Applied General Topology
© Universidad Politécnica de Valencia
Volume 4, No. 1, 2003
pp. 15–24

# The quasitopos hull of the construct of closure spaces

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Abstract. In the list of convenience properties for topological constructs the property of being a quasitopos is one of the most interesting ones for investigations in function spaces, differential calculus, functional analysis, homotopy theory, etc. The topological construct **Cls** of closure spaces and continuous maps is not a quasitopos. In this article we give an explicit description of the quasitopos topological hull of **Cls** using a method of F. Schwarz: we first describe the extensional topological hull of **Cls** and of this hull we construct the cartesian closed topological hull.

2000 AMS Classification: 18A35, 18B25, 18D15, 54A05, 54C35
Keywords: Topological construct, closure space, extensional topological construct, quasitopos, cartesian closed category, cartesian closed topological hull.

## 1. Introduction

Cartesian closedness is an interesting property for topological constructs. It guarantees the existence of nice function spaces in the construct. This property has been studied extensively in the literature.

A *closure space* is a set $X$ endowed with a *closure operator*, i.e. a map $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfying the following conditions: $\mathrm{cl}\,\varnothing = \varnothing$, $A \subset \mathrm{cl}\,A$, $A \subset B \Rightarrow \mathrm{cl}\,A \subset \mathrm{cl}\,B$ and $\mathrm{cl}\,(\mathrm{cl}\,A) = \mathrm{cl}\,A$. One notes that the closure is allowed to be non-additive. A map $f : X \to Y$ is said to be *continuous* if $f(\mathrm{cl}\,A) \subset \mathrm{cl}\,(f(A))$ for all subsets $A \subset X$. The construct of closure spaces and continuous maps is denoted by **Cls**. **Cls** is known to be a well-fibred topological construct. It is an interesting construct since non-additive closures arise in different fields of mathematics, in particular in algebra, geometry and analysis. Perhaps the best known example is the convex hull in vector spaces. Other examples are listed in the introductory chapter of [6]. There is also a strong relation between closures and complete lattices [4]. In recent years closures have even been used in connection with quantum logic and in the representation theory of physical systems [13, 14, 19].

The construct **Cls** is not cartesian closed. In [3] the cartesian closed topological hull of **Cls** was described. In this paper we will describe an even more convenient hull: the quasitopos hull. A topological construct is a *quasitopos* if it is both extensional and cartesian closed. The property of extensionality ensures the existence of one-point extensions in the topological construct, or, equivalently, that quotients and coproducts are preserved by pullbacks along embeddings. This property is extensively treated in [20]. The stronger concept topos is not interesting for topological categories, since the only topological topoi are isomorphic to the category **Set** [1, 22]. Quasitopoi (also called *topological universes* for topological categories) were introduced by J. Penon [18] as a generalization of topoi. This generalization is broad enough to allow topological examples, but not too broad for losing most useful properties of topoi. Topological universes are used for functional analysis and differential calculus (see e.g. [15, 16, 17]) and for a theory of holomorphic maps (see e.g. [5]). Also for a topological construct being a quasitopos is equivalent to being *locally cartesian closed* i.e. all comma-categories being cartesian closed. It is well known that the construct **Top** of topological spaces and continuous maps is not a quasitopos: it is neither extensional nor cartesian closed. Schwarz [21] showed that the topological quasitopos hull of a construct –if it exists– can be described as the cartesian closed topological hull of the extensional topological hull. Therefore we first construct the extensional topological hull of **Cls** and then the cartesian closed topological hull of this extensional hull.

Categorical terminology follows [1]. We will only consider constructs that are well-fibred.

## 2. The extensional topological hull of Cls

In this section, we construct the extensional topological hull of **Cls**. We do this in the same way as has been done for **Top**: we weaken the axioms of the closure operator.

**Definition 2.1.** [10, 11, 20] A topological construct **A** is called extensional if it has representable partial morphisms to all **A**-objects, where

- A partial morphism from $X$ to $Y$ is a morphism $f : Z \to Y$ whose domain $Z$ is a subspace of $X$.
- Partial morphisms to $Y$ are representable provided $Y$ can be embedded via the addition of a single point $\infty_Y$ into an object $Y^\sharp$ with the property that for every partial morphism $f : Z \to Y$ from $X$ to $Y$, the map $f^X : X \to Y^\sharp$, defined by $f^X(x) = f(x)$ if $x \in Z$, $f^X(x) = \infty_Y$ if $x \in X \setminus Z$, is a morphism. The object $Y^\sharp$ is called the one-point extension of $Y$.

