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Holonomy, extendibility, and the star universal cover of a topological groupoid

Osman Mucuk and İlhan İçen

Abstract. Let G be a groupoid and W be a subset of G which contains all the identities and has a topology. With some conditions on G and W, the pair (G, W) is called a locally topological groupoid. We explain a criterion for a locally topological groupoid to be extendible to a topological groupoid. In this paper we apply this result to get a topology on the monodromy groupoid MG which is the union of the universal covers of G_x's.

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1. Introduction

A groupoid is a small category in which each morphism is an isomorphism. Thus a group is a particular example of a groupoid. There is considerable evidence (see for example [9]) that the extension from groups to groupoids is widely useful in mathematics, and is one way of encoding many of the intuitions and methods of Sophus Lie which are difficult to encode in the language purely of group theory. For this encoding, we need the notion of topological (and of Lie groupoid) and so it is important to examine the extent to which standard constructions on topological groups are available for topological groupoids. The book [9] gives considerable information on this. In this paper we give an exposition of the construction of an analogue of the classical universal cover of a connected topological group, and which we call the monodromy groupoid, following Pradines [12].

The ideas for this are taken from [10,6] but we use a result from [6] to give a more direct proof of the construction than in [6], although in this way we do lose some power, notably the monodromy principle as given in [6].

We again emphasise the use of the holonomy groupoid construction, as first developed by Pradines in [12], which however contains no details. Full details were first given in [1], as explained there. We feel it important to stress the construction of Pradines as expressing well the intuitive idea of non trivial holonomy as dealing with an 'iteration of local procedures which returns to the starting position but not the starting value'. Thus we see the concept of groupoid as adding to the concept of group an extra notion of 'position', through the set or space of objects, and of 'transition', through the arrows between objects. This extension has proved to be generally powerful.

Let G be a groupoid and W a subset of G containing all the identities in G. Suppose that W has a topology. For certain conditions on W the pair (G,W) is called a *locally topological groupoid*. The topology on W does not in general extend to a topological groupoid structure on G which restricts to that on W, but there is a topological groupoid H, called the *holonomy groupoid*, with a morphism $H \to G$ such that H contains W as a subspace and H has a universal property. The full details of this result are given by Aof and Brown in [1].

A locally topological groupoid (G,W) is called *extendible* if there is a topology on G such that G is a topological groupoid with this topology and W is open in G. A locally topological groupoid is not in general extendible. It is proven by Brown and Mucuk in [7] that the charts of a foliated manifold may be chosen so that they give rise to a locally topological groupoid which in general is not extendible. We have also examples of locally topological groupoids, due to Pradines and explained in [1], which are not extendible. A full account of the monodromy groupoids was given in [10] and published in [6].

Let G be a topological groupoid in which each star G_x has a universal cover. Then the monodromy groupoid GM is constructed by Mackenzie in [9] as the union over x in O_G of the universal covers based at 1_x of the stars G_x. In the locally trivial case in [9], the groupoid MG is given a topology such that MG is a topological groupoid with this topology.

Let G be a locally sectionable topological groupoid and W an open subset containing all the identities. In this paper we use a criterion obtained from holonomy to prove that the monodromy groupoid MG has a structure of topological groupoid such that each star $(MG)_x$ is a universal cover of G_x. In [6] the groupoid associated with a pregroupoid is used to verify a monodromy property for MG, namely extendibility to MG of a local morphism on G.

2. Groupoids and topological groupoids

A *groupoid* G on O_G is a small category in which each morphism is an isomorphism. Thus G has a set of morphisms, which we call just *elements* of G, a set O_G of *objects* together with functions $\alpha, \beta \colon G \to O_G$, $\epsilon \colon O_G \to G$ such that $\alpha\epsilon = \beta\epsilon = 1_{O_G}$, the identity map. The functions α, β are called *initial* and *final* point maps respectively. If $a, b \in G$ and $\beta(a) = \alpha(b)$, then the *product* or *composite* ba exists such that $\alpha(ba) = \alpha(a)$ and $\beta(ba) = \beta(b)$. Further, this composite is associative, for $x \in O_G$ the element $\epsilon(x)$ denoted by 1_x acts as the identity, and each element a has an inverse a^{-1} such that $\alpha(a^{-1}) = \beta(a)$,

$\beta(a^{-1}) = \alpha(a)$, $aa^{-1} = (\epsilon\beta)(a)$, $a^{-1}a = (\epsilon\alpha)(a)$. The map $G \to G$, $a \mapsto a^{-1}$, is called the *inversion*.

In a groupoid G for $x, y \in O_G$ we write $G(x, y)$ for the set of all morphisms with initial point x and final point y. For $x \in O_G$ we denote the star $\{a \in G \colon \alpha(a) = x\}$ of x by G_x and the *costar* $\{a \in G \colon \beta(a) = x\}$ of x by G^x. In G the set O_G is mapped bijectively to the set of identities by $\epsilon \colon O_G \to G$. So we sometimes write O_G for the set of identities. Let G be a groupoid and W a subset of G such that $O_G \subseteq W$. We say G is *generated* by W if each element of G may be written as a product of elements of W.

Let G be a groupoid. A *subgroupoid* of G is a pair of subsets $H \subseteq G$ and $O_H \subseteq O_G$ such that $\alpha(H) \subseteq O_H$, $\beta(H) \subseteq O_H$, $1_x \in H$ for each $x \in O_H$ and H is closed under the partial multiplication and the inversion in G.

A *morphism* of groupoids H and G is a functor, that is, it consists of a pair of functions $f \colon H \to G$ and $O_f \colon O_H \to O_G$ preserving all the structures.

Definition 2.1. A *topological groupoid* is a groupoid G on O_G, together with topologies on G and O_G, such that the maps which define the groupoid structure are continuous, namely the initial and final point maps $\alpha, \beta \colon G \to O_G$, the object inclusion map $\epsilon \colon O_G \to G$, $x \mapsto \epsilon(x)$, the inversion $G \to G$, $a \mapsto a^{-1}$ and the partial multiplication $G_\alpha \times_\beta G \to G, (b, a) \mapsto ba$, where the pullback

$$G_\alpha \times_\beta G = \{(b, a) \in G \times G \colon \alpha(b) = \beta(a)\}$$

has the subspace topology from $G \times G$.

A *morphism* of topological groupoids $f \colon H \to G$ is a morphism of groupoids in which both maps $f \colon H \to G$ and $O_f \colon O_H \to O_G$ are continuous.

Note that in this definition the partial multiplication $G_\alpha \times_\beta G \to G, (b, a) \mapsto ba$ and the inversion map $G \to G, a \mapsto a^{-1}$ are continuous if and only if the map $\delta \colon G \times_\alpha G \to G, (b, a) \mapsto ba^{-1}$, called the groupoid difference map, is continuous, where the pullback

$$G \times_\alpha G = \{(b, a) \in G \times G \colon \alpha(b) = \alpha(a)\}$$

has the subspace topology from $G \times G$. Again if one of the maps α, β and the inversion are continuous, then the other map is continuous.

Let X be a topological space. Then $G = X \times X$ is a topological groupoid on X, in which each pair (y, x) is a morphism from x to y and the groupoid composite is defined by $(z, y)(y, x) = (z, x)$. The inverse of (y, x) is (x, y) and the identity at 1_x is the pair (x, x).

Note that in a topological groupoid, G, for each $a \in G(x, y)$ right translation $R_a \colon G_y \to G_x, b \mapsto ba$ and left translation $L_a \colon G^x \to G^y$, $b \mapsto ab$ are homeomorphisms.

A groupoid G in which each star G_x has a topology such that for each $a \in G(x, y)$ the right translation $R_a \colon G_y \to G_x, b \mapsto ba$ (and hence also the left translation $L_a \colon G^x \to G^y, b \mapsto ab$) is a homeomorphism, is called a *star topological groupoid*

3. Locally topological groupoids and Extendibility

The following definition is due to Ehresmann [8].

Definition 3.1. Let G be a groupoid and let $X = O_G$ be a topological space. An *admissible local section* of G is a function $\sigma : U \to G$ from an open set in X such that $\alpha\sigma(x) = x$ for all $x \in U$; $\beta\sigma(U)$ is open in X, and $\beta\sigma$ maps U homeomorphically to $\beta\sigma(U)$.

Let W be a subset of G containing O_G and let W have the structure of a topological space. We say that (α, β, W) is *locally sectionable* if for each $w \in W$ there is an admissible local section $\sigma : U \to G$ of G such that (i) $\sigma\alpha(w) = w$, (ii) $\sigma(U) \subseteq W$ and (iii) σ is continuous as a function from U to W. Such a σ is called a *continuous admissible local section.*

The following definition is due to Pradines [12] under the name "*morceau de groupoide différentiables*".

Definition 3.2. A *locally topological groupoid* is a pair (G, W) consisting of a groupoid G and a topological space W such that:

- i) $O_G \subseteq W \subseteq G$;
- ii) $W = W^{-1}$;
- iii) the set $W(\delta) = (W \times_\alpha W) \cap \delta^{-1}(W)$ is open in $W \times_\alpha W$ and the restriction of δ to $W(\delta)$ is continuous;
- iv) the restrictions to W of the source and target maps α and β are continuous, and the triple (α, β, W) is locally sectionable;
- v) W generates G as a groupoid.

Note that in this definition, G is a groupoid but does not need to have a topology. The locally topological groupoid (G, W) is said to be *extendible* if there can be found a topology on G making it a topological groupoid and for which W is an open subset. In general, (G, W) is not extendible, but there is a holonomy groupoid $Hol(G, W)$ and a morphism $\psi : Hol(G, W) \to G$ such that $Hol(G, W)$ admits the structure of topological groupoid and is the "minimal" such overgroupoid of G. The construction is given in detail in [1] and is outlined below.

It is easiest to picture locally topological groupoids (G, W) for groupoids G such that $\alpha = \beta$, so that G is just a bundle of groups. Here is a specific such example of a locally topological groupoid [1], which is not extendible.

Example 3.3. Let F be the bundle of groups $p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers and p is the first projection. The usual topology on $\mathbb{R} \times \mathbb{R}$ gives F the structure of a topological groupoid in which each $p^{-1}(x)$ is isomorphic as an additive group to $\mathbb{R}$.

Let N be the subbundle of F given by the union of the sets $\{(x, 0)\}$ if $x < 0$ and $\{x\} \times \mathbb{Z}$ if $x \geq 0$, where $\mathbb{Z}$ is the set of integers. Let G be the quotient bundle F/N and let $q : F \to G$ be the quotient morphism. Then the source map $\alpha : G \to \mathbb{R}$ has $\alpha^{-1}(x)$ isomorphic to $\mathbb{R}$ for $x < 0$ and to $\mathbb{R}/\mathbb{Z}$ for

$x \geq 0$. Let W' be the subset $\mathbb{R} \times (-\frac{1}{4}, \frac{1}{4})$ of F. Then q maps W' bijectively to $W = q(W')$; let W have the topology in which this map is a homeomorphism. It is easily checked that (G, W) is a locally topological groupoid. Suppose this locally topological groupoid is extended to a topological groupoid structure on G. Let s' be the section of p in which $x \mapsto (x, \frac{1}{8})$, and let $s = qs'$. Then s is an admissible section of α but $t = 9s$ is not. However $t(0) = q(0, \frac{1}{8})$. Let U be an open neighbourhood of $(\frac{1}{8}, 0)$ in $\mathbb{R}^2$ such that U is contained in W'. Then $p(U)$ is contained in W and is a neighbourhood of $t(0)$. But $t^{-1}q(U)$ is contained in $[0, \infty)$, so that t is not continuous. This gives a contradiction, and shows that the locally topological groupoid (G, W) is not extendible. By contrast, if we proceed as before but replace N by N_1, which is the union of the sets $\{(x, 0)\}$ for $x \leq 0$ and $\{x\} \times \mathbb{Z}$ for $x > 0$, then the resulting locally topological groupoid (G_1, W_1) is extendible.

Example 3.4. There is a variant of the last example in which F is as before, but this time N is the union of the groups $\{x\} \times (1 + |x|)\mathbb{Z}$ for all $x \in \mathbb{R}$. If one takes W' as before, and W is the image of W' in $G = F/N$, then the locally topological groupoid (G, W) can be extended to give a topological groupoid structure on G. However, now consider W as a differential manifold. The differential structure cannot be extended to make G a differential groupoid with W as submanifold. The reason is analogous to that given in the previous example, namely that such a differential structure would entail the existence of a local differentiable admissible section whose sum with itself is not differentiable, thus giving a contradiction.

Example 3.5. ([7]) Let X be a paracompact foliated manifold. Then there is an equivalence relation, written R_F, on X determined by the leaves. So R_F is a subspace of $X \times X$ and becomes a topological groupoid on X with the usual multiplication $(z, y)(y, x) = (z, x)$, for $(y, x), (z, y) \in R_F$.

For any subset U of X we write $R_F(U)$ for the equivalence relation on U whose classes are the plaques of U. If $\Lambda = \{(U_\lambda, \phi_\lambda)\}$ is a foliated atlas for X, we write $W(\Lambda)$ for the union of the sets $R_F(U_\lambda)$ for all domains U_λ of charts of Λ. Let $W(\Lambda)$ have its topology as a subspace of R_F and so of $X \times X$.

In [7] it is proved that the pair (R_F, W'), where W' derives from a refinement of Λ, is a locally topological groupoid. Some special foliated manifolds are given in which the locally topological groupoid (R_F, W) is not extendible.

There is a main globalisation theorem for a locally topological groupoid due to Aof-Brown [1], and a Lie version of this is stated in Brown-Mucuk [6]; it shows how a locally topological groupoid gives rise to its holonomy groupoid, which is a topological groupoid satisfying a universal property. This theorem gives a full statement and proof of a part of Théorème 1 of [12].

Theorem 3.6. (Globalisation Theorem) *Let (G, W) be a locally topological groupoid. Then there is a topological groupoid H, a morphism $\phi : H \to G$ of groupoids, and an embedding $i : W \to H$ of W to an open neighbourhood of O_H, such that:*

i) *ϕ is the identity on objects, $\phi i = id_W$, $\phi^{-1}(W)$ is open in H, and the restriction $\phi_W : \phi^{-1}(W) \to W$ of ϕ is continuous;*

ii) (universal property) *If A is a topological groupoid and $\zeta : A \to G$ is a morphism of groupoids such that:*

   a) *ζ is the identity on objects;*

   b) *The restriction $\zeta_W : \zeta(W) \to W$ of ζ is continuous and $\zeta^{-1}(W)$ is open in A and generates A;*

   c) *The triple (α_A, β_A, A) has enough continuous admissible local sections,*

   then there is a unique morphism $\zeta' : A \to H$ of topological groupoids such that $\phi\zeta' = \zeta$ and $\zeta' a = i\zeta a$ for $a \in \zeta^{-1}(W)$.

The groupoid H is called the *holonomy groupoid* $\mathrm{Hol}(G, W)$ of the locally topological groupoid (G, W); its essential uniqueness follows from the condition (ii) above.

We outline the proof of which full details are given in [1]. Some details of part of the construction are needed for Proposition 3.7.

Proof. (Outline) Let $\Gamma(G)$ be the set of all admissible local sections of G. Define a product on $\Gamma(G)$ by

$$(ts)x = (t\beta sx)(sx)$$

for two admissible local sections s and t. If s is an admissible local section then write s^{-1} for the admissible local section $\beta s\mathcal{D}(s) \to G, \beta sx \mapsto (sx)^{-1}$. With this product $\Gamma(G)$ becomes an inverse semigroup. Let $\Gamma^c(W)$ be the subset of $\Gamma(G)$ consisting of admissible local sections which have values in W and are continuous. Let $\Gamma^c(G, W)$ be the subsemigroup of $\Gamma(G)$ generated by $\Gamma^c(W)$. Then $\Gamma^c(G, W)$ is again an inverse semigroup. Intuitively, it contains information on the iteration of local procedures.

Let $J(G)$ be the sheaf of germs of admissible local sections of G. Thus the elements of $J(G)$ are the equivalence classes of pairs (x, s) such that $s \in \Gamma(G), x \in \mathcal{D}(s)$, and (x, s) is equivalent to (y, t) if and only if $x = y$ and s and t agree on a neighbourhood of x. The equivalence class of (x, s) is written $[s]_x$. The product structure on $\Gamma(G)$ induces a groupoid structure on $J(G)$ with X as the set of objects, and source and target maps $[s]_x \mapsto x, [s]_x \mapsto \beta sx$. Let $J^c(G, W)$ be the subsheaf of $J(G)$ of germs of elements of $\Gamma^c(G, W)$. Then $J^c(G, W)$ is generated as a subgroupoid of $J(G)$ by the sheaf $J^c(W)$ of germs of elements of $\Gamma^c(W)$. Thus an element of $J^c(G, W)$ is of the form

$$[s]_x = [s_n]_{x_n} \cdots [s_1]_{x_1}$$

where $s = s_n \cdots s_1$ with $[s_i]_{x_i} \in J^c(W), x_{i+1} = \beta s_i x_i, i = 1, \ldots, n$ and $x_1 = x \in \mathcal{D}(s)$.

Let $\psi : J(G) \to G$ be the final map defined by $\psi([s]_x) = s(x)$, where s is an admissible local section. Then $\psi(J^c(G, W)) = G$. Let $J_0 = J^c(W) \cap \ker \psi$. Then J_0 is a normal subgroupoid of $J^c(G, W)$; the proof is in [1] Lemma 2.2. The holonomy groupoid $\mathrm{Hol} = \mathrm{Hol}(G, W)$ is defined to be the quotient

$J^c(G,W)/J_0$. Let $p : J^c(G,W) \to \mathrm{Hol}$ be the quotient morphism and let $p([s]_x)$ be denoted by $\langle s\rangle_x$. Since $J_0 \subseteq \ker\psi$ there is a surjective morphism $\phi : \mathrm{Hol} \to G$ such that $\phi p = \psi$.

The topology on the holonomy groupoid Hol such that Hol with this topology is a topological groupoid, is constructed as follows. Let $s \in \Gamma^c(G,W)$. A partial function $\sigma_s : W \to \mathrm{Hol}$ is defined as follows. The domain of σ_s is the set of $w \in W$ such that $\beta w \in \mathcal{D}(s)$. A continuous admissible local section f through w is chosen and the value $\sigma_s w$ is defined to be

$$\sigma_s w = \langle s\rangle_{\beta w}\langle f\rangle_{\alpha w} = \langle sf\rangle_{\alpha w}.$$

It is proven that $\sigma_s w$ is independent of the choice of the local section f and that these σ_s form a set of charts. Then the initial topology with respect to the charts σ_s is imposed on Hol. With this topology Hol becomes a topological groupoid. The proof is in Aof-Brown [1]. □

From the construction of the holonomy groupoid we easily obtain the following extendibility condