZFC**. Recently **SH** has been applied by Vohra [35] in order to prove in **ZFC+SH** a general continuous utility representation theorem. It is easily to be seen that **SH** is equivalent to the assertion that every chain $(Z, <)$ that satisfies **ccc** and only has countably many jumps can be order-embedded into the real line.

The particular relevance of Lemma 4.7 is based upon the following theorem that in combination with Proposition 4.2 and Lemma 4.7 is the main result of this section.

Theorem 4.8. *The following assertions are equivalent:*

(i) **SH** *holds.*

(ii) *For every set* $\mathbf{X}$ *and any topology* t *on* $\mathbf{X}$ *the concepts* t *to be countably isolated and* t *to be completely useful are equivalent.*

Proof. (i) $\Rightarrow$ (ii): Let $\mathbf{X}$ be an arbitrary set and let t be some countably isolated topology on $\mathbf{X}$. In order to show that t is completely useful we consider some upper semicontinuous linear preorder $\precsim$ on $\mathbf{X}$. Then restricting our considerations to equivalence classes a routine and well known argument allows us to assume that $\precsim$, actually, is an order on $\mathbf{X}$. The validity of **SH** implies together with the Representation Lemma (**RL**) that assertion (ii) will follow if we are able to prove that $(\mathbf{X}, \precsim)$ satisfies **ccc** and only has countably many jumps. Therefore, we choose the set $\mathbf{L} := \{L(x) \mid x \in \mathbf{X}\}$ (cf. the proof of the implication "(iv) $\Rightarrow$ (i)" of Lemma 4.7). Since every set $\mathbf{O} \in \mathcal{O}$ only contains countably many isolated sets the proof of Lemma 3.1 implies that $(\mathbf{X}, \precsim)$ only has countably many jumps, and it remains to verify that $(\mathbf{X}, \precsim)$ satisfies **ccc**. Let us, thus, assume in contrast that $(\mathbf{X}, \precsim)$ does not satisfy **ccc**. Then there exists an uncountable family $\{]x_i, y_i[\}_{i \in I}$ of pairwise disjoint non-empty open intervals of $(\mathbf{X}, \precsim)$. But this means that the set $\mathbf{L}' := \{L(y_i) \mid i \in I\}$ contains uncountably many isolated sets. Indeed, it follows from the disjointness of the open intervals $]x_i, y_i[$ of $(\mathbf{X}, \precsim)$ that every set $L(y_i) \in \mathbf{L}'$ is isolated (cf. the characterization of the jumps of $\precsim$ by means of the sets $L(y)$ in the proofs of Theorem 3.2 and Lemma 4.7). This contradiction implies that $(\mathbf{X}, \precsim)$ satisfies **ccc** and, thus, finishes the proof of assertion (ii).

(ii) $\Rightarrow$ (i): This implication will be proved by contraposition. Let us, therefore, assume that **SH** does not hold. Then there exists some linearly ordered set $(Z, <)$ which satisfies **ccc**, only has countably many jumps and is not representable by a real-valued order-preserving function. Therefore, we set $\mathbf{X} := Z$ and consider the linearly ordered topology $t := t^{\mathbf{L}}$ on $\mathbf{X}$ that is induced by the set $\mathbf{L} := \{L(x) \mid x \in \mathbf{X}\}$ (cf. the proof of the implication "(i) $\Rightarrow$ (ii)" of Theorem 3.2). In order to now verify the desired implication it suffices to show that every set $\mathbf{O} \in \mathcal{O}$ only contains countably many isolated sets. Let some set $\mathbf{O} \in \mathcal{O}$ be arbitrarily chosen. We choose the upper semicontinuous linear (pre)order $\precsim_{\mathbf{O}}$ on $\mathbf{X}$ that is induced by $\mathbf{O}$ (cf. the proof of Lemma 3.1). The definition of t implies that for all points $x, y \in \mathbf{X}$ the implication "$x \prec_{\mathbf{O}} y \Rightarrow x < y$" holds. If we, thus, assume that $\mathbf{O}$ contains uncountably many isolated sets, then it follows with help of the construction of the function ψ in the proof of Lemma 3.1 that $(Z, <)$ does not satisfy **ccc** or has uncountably many jumps. This contradiction proves the validity of the implication "(ii) $\Rightarrow$ (i)" and, therefore, finishes the proof of the theorem. □

Theorem 4.8 implies in combination with Lemma 3.1 that the following equivalence holds.

Corollary 4.9. *Let t be an arbitrary topology on* $\mathbf{X}$*. Then in* **ZFC+SH** *the following assertions are equivalent:*

(i) *t is completely useful.*
(ii) *t is countably isolated.*

We want to finish this section by proving a theorem that in some sense may be considered as the strengthening of Corollary 4.9 in **ZFC**. We hope that in combination with Theorem 3.2 and the preceding results of this section it presents a quite satisfactory characterization of completely useful topologies.

Indeed, the concept of a gap allows us to strengthen the equivalent concepts of Lemma 4.7.

A topology t on $\mathbf{X}$ is said to be *strictly thin* if the following construction by transfinite induction, that will be abbreviated by **TIP** (transfinite induction procedure) always leads to a countable set $\mathbf{O} \in \mathcal{O}$.

In the first step we choose arbitrary open subsets $O_0 \subsetneq O'_0$ of $\mathbf{X}$ and set $\mathbf{O_0} := \{O_0, O'_0\}$.

At non-limit steps α we interpose between any pair of subsets $O \subsetneq B$ of $\mathbf{X}$ that define a gap of $\mathbf{O}_{\alpha-1}$ some open set $O^+ \in t \setminus \mathbf{O}_{\alpha-1}$ such that $O \subsetneq O^+ \subsetneq B$. Then $\mathbf{O}_\alpha$ is the union of $\mathbf{O}_{\alpha-1}$ with the family of these additional sets O^+. In case that $\mathbf{O}_{\alpha-1}$ has no gaps the transfinite induction process stops.

At limit steps α we set $\mathbf{O}_\alpha := \bigcup_{\beta<\alpha} \mathbf{O}_\beta$.

Finally we set $\mathbf{O} := \bigcup_{\alpha} \mathbf{O}_\alpha$.

In order to make the concept of a thin topology on $\mathbf{X}$ more transparent we still need the following notation:

t is said to be *locally strictly thin* if for every open subset O of $\mathbf{X}$ and every gap free set $\mathbf{O} \in \mathcal{O}$ such that $O \subsetneq O'$ for each $O' \in \mathbf{O}$ there exists some set $O'' \in \mathbf{O}$ such that starting with $\mathbf{O'}_0 := \{O, O''\}$ **TIP** always leads to a countable set $\mathbf{O'} \in \mathcal{O}$.

t is said to be *countably dense* if every gap free set $\mathbf{O} \in \mathcal{O}$ contains some countable subset $\mathbf{O'}$ such that every set $O \in \mathbf{O}$ is the union or meet of sets $O' \in \mathbf{O'}$.

t is said to be *linear separable* if for every set $\mathbf{O} \in \mathcal{O}$ there exists a countable subset Y of $\mathbf{X}$ such that $Y \cap (O' \setminus O) \neq \varnothing$ for every pair of sets $O, O' \in \mathbf{O}$ such that $O \subsetneq O'$ and (O, O') is a gap of $\{O, O'\}$.

With help of **TIP** *the reader will have no effort in order to conclude that t is linear separable if and only if for every* **gap free** *set $\mathbf{O} \in \mathcal{O}$ there exists a countable subset Y of $\mathbf{X}$ such that $Y \cap (O' \setminus O) \neq \varnothing$ for every pair of sets $O, O' \in \mathbf{O}$ such that $O \subsetneq O'$ and (O, O') is a gap of $\{O, O'\}$.*

t is said to be *locally linear separable* if for every open subset O of $\mathbf{X}$ and every gap free set $\mathbf{O} \in \mathcal{O}$ such that $O \subsetneq O'$ for every set $O' \in \mathbf{O}$ there exists some set O'' and some countable subset Y of $\mathbf{X}$ such that $Y \cap (O^{++} \setminus O^+) \neq \varnothing$ for every pair of sets $O^+, O^{++} \in \mathbf{O}$ such that $O^+ \subsetneq O^{++} \subset O''$ and (O^+, O^{++}) is a gap of $\{O^+, O^{++}\}$.

In order to prove in **ZFC** the desired characterization of completely useful topologies we need the following lemma. Its proof is essentially based upon **ccc**. Since its proof is straightforward it will be omitted for the sake of brevity.

Lemma 4.10. *Let $(Z, <)$ be some chain that satisfies* **ccc** *but is not representable by a real-valued order-preserving function. Then there exists some point $z \in Z$ such that no non-degenerate (non-trivial) interval I of $(Z, <)$ that contains z is representable by a real-valued order-preserving function.*

Now we are able to prove the following theorem.

Theorem 4.11. *Let t be a topology on $\mathbf{X}$. Then the following assertions are equivalent:*

(i) *t is completely useful.*
(ii) *t is strictly thin.*
(iii) *t is countably dense.*
(iv) *t is short and locally strictly thin.*
(v) *t is linear separable.*
(vi) *t is short and locally linear separable.*

Proof. (i) $\Rightarrow$ (ii): Let us assume, in contrast, that t is not strictly thin. Then there exists an uncountable set $\mathbf{O} \in \mathcal{O}$ that can be constructed by **TIP**. Let $\mathbf{O}_0 \subsetneq \mathbf{O}_1 \subsetneq \mathbf{O}_2 \subsetneq \ldots \subsetneq \mathbf{O}_\alpha \subsetneq \ldots \subsetneq$ be the corresponding chain of sets $\mathbf{O}_\alpha \in \mathcal{O}$ and let Γ be the corresponding set of ordinals that appear as indexes of these sets. We consider the upper semicontinuous linear preorder $\precsim_{\mathbf{O}}$ on $\mathbf{X}$ that is induced by $\mathbf{O}$ (cf. the proof of Lemma 3.1) and the corresponding set $\mathbf{L} := \{L(x) \mid x \in \mathbf{X}\}$. Assertion (i) implies with help of Lemma 4.7(iv) that the topology $t^{\mathbf{L}}$ of t that is induced by $\mathbf{L}$ is second countable. On the other hand, the particular construction of the sets $\mathbf{O}_\alpha$ $(\alpha \in \Gamma)$ implies that for non-limit ordinals α no set $O \in \mathbf{O}_\alpha \setminus \mathbf{O}_{\alpha-1}$ is the union of sets $O' \in \mathbf{O}_{\alpha-1}$. Hence, the second countability of $t^{\mathbf{L}}$ implies that there must exist some countable ordinal α such that $\mathbf{O}_\alpha$ has no gaps. This conclusion contradicts the uncountability of Γ and, thus, proves assertion (ii).

(ii) $\Rightarrow$ (iii): Let $\mathbf{O} \in \mathcal{O}$ be some gap free set. Then starting with $\mathbf{O}_0 := \{\varnothing, \mathbf{X}\}$ **TIP** allows us to construct a countable gap free subset $\mathbf{O}'$ of $\mathbf{O}$. Since $\mathbf{O}_0 = \{\varnothing, \mathbf{X}\}$ it follows that $\mathbf{O}'$ can be constructed in such a way that every set $O \in \mathbf{O}$ is the union or meet of sets $O' \in \mathbf{O}'$, which means that t is countably dense.

(iii) $\Rightarrow$ (i): Assertion (iii) immediately implies that t is countably isolated. In order to, therefore, apply assertion (iii) of Theorem 3.2 we consider a linearly ordered subtopology t^l of t that only contains countably many left isolated sets. In order to show that t^l is second countable we may assume without loss of generality that t^l is gap free. Indeed, **TIP** allows us to construct some gap free linearly ordered subtopology t'^l of t that only contains countably many left isolated sets and contains t^l. Since t^l is a linearly ordered subtopology of t'^l then the second countability of t'^l implies the second countability of t^l. If t^l is gap free assertion (iii) guarantees the existence of some countable subset b_l of t^l such that every set $O \in t^l$ is the union or meet of sets $O' \in b_l$. Hence, the assumption that t^l only has countably many left isolated sets implies that the union of b_l with the set of all left isolated sets of t^l is a countable base of t^l. Thus, Theorem 3.2(iii) can be applied and assertion (i) follows.

(ii) $\Rightarrow$ (iv): Obviously, a strictly thin topology on $\mathbf{X}$ must be thin. Hence, assertion (ii) implies with help of Lemma 4.7 that t must be short. Since t is strictly thin it follows, in particular, that t is locally strictly thin, which proves the desired assertion.

(iv) $\Rightarrow$ (i): Let $\precsim$ be an arbitrary upper semicontinuous linear preorder on $\mathbf{X}$. With help of the Lemma of Zorn we may conclude that there exists a maximal upper semicontinuous linear preorder $\lesssim$ on $\mathbf{X}$ such that $\prec \subset <$, i.e. for every upper semicontinuosu linear preorder $\precsim'$ on $\mathbf{X}$ such that $< \subset \prec'$ the equality of $<$ and $\prec'$ holds. Obviously, $\precsim$ has an upper semicontinuous utility representation if $\lesssim$ has an upper semicontinuous utility representation. Therefore, the desired implication follows if we are able to verify that $\lesssim$ has an upper semicontinuous utility representation. Let, thus, $\mathbf{L}$ be the family of all sets $L(x)$ $(x \in \mathbf{X})$ that are defined with respect to $\lesssim$. Then the maximality of $\lesssim$ implies that $\mathbf{L}$ has no gaps. Let us abbreviate this observation by (*). Assertion (iv) implies that t satisfies the assumptions of assertion (iv) of Lemma 4.7. Hence, we may conclude with help of Lemma 4.7 and the proof of Theorem 4.8 that $\lesssim$ satisfies **ccc**. Let us now assume, in contrast, that $\lesssim$ has no upper semicontinuous utility representation. As in the implication "(i) $\Rightarrow$ (ii)" of the proof of Theorem 4.8 we may assume without loss of generality that $\lesssim$ is an order on $\mathbf{X}$. Then **RL** implies that $(\mathbf{X}, \lesssim)$ is not order-embeddable into $(\mathbb{R}, \leq)$. Thus, Lemma 4.10 allows us to conclude that there exists some point $x \in \mathbf{X}$ such that no non-degenerate interval I of $(\mathbf{X}, \lesssim)$ that contains x is order-embeddable into $(\mathbb{R}, \leq)$. Since t is locally strictly thin there exists some set $L(x) \subsetneq L(y) \in \mathbf{L}$ such that starting with $\mathbf{L}'_0 := \{L(x), L(y)\}$ **TIP** leads to a countable subset $\mathbf{O} \in \mathcal{O}$. Because of (*) and the arguments of the proof of the implication "(ii) $\Rightarrow$ (iii)" it follows that $\mathbf{O} \in \mathcal{O}$ can be constructed by **TIP** in such a way that $\mathbf{O} \subset \mathbf{L}$ and that every set $L(u) \in \mathbf{L}$ is the union or meet of sets $O \in \mathbf{O}$. We abbreviate this conclusion by (**). Now we consider the set Z of all points $z \in \mathbf{X}$ such that $L(z) \in \mathbf{O}$. The countability of $\mathbf{O}$ implies that Z is a countable subset of $\mathbf{X}$. Because of conclusion (**) it follows, in addition, that Z is an order-dense subset of the non-degenerate interval $[x, y]$ of $(\mathbf{X}, \lesssim)$ that contains x. This means that every point $u \in [x, y]$ is the greatest upper bound, respectively smallest lower bound, of some subset U of Z. This property of Z implies that the interval $([x, y], \lesssim)$ can be order-embedded into $(\mathbb{R}, \leq)$, a contradiction. Therefore, $(\mathbf{X}, \lesssim)$ has an upper semicontinuous utility representation and the validity of assertion (i) follows.

(iii) $\Rightarrow$ (v): Let some gap free set $\mathbf{O} \in \mathcal{O}$ be arbitrarily chosen. Then assertion (iii) implies the existence of some countable subset $\mathbf{O}'$ of $\mathbf{O}$ such that every set $O \in \mathbf{O}$ is the union or meet of sets $O' \in \mathbf{O}'$. Hence, we choose in every set $O'' \setminus O'$ such that $(O', O'') \in \mathbf{O}' \times \mathbf{O}'$ and $O' \subsetneq O''$ some point $y \in O'' \setminus O'$. The set Y of all in this way chosen points is countable. Furthermore, the properties of $\mathbf{O}'$ imply that $Y \cap (O^{++} \setminus O^{+}) \neq \varnothing$ for every pair of sets $O^{+}, O^{++} \in \mathbf{O}$ such that $O^{+} \subsetneq O^{++}$ and (O^{+}, O^{++}) is a gap of $\{O^{+}, O^{++}\}$, which means that assertion (v) holds.

(v) $\Rightarrow$ (vi): This implication can be proved in a similar way as the implication "(ii) $\Rightarrow$ (iv)".

(vi) $\Rightarrow$ (i): This implication follows by analogous arguments as have been applied in the proof of the implication "(iv) $\Rightarrow$ (i)". For the sake of brevity we, thus, may omit the details of the proof. □

Remark 4.12. A comparison of Lemma 4.7 and Theorem 4.8 with Theorem 4.11 implies, in particular, the equivalence of **SH** with the statement that for every set $\mathbf{X}$ and every topology t on $\mathbf{X}$ the transfinite induction process **TIP** stops after countably many steps if and only if every set $\mathbf{O} \in \mathcal{O}$ has at most countably many gaps.

Corollary 4.13. *Let κ be an arbitrary ordinal number and let t be a completely useful topology on $\mathbf{X}$ that contains at least three open sets. Then the following assertions are equivalent:*

(i) *The topological product $(\mathbf{X}^\kappa, t_{prod})$ is completely useful.*
(ii) *κ is countable.*

Proof. (i) $\Rightarrow$ (ii): Since $| t |> 2$ there exists some pair of points $x, y \in \mathbf{X}$ for which there exists an open subset O of $\mathbf{X}$ such that $x \in O$ and $y \in \mathbf{X} \setminus O$. We abbreviate this observation by (*). Let us now assume, in contrast, that κ is not countable. Then we choose the subset Z of $\mathbf{X}^\kappa$ that consists of all tuples $(z_\alpha)_{\alpha<\kappa}$ for which there exists a unique ordinal β such that $z_\beta = y$ and $z_\alpha = y$ for all ordinals $\alpha < \kappa$ that are different from β. Because of observation (*) it follows that $(Z, t_{prod|Z})$ is not separable, which contradicts Proposition 4.2 and, thus, proves assertion (ii).

(ii) $\Rightarrow$ (i): Let us assume, in contrast, that there exists some set $\mathbf{O} \in \mathcal{O}$ of open subsets O of $\mathbf{X}^\kappa$ that has been constructed by **TIP** but is not countable. Since for every ordinal $\alpha < \kappa$ the projection p_α of $\mathbf{X}^\kappa$ onto $\mathbf{X}$ is open and since κ is a countable ordinal number a routine cardinality argument implies that there exists at least one ordinal $\alpha < \kappa$ such that the set $p_\alpha(\mathbf{O}) := \{p_{alpha}(O) \mid O \in \mathbf{O}\}$ is not countable. Because $\mathbf{O}$ has been constructed by **TIP** we may conclude that $p_\alpha(\mathbf{O})$ can be constructed by **TIP**. Hence, $\mathbf{X}$ is not completely useful. This contradiction implies the validity of assertion (i). □

Remark 4.14. Of course, Corollary 4.13 can be generalized to products of completely useful topologies t_α on sets $\mathbf{X}_\alpha$.

5. The cardinality of the set of all useful and the cardinality of the set of all completely useful topologies on $\mathbf{X}$

Let $\mathbf{X}$ be an infinite set. Then we denote by $Top(\mathbf{X})$ the set of all topologies on $\mathbf{X}$, by $U - Top(\mathbf{X})$ the set of all useful and by $CU - Top(\mathbf{X})$ the set of all completely useful topologies on $\mathbf{X}$. In addition, we denote by $\rho(\mathbf{X})$ the cardinality of the power set $P(\mathbf{X})$ of $\mathbf{X}$. It is well known that $| Top(\mathbf{X}) |= 2^{\rho(\mathbf{X})}$ (cf. Herrlich [21]).

Let c be the cardinality of the real line and let ω be the first infinite ordinal (cardinal). We set $\gamma := min\{\kappa \mid \kappa\ is\ a\ cardinal\ number\ and\ 2^{2^\kappa} > 2^c\}$. Unfortunately, the only inequality that is known in **ZFC** about γ means that

$\gamma > \omega$. In order to illustrate up to which degree useful topologies on $\mathbf{X}$ generalize completely useful topologies on $\mathbf{X}$ we still want to prove the following theorem.

Theorem 5.1. *The following assertions hold:*

(i) *If* $|\mathbf{X}| < \gamma$, *then* $|U - Top(\mathbf{X})| = |CU - Top(\mathbf{X})| = 2^{\rho(\mathbf{X})} = 2^c$.
(ii) *If* $\gamma \leq |\mathbf{X}| \leq c$, *then* $|U - Top(\mathbf{X})| = 2^{\rho(\mathbf{X})} > |CU - Top(\mathbf{X})| = 2^c$.
(iii) *If* $|\mathbf{X}| \geq c$, *then* $|U - Top(\mathbf{X})| = 2^{\rho(\mathbf{X})} > |CU - Top(\mathbf{X})| = \rho(\mathbf{X})$.
(iv) *If* $|\mathbf{X}| > \omega$, *then* $|U - Top(\mathbf{X}) \setminus CU - Top(\mathbf{X})| = 2^{\rho(\mathbf{X})}$.

Proof. (i): On a countable set $\mathbf{X}$ the sets $Top(\mathbf{X})$, $U-Top(\mathbf{X})$ and $CU-Top(\mathbf{X})$ coincide. In case that $\omega \leq |\mathbf{X}| < \gamma$ we, thus, may conclude that $2^{2^\omega} = 2^c \leq |CU - Top(\mathbf{X})| \leq |U - Top(\mathbf{X})| \leq |Top(\mathbf{X})| = 2^{\rho(\mathbf{X})} = 2^{2^{|\mathbf{X}|}} = 2^c$, which already proves assertion (i).

(ii): Let $\mathbf{X}$ be an arbitrary infinite set. Then we choose some fixed point $x \in \mathbf{X}$, set $\mathbf{X}' := \mathbf{X} \setminus \{x\}$ and consider some bijection $\phi : \mathbf{X} \to \mathbf{X}'$. ϕ induces a bijection $\Phi : TOP(\mathbf{X}) \to TOP(\mathbf{X}')$. In addition, every topology t' on $\mathbf{X}'$ induces a connected and separable and, thus, useful topology t on $\mathbf{X}$ by setting $t := \{\varnothing\} \cup \{O \cup \{x\} \mid O \in t'\}$. Hence, we may conclude that $|U - Top(\mathbf{X})| = |Top(\mathbf{X})| = 2^{\rho(\mathbf{X})}$.

Because of the definition of γ the proof of assertion (i) will be complete if we are able to show that $|CU - Top(\mathbf{X})| = 2^c$. Let, therefore, $\gamma \leq |\mathbf{X}| \leq c$ and t be a completely useful topology on $\mathbf{X}$. Then Proposition 4.2 implies that t is hereditarily separable. For every open subset O of $\mathbf{X}$ we, thus, may (uniquely) choose some countable subset C_O of $\mathbf{X} \setminus O$ such that $\overline{C_O} = \mathbf{X} \setminus O$. Since $C_O \neq C_{O'}$ for every pair of different open subsets O, O' of $\mathbf{X}$, the correspondence $O \to C_O$ defines an injection from t into the set of all countable subsets of $\mathbf{X}$. Hence, it follows that $|t| \leq |\{C \subset \mathbf{X} \mid C \textit{ is countable}\}| \leq |\mathbf{X}|^\omega \leq c^\omega \leq c$. These inequalities imply that $|CU-Top(\mathbf{X})| \leq |\{S \subset P(\mathbf{X}) \mid |S| \leq |\mathbf{X}|^\omega\}| \leq (2^c)^c = 2^{c^2} = 2^c$. Summarizing these last inequalities and equalities we may conclude that $|CU-Top(\mathbf{X})| \leq 2^c$. Conversely, it follows with help of assertion (i) that $2^c \leq |CU-Top(\mathbf{X})|$. Hence, the desired equation $|CU-Top(\mathbf{X})| = 2^c$ holds and assertion (ii) follows.

(iii): Let $|\mathbf{X}| \geq c$. Because of the first part of the proof of assertion (ii) it suffices to verify that $|CU - Top(\mathbf{X})| = \rho(\mathbf{X})$. The inequality $|\mathbf{X}| \geq c$ implies with the help of the same arguments that have been applied in the second part of the proof of assertion (ii) that the inequality and equality $|CU - Top(\mathbf{X})| \leq |\{S \subset P(\mathbf{X}) \mid |S| \leq |\mathbf{X}|\}| = \rho(\mathbf{X})$ hold, which means that $|CU-Top(\mathbf{X})| \leq \rho(\mathbf{X})$. On the other hand, it is well known and easily verified that the cardinality of the set of all second countable topologies on $\mathbf{X}$ is $\rho(\mathbf{X})$. This means because of **RT** that $\rho(\mathbf{X}) \leq |CU-Top(\mathbf{X})| \leq \rho(\mathbf{X})$ and the desired equation $|CU - Top(\mathbf{X})| = \rho(\mathbf{X})$ follows, which finishes the proof of assertion (iii).

(iv): Let $|\mathbf{X}| > \omega$. We choose some fixed point $z \in \mathbf{X}$ and divide $\mathbf{X}$ into three pairwise disjoint sets $\mathbf{X}'$, $\mathbf{Y}$ and $\{z\}$ such that $|\mathbf{X}| = |\mathbf{X}'| = |\mathbf{Y}|$.

Let $\phi : \mathbf{X} \to \mathbf{X}'$ be an arbitrary bijection. As in the first part of the proof of assertion (ii) it follows that ϕ induces a bijection Φ between $TOP(\mathbf{X})$ and $TOP(\mathbf{X}')$. Furthermore, for every topology t' on $\mathbf{X}'$ the set $b' := \{\varnothing\} \cup \{O \cup \{z\} \mid O \in t'\} \cup \{V \cup \{z\} \mid V \in P(\mathbf{Y})\}$ is a base of a topology t'' on $\mathbf{X}$. Of course, $t_1'' \neq t_2''$ for every pair of different topologies t_1', t_2' on $\mathbf{X}'$. t'' is a connected and separable and, thus, useful topology on $\mathbf{X}$. Since $\mid \mathbf{Y} \mid = \mid \mathbf{X} \mid > \omega$ we may conclude, on the other hand, that t'' is not short, which means that t'' is not completely useful. Summarizing these considerations it follows that $2^{\rho(\mathbf{X})} = \mid Top(\mathbf{X}) \mid \leq \mid U - Top(\mathbf{X}) \setminus CU - Top(\mathbf{X}) \mid \leq 2^{\rho(\mathbf{X})}$, which implies the validity of assertion (iv). □

Remark 5.2. The characterization of useful topologies t on $\mathbf{X}$ is based upon the concept of a separable system on $\mathbf{X}$ (cf. Herden and Pallack [20]). The reader may recall that a subset $\mathcal{E}$ of t is said to be a separable system on $\mathbf{X}$ if there exist (open) sets $E \subset E' \in \mathcal{E}$ such that $\overline{E} \subset E'$ and if for every pair of (open) sets $E \subset E' \in \mathcal{E}$ such that $\overline{E} \subset E'$ there exists some (open) set $E'' \in \mathcal{E}$ such that $E \subset \overline{E} \subset E'' \subset \overline{E}'' \subset E'$. Indeed, if one replaces in the results of this paper the concept of an open set by the concept of a separable system on $\mathbf{X}$, then one often (not always) obtains characterizations of useful topologies t on $\mathbf{X}$. The main difference between completely useful and useful topologies t on $\mathbf{X}$, thus, means that in order to characterize completely useful topologies t on $\mathbf{X}$ we may restrict our considerations on open subsets of $\mathbf{X}$ instead of considering the much more complicated concept of a separable system on $\mathbf{X}$.

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