Extensional was called hereditary in [10] and [20]. In [20] Schwarz proved that the one-point extension of an object $Y$ (if it exists) carries the smallest (i.e. coarsest) structure that makes $Y$ a subspace. Thus the one-point extension is unique (up to isomorphism). There is no smallest closure on $\{0, 1, \infty\}$ which makes the Sierpinski space **2** a subspace, so the construct **Cls** can not be extensional. We shall now obtain an extensional supercategory of **Cls** by dropping the idempotency of the closure operator.

**Definition 2.2.** A preclosure space $(X, \mathrm{cl})$ is a set $X$ structured by a preclosure operator $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfying the following conditions. For $A, B \subset X$ :

$(C_1)$ $\mathrm{cl}\,\varnothing = \varnothing$,
$(C_2)$ $A \subset \mathrm{cl}\,A$,
$(C_3)$ $A \subset B \Rightarrow \mathrm{cl}\,A \subset \mathrm{cl}\,B$.

A function $f : (X, \mathrm{cl}_X) \to (Y, \mathrm{cl}_Y)$ between preclosure spaces is continuous iff $f(\mathrm{cl}_X A) \subset \mathrm{cl}_Y f(A)$ for all $A \subset X$. The construct of preclosure spaces and continuous maps is denoted by **PrCls**.

A preclosure space can also be described in an isomorphic way using neighborhoods. For a preclosure space $(X, \mathrm{cl})$ and $x \in X$, the collection of neighborhoods $\mathcal{V}(x) = \{V \subset X \mid x \notin \mathrm{cl}\,(X \setminus V)\}$ satisfies the following three conditions:

$(V_1)$ $X \in \mathcal{V}(x)$,
$(V_2)$ $\forall\, V \in \mathcal{V}(x) :\ x \in V$,
$(V_3)$ $\forall\, V \in \mathcal{V}(x) :\ V \subset W \Rightarrow W \in \mathcal{V}(x)$.

Conversely, if the family $(\mathcal{V}(x))_{x \in X}$ satisfies the conditions $(V_1), (V_2)$ and $(V_3)$ for each $x \in X$, then a unique preclosure operator cl on $X$ exists, such that for each $x \in X$, $\mathcal{V}(x)$ are the neighborhoods of $x$. This closure operator is defined by: $\mathrm{cl}\,A = \{x \in X \mid \forall\, V \in \mathcal{V}(x) : V \cap A \neq \varnothing\}$ for all $A \subset X$. A function $f : (X, \mu) \to (Y, \eta)$ is continuous iff $f^{-1}(\eta(f(x))) \subset \mu(x)$ for every $x \in X$.

From the definition of **PrCls** follows immediately that **Cls** is a full subcategory of **PrCls**.

**Proposition 2.3.** **PrCls** *is a topological construct.*

*Proof.* For a structured source $(f_i : X \to (Y_i, \mathrm{cl}_i))_{i \in I}$ in **PrCls**, the initial preclosure operator on $X$ is defined by: $\mathrm{cl}A = \bigcap_{i \in I} f_i^{-1}(\mathrm{cl}_i(f_i(A)))$ for each $A \subset X$ or with neighborhoods: $\mathcal{V}(x) = \{V \subset X \mid \exists\, i \in I,\ \exists\, W \in \mathcal{V}_i(f_i(x)) : f_i^{-1}(W) \subset V\}$. □

It is clear that **Cls** is closed under formation of initial structures in **PrCls**, and thus we have:

**Proposition 2.4.** **Cls** *is a bireflective subconstruct of* **PrCls**.

**Theorem 2.5.** **PrCls** *is extensional.*

*Proof.* Analogously as for **PrTop** [11]. □

**Definition 2.6.** [9, 11] An extensional topological construct **B** is called an extensional topological hull of a construct **A** if **B** is a finally dense extension of **A** with the property that any finally dense embedding of **A** into an extensional topological construct can be uniquely extended to **B**.

The extensional topological hull of a construct – if it exists – is unique up to isomorphism.

**Theorem 2.7.** [10, 11] *The extensional topological hull* **B** *of a construct* **A** *is characterized by the following properties:*

(1) **B** *is an extensional topological construct.*
(2) **A** *is finally dense in* **B**.
(3) $\{Y^\sharp \mid Y \in |\mathbf{A}|\}$ *is initially dense in* **B**.

The proofs of the following propositions are similar to those in [11].

**Proposition 2.8.** **Cls** *is a finally dense subcategory of* **PrCls**.

**Definition 2.9.** The one-point extension $\mathbf{2}^\sharp$ of the Sierpinski space is the preclosure space **3** with underlying set $\{0, 1, 2\}$ (we denote $\infty$ by 2) and neighborhoods: $\mathcal{V}(0) = \mathcal{V}(2) = \dot{0} \cap \dot{1} \cap \dot{2}$ and $\mathcal{V}(1) = \dot{1} \cap \dot{2}$.

**Proposition 2.10.** $\{\mathbf{3}\}$ *is initially dense in* **PrCls**.

Now from Theorem 2.5, Proposition 2.8, Proposition 2.10 and Theorem 2.7 we have the following result:

**Theorem 2.11.** **PrCls** *is the extensional topological hull of* **Cls**.

## 3. The cartesian closed hull of PrCls

An object $X$ in a category with finite products is *exponential* if $X \times -$ has a right adjoint. In a well-fibred topological construct **D**, this notion can be characterized as follows: $X$ is exponential in **D** iff for each **D**-object $Y$ the set $\mathrm{Hom}_{\mathbf{D}}(X, Y)$ can be supplied with the structure of a **D**-object – a function space or a power object $Y^X$ – such that

(1) the evaluation map $\mathrm{ev} : X \times Y^X \to Y$ is a **D**-morphism, and
(2) for each **D**-object $Z$ and each **D**-morphism $f : X \times Z \to Y$ the map $f^* : Z \to Y^X$ defined by $f^*(z)(x) = f(x, z)$ is a **D**-morphism.

It is well known that in the setting of a topological construct **D**, an object $X$ is exponential in **D** iff $X \times -$ preserves final episinks [7, 8]. Moreover, small fibredness of **D** ensures that this is equivalent to the condition that $X \times -$ preserves quotients and coproducts. A well-fibred topological construct **D** is said to be *cartesian closed* (or to have function spaces) if every object is exponential. It was shown in [3] that **Cls** is not cartesian closed. In fact, the class of exponential objects consists precisely of all indiscrete closure spaces. We show that exponential objects are unchanged if we replace **Cls** by **PrCls**.

**Proposition 3.1.** *A preclosure space* $X$ *is an exponential object in* **PrCls** *if and only if* $X$ *is indiscrete.*

*Proof.* Suppose $\mu$ is an admissible **PrCls**-structure on $\mathbf{PrCls}(X, \mathbf{3})$. Take $x \in X$ and $V \in \mathcal{V}_X(x)$. Then $f : X \to \mathbf{3}$ defined by $f^{-1}(1) = \{x\}, f^{-1}(\{1, 2\}) = V$ is a **PrCls**-morphism. By continuity of the evaluation map ev in $(x, f)$ there exists $B \in \mathcal{V}_\mu(f)$ such that $\mathrm{ev}(X \times B) \subset \{1, 2\}$. This implies $X = f^{-1}(\{1, 2\})$

and so $V = X$. If $X$ is indiscrete, then the **PrCls**-structure $\mu$ on $\mathbf{PrCls}(X, \mathbf{3})$ given by $\mathcal{V}_\mu(f) = \{\mathbf{PrCls}(X, \mathbf{3})\}$ if $f^{-1}(1) = \varnothing$ and $\mathcal{V}_\mu(f) = \text{stack}\,\{\{g \in \mathbf{PrCls}(X, \mathbf{3}); g(X) \subset \{1, 2\}\}\}$ if $f^{-1}(1) \neq \varnothing$ is admissible and proper. □

We first recall the definitions of CCT hull, multimorphism, strictly dense subcategory, power-closed collection and the construction of the CCT hull presented by J. Adámek, J. Reiterman and G. E. Strecker [2]. Then we use this method to construct the CCT hull of **PrCls**.

**Definition 3.2.** [12] A cartesian closed topological construct $\mathbf{B}$ is called a cartesian closed topological hull (CCT hull) of a construct $\mathbf{A}$ if $\mathbf{B}$ is a finally dense extension of $\mathbf{A}$ with the property that any finally dense embedding of $\mathbf{A}$ into a cartesian closed topological construct can be uniquely extended to $\mathbf{B}$.

**Definition 3.3.** [2] Let $\mathbf{K}$ be a construct and let $H, K$ be $\mathbf{K}$-objects and $X$ a set. A function $h : X \times H \to K$ is called a multimorphism if for each $x \in X$, $h(x, -) : H \to K$ defined by $h(x, -)(y) = h(x, y)$ is a morphism.

**Definition 3.4.** [2] Let $\mathbf{K}$ be a construct with quotients and finite products. A full subcategory $\mathbf{H}$ of $\mathbf{K}$ is said to be strictly dense in $\mathbf{K}$ provided that :

(1) for each object $K \in |\mathbf{K}|$ there exists a productively final sink $(H_i \xrightarrow{h_i} K)_{i \in I}$ with $H_i \in |\mathbf{H}|$, i.e., a final sink such that for each $L \in |\mathbf{K}|$ the sink $(H_i \times L \xrightarrow{h_i \times 1_L} K \times L)_{i \in I}$ is final as well.

(2) $\mathbf{H}$ is well-fibred, closed under quotients, and has productive quotients (i.e., for each quotient $e : A \to B$ with $A \in |\mathbf{H}|$, we have $B \in |\mathbf{H}|$ and $e \times 1_H : A \times H \to B \times H$ is a quotient for each $H \in |\mathbf{H}|$).

**Definition 3.5.** [2] Let $\mathbf{K}$ be a construct with quotients and finite products and let $\mathbf{H}$ be strictly dense in $\mathbf{K}$. A collection $\mathsf{A}$ of $\mathbf{H}$-objects $(A, \alpha)$ with $A \subset X$ is said to be power-closed in $X$ provided that $\mathsf{A}$ contains each $\mathbf{H}$-object $(A_0, \alpha_0)$, $A_0 \subset X$, with the following property:

Given a multimorphism $h : X \times H \to K$ with $H \in |\mathbf{H}|$ and $K \in |\mathbf{K}|$ such that for each $(A, \alpha) \in \mathsf{A}$ the restriction $h|_A : (A, \alpha) \times H \to K$ is a morphism, then the restriction $h|_{A_0} : (A_0, \alpha_0) \times H \to K$ is also a morphism.

We denote by $\mathsf{PC}_{\mathbf{H}}(\mathbf{K})$ the category of power-closed collections in $\mathbf{H}$. Objects are pairs $(X, \mathsf{A})$, where $X$ is a set and $\mathsf{A}$ is a power-closed collection of $\mathbf{H}$-objects in $X$. Morphisms $f : (X, \mathsf{A}) \to (Y, \mathsf{B})$ are functions from $X$ to $Y$ such that for each $(A, \alpha) \in \mathsf{A}$ the final object of the restriction $f_A : (A, \alpha) \to f(A)$ is in $\mathsf{B}$.

If $\mathbf{H} = \mathbf{K}$ then we simply write $\mathsf{PC}(\mathbf{K})$.

**Theorem 3.6.** [2] *Any construct* $\mathbf{K}$ *which has quotients and finite products that are preserved by the forgetful functor, and which has a strictly dense subcategory* $\mathbf{H}$*, has a CCT hull. Moreover, this hull is precisely the category of power-closed collections in* $\mathbf{H}$.

**Proposition 3.7.** *In* **PrCls** *arbitrary products of quotients are quotients.*

*Proof.* Let $X_i \xrightarrow{f_i} Y_i$ be a quotient in **PrCls** for any $i \in I$, which means that for every $A_i \subset Y_i : \mathrm{cl}_{Y_i} A_i = f_i(\mathrm{cl}_{X_i}(f_i^{-1}(A_i)))$.
Let $f = \prod_{i\in I} f_i : \prod_{i\in I} X_i \to \prod_{i\in I} Y_i$ and $A \subset \prod_{i\in I} Y_i, (y_i)_{i\in I} \in \mathrm{cl}A$.
Since all $f_i : X_i \to Y_i$ are quotients, we have:
$y_i \in \mathrm{cl}_{Y_i}(\mathrm{pr}_{Y_i} A) = f_i(\mathrm{cl}_{X_i} f_i^{-1}(\mathrm{pr}_{Y_i} A)) = f_i(\mathrm{cl}_{X_i} \mathrm{pr}_{X_i}(f^{-1}(A))$ for all $i \in I$.
This implies: $(y_i)_{i\in I} \in f(\mathrm{cl} f^{-1}(A))$. □

From the previous proposition follows that the construct **PrCls** is strictly dense in itself. Our aim is to give an explicit description of the objects of the construct $\mathsf{PC}$(**PrCls**) in terms of preclosures.

**Proposition 3.8.** *If $X$ is a set and $\mathsf{C}$ is a power-closed collection in $X$, then there exists a unique collection $\mathcal{A} \subset \mathcal{P}(X)$ and for all $A \in \mathcal{A}$ a unique* **PrCls**-*structure $\alpha_A$ on $A$ such that $\mathsf{C} = \{(A, \beta) \in |\mathbf{PrCls}| : A \in \mathcal{A},\ \alpha_A \leqslant \beta\}$.*

*Proof.* For $\mathcal{A}$ we take the set of underlying sets of objects in $\mathsf{C}$ and for $A \in \mathcal{A}$ we set $\alpha_A$ the final **PrCls**-structure on $A$ for the sink $(1_A : (A, \beta) \to A)_{(A,\beta)\in\mathsf{C}}$. It remains to prove that $(A, \alpha_A) \in \mathsf{C}$ for all $A \in \mathcal{A}$. Take $A \in \mathcal{A}$ and take a multimorphism $h : X \times (H, \gamma) \to \mathbf{3}$, with $(H, \gamma) \in |\mathbf{PrCls}|$, such that for all $(C, \delta) \in \mathsf{C}$ the restriction $h|_C : (C, \delta) \times (H, \gamma) \to \mathbf{3}$ is a **PrCls**-morphism. Then $h|_A : (A, \beta) \times (H, \gamma) \to \mathbf{3}$ is a **PrCls**-morphism for all $(A, \beta) \in \mathsf{C}$ with underlying set $A$. So if $(a, y) \in A \times H$ satisfies $h(a, y) = 1, (h|_A)^{-1}(\{1, 2\})$ is a neighborhood of $(a, y)$ in $(A, \beta) \times (H, \gamma)$ for all $(A, \beta) \in \mathsf{C}$, and one of the following cases arises:

(1) $\exists\, (A, \beta) \in C,\ \exists\, W \in \mathcal{V}_\gamma(y)$ with $A \times W \subset (h|_A)^{-1}(\{1, 2\})$.
(2) $\forall\, (A, \beta) \in C,\ \exists\, V_\beta \in \mathcal{V}_\gamma(a)$ with $V_\beta \times H \subset (h|_A)^{-1}(\{1, 2\})$.

In case (1), $A \times W$ is a neighborhood of $(a, y)$ in $(A, \alpha_A) \times (H, \gamma)$.
In case (2), $V = \bigcup\{V_\beta \mid (A, \beta) \in \mathsf{C}\}$ is a neighborhood of $a$ in $\alpha_A$ and $V \times H \subset (h|_A)^{-1}(\{1, 2\})$.
We conclude that $h|_A : (A, \alpha_A) \times (H, \gamma) \to \mathbf{3}$ is a **PrCls**-morphism. □

**Proposition 3.9.** *With the notation of Proposition 3.8 the collection $\mathcal{A}$ has the properties*

(A1) $\forall\, x \in X,\ \{x\} \in \mathcal{A}$,
(A2) $A' \subset A \in \mathcal{A} \implies A' \in \mathcal{A}$,

*and the* **PrCls**-*structures $\alpha_A$ satisfy*

(*B*1) $A' \subset A \in \mathcal{A} \implies \alpha_{A'} \leqslant \alpha_A|_{A'}$,
(*B*2) $\forall\, A \in \mathcal{A},\ \forall\, x \in A,\ \forall\, V \in \mathcal{V}_{\alpha_A}(x),\ \exists\, T \in \mathcal{V}_{\alpha_A}(x)$ *with* $T \subset V$ *and* $\{x\} \in \mathcal{V}_{\alpha_{(A\setminus T)\cup\{x\}}}(x)$.

*Proof.* For the difficult part take $x \in A \in \mathcal{A}$ and $V \in \mathcal{V}_{\alpha_A}(x)$ and suppose $V \neq A$ (otherwise we can take $T = V = A$). A **PrCls**-structure $\mu$ on $A$ is given by

$$\mathcal{V}_\mu(x) = \mathcal{V}_{\alpha_A}(x) \setminus \{W \in \mathcal{P}(A) \mid W \subset V\},$$
$$\mathcal{V}_\mu(y) = \mathcal{V}_{\alpha_A}(y) \text{ if } y \neq x.$$

Clearly $(A,\mu)$ does not belong to $\mathsf{C}$ and so there exists a multimorphism $h : X\times(H,\gamma)\to\mathbf{3}$, with $(H,\gamma)\in|\mathbf{PrCls}|$, such that for all $(C,\delta)\in\mathsf{C}$, $h|_C : (C,\delta)\times(H,\gamma)\to\mathbf{3}$ is a **PrCls**-morphism and $h|_A : (A,\mu)\times(H,\gamma)\to\mathbf{3}$ is not a **PrCls**-morphism. There exists $y\in H$ such that $h(x,y)=1$ and $h|_A : (A,\mu)\times(H,\gamma)\to\mathbf{3}$ is not continuous at $(x,y)$. Since $h|_A : (A,\alpha_A)\times(H,\gamma)\to\mathbf{3}$ is continuous one of the following cases arises:

(1) $\exists\, W\in\mathcal{V}_\gamma(y)$ with $A\times W\subset(h|_A)^{-1}(\{1,2\})$.

(2) $\exists\, T'\in\mathcal{V}_{\alpha_A}(x)$ with $T'\times H\subset(h|_A)^{-1}(\{1,2\})$.

Since $h|_A : (A,\mu)\times(H,\gamma)\to\mathbf{3}$ is not continuous at $(x,y)$ we can suppose (2). Then

$$T=\{t\in A \mid \{t\}\times H\subset(h|_A)^{-1}(\{1,2\})\}$$

belongs to $\mathcal{V}_{\alpha_A}(x)$ and satisfies $T\times H\subset(h|_A)^{-1}(\{1,2\})$ and $T\subset V$. Now

$$h|_{(A\setminus T)\cup\{x\}} : ((A\setminus T)\cup\{x\},\alpha_{(A\setminus T)\cup\{x\}})\times(H,\gamma)\to\mathbf{3}$$

is continuous at $(x,y)$ and so one of the following cases arises:

(1) $\exists\, W'\in\mathcal{V}_\gamma(y)$ with $((A\setminus T)\cup\{x\})\times W'\subset(h|_{(A\setminus T)\cup\{x\}})^{-1}(\{1,2\})$.

(2) $\exists\, T''\in\mathcal{V}_{\alpha_{(A\setminus T)\cup\{x\}}}(x)$ with $T''\times H\subset(h|_{(A\setminus T)\cup\{x\}})^{-1}(\{1,2\})$.

In the first case $A\times W'$ would be a neighborhood of $(x,y)$ in $(A,\mu)\times(H,\gamma)$ with $A\times W'\subset(h|_A)^{-1}(\{1,2\})$, so we can suppose (2). For $T''$ we have $T''\subset(A\setminus T)\cup\{x\}$ as well as $T''\subset T$, so $T''=\{x\}$. □

**Proposition 3.10.** *If $X$ is a set, $\mathcal{A}\subset\mathcal{P}(X)$ a collection of subsets satisfying the conditions (A1) and (A2); and if for each $A\in\mathcal{A}$ a* **PrCls***-structure $\alpha_A$ is given such that (B1) and (B2) are satisfied, then the set $\mathsf{C}=\{(A,\beta)\in|\mathbf{PrCls}| : A\in\mathcal{A},\ \alpha_A\leqslant\beta\}$ is a power-closed collection in $X$.*

*Proof.* Take a **PrCls**-object $(A_0,\alpha_0)$ with $A_0\subset X$ that does not belong to $\mathsf{C}$. Then either $A_0\notin\mathcal{A}$ or both $A_0\in\mathcal{A}$ and $\alpha_{A_0}\nleqslant\alpha_0$. In both cases we give a multimorphism $h : X\times(H,\gamma)\to\mathbf{3}$, with $(H,\gamma)\in|\mathbf{PrCls}|$, such that for all $(A,\beta)\in\mathsf{C}$, $h|_A : (A,\beta)\times(H,\gamma)\to\mathbf{3}$ is a morphism and $h|_{A_0} : (A_0,\alpha_0)\times(H,\gamma)\to\mathbf{3}$ is not a morphism.

First suppose $A_0\notin\mathcal{A}$ and take $x_0\in A_0$. The **PrCls**-object $(H,\gamma)$ and multimorphism $h : X\times(H,\gamma)\to\mathbf{3}$ are defined as follows: $H=\mathcal{A}\cup\{\infty\}$ $(\infty\notin\mathcal{A})$, $\mathcal{V}_\gamma(y)=\{H\}$ if $y\in\mathcal{A}$, $\mathcal{V}_\gamma(\infty)=\mathrm{stack}_H\{\{\infty,\mathcal{A}\}\mid x_0\in A\in\mathcal{A}\}$,

$$h : X\times H\to\mathbf{3} : (x,y)\mapsto\begin{cases}1 & \text{if } (x,y)=(x_0,\infty),\\ 2 & \text{if } (x,y)\in\displaystyle\bigcup_{x_0\in A\in\mathcal{A}} A\times\{A,\infty\}\setminus\{(x_0,\infty)\},\\ 0 & \text{in all other cases.}\end{cases}$$

For all $(A,\beta)\in\mathsf{C}$, $h|_A : (A,\beta)\times(H,\gamma)\to\mathbf{3}$ is continuous since if $x_0\in A$ the neighborhood $A\times\{A,\infty\}$ of $(x_0,\infty)$ in $(A,\beta)\times(H,\gamma)$ satisfies $A\times\{A,\infty\}\subset(h|_A)^{-1}(\{1,2\})$. However $h|_{A_0} : (A_0,\alpha_0)\times(H,\gamma)\to\mathbf{3}$ is not continuous at $(x_0,\infty)$ since for $V\in\mathcal{V}_{\alpha_0}(x_0)$ we have $V\times H\not\subset(h|_{A_0})^{-1}(\{1,2\})$ (because $(x_0,\phi)\in V\times H$ but $h(x_0,\phi)=0$) and for $x_0\in A\in\mathcal{A}$ we have $A_0\times\{A,\infty\}\not\subset(h|_{A_0})^{-1}(\{1,2\})$ (for $x\in A_0\setminus A$ the pair $(x,A)$ satisfies $(x,A)\in A_0\times\{A,\infty\}$ and $h(x,A)=0$). Now suppose $A_0\in\mathcal{A}$ and $\alpha_{A_0}\nleqslant\alpha_0$. Then there exist

$x_0 \in A_0$ and $V \in \mathcal{V}_{\alpha_{A_0}}(x_0) \setminus \mathcal{V}_{\alpha_0}(x_0)$. Take $T \in \mathcal{V}_{\alpha_{A_0}}(x_0)$, $T \subset V$ such that $\{x_0\} \in \mathcal{V}_{\alpha_{A_0 \setminus T \cup \{x_0\}}}(x_0)$. If $(A_0 \setminus T) \cup \{x_0\} \subset A \in \mathcal{A}$, then using $(B1)$ we can choose $V_A \in \mathcal{V}_{\alpha_A}(x_0)$ with $V_A \cap ((A_0 \setminus T) \cup \{x_0\}) = \{x_0\}$. So we can define the map $h : X \times H \to \mathbf{3}$ (with $(H, \gamma)$ as in the case $A_0 \notin \mathcal{A}$) as follows:

$$h(x,y) = \begin{cases} 1 & \text{if } (x,y) = (x_0, \infty), \\ 2 & \text{if } (x,y) \in \bigcup\{V_A \times H \mid (A_0 \setminus T) \cup \{x_0\} \subset A \in \mathcal{A}\} \cup \\ & \bigcup\{A \times \{A, \infty\} \setminus \{(x_0, \infty)\} \mid x_0 \in A \in \mathcal{A}, (A_0 \setminus T) \cup \{x_0\} \not\subset A\}, \\ 0 & \text{in all other cases.} \end{cases}$$

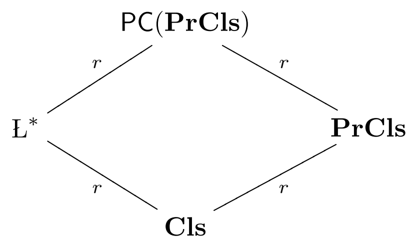
Then $h : X \times (H, \gamma) \to \mathbf{3}$ is a multimorphism. Now for all $(A, \beta) \in \mathsf{C}$, $h|_A : (A, \beta) \times (H, \gamma) \to \mathbf{3}$ is a **PrCls**-morphism: if $(A_0 \setminus T) \cup \{x_0\} \subset A$ then $V_A \times H$ is a neighborhood of $(x_0, \infty)$ in $(A, \beta) \times (H, \gamma)$ and $V_A \times H \subset (h|_A)^{-1}(\{1, 2\})$ and if $(A_0 \setminus T) \cup \{x_0\} \not\subset A$ and $x_0 \in A$ then $A \times \{A, \infty\}$ is a neighborhood of $(x_0, \infty)$ in $(A, \beta) \times (H, \gamma)$ which is contained in $(h|_A)^{-1}(\{1, 2\})$. Now we only have to prove that $h|_{A_0} : (A_0, \alpha_0) \times (H, \gamma) \to \mathbf{3}$ is not continuous at $(x_0, \infty)$. Therefore we show that for $x_0 \in A \in \mathcal{A}$, $A_0 \times \{A, \infty\} \not\subset (h|_{A_0})^{-1}(\{1, 2\})$ and for $W \in \mathcal{V}_{\alpha_0}(x_0)$ we have $W \times H \not\subset (h|_{A_0})^{-1}(\{1, 2\})$. Take $x_0 \in A \in \mathcal{A}$. If $(A_0 \setminus T) \cup \{x_0\} \subset A$ then for $x \in A_0 \setminus T$ we have $h(x, A) = 0$. If $(A_0 \setminus T) \cup \{x_0\} \not\subset A$ then $h(x, A) = 0$ for $x \in ((A_0 \setminus T) \cup \{x_0\}) \setminus A$. Finally take $W \in \mathcal{V}_{\alpha_0}(x_0)$. Then $h(x, \{x_0\}) = 0$ for $x \in W \setminus T$. □

**Definition 3.11.** [21] A quasitopos **B** is called a quasitopos hull of a construct **A** if **B** is a finally dense extension of **A** with the property that any finally dense embedding of **A** into a quasitopos can be uniquely extended to **B**.

F. Schwarz [21] proved that the quasitopos hull of a construct (if it exists) can be described as the cartesian closed topological hull of the extensional topological hull. Using this we have the following proposition.

**Proposition 3.12.** *The quasitopos hull of* **Cls** *is the construct which has as objects the pairs* $(X, \{(A, \beta) \in |\mathbf{PrCls}| : A \in \mathcal{A}, \alpha_A \leqslant \beta\})$, *where* $X$ *is a set,* $\mathcal{A} \subset \mathcal{P}(X)$ *is a collection of subsets of* $X$ *satisfying* $(A1)$ *and* $(A2)$ *(see Proposition 3.9), and for each* $A \in \mathcal{A}$, $\alpha_A$ *is a* **PrCls**-*structure on* $A$ *such that the properties* $(B1)$ *and* $(B2)$ *(see Proposition 3.9) are fulfilled. Morphisms* $f : (X, \mathsf{A}) \to (Y, \mathsf{B})$ *are functions* $f : X \to Y$ *such that for each* $(A, \beta) \in \mathsf{A}$ *the final* **PrCls**-*object of the restriction* $f|_A : (A, \beta) \to f(A)$ *is in* $\mathsf{B}$.

In [3] we constructed the cartesian closed topological hull of **Cls**. This construct was denoted by Ł*. The following diagram shows the bireflective inclusions into the three hulls of **Cls** under discussion. **PrCls** is not a subcategory of Ł*, otherwise Ł* would be the CCT hull of **PrCls**. There is no concrete embedding from Ł* into **PrCls**.

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Received December 2001
Revised December 2002

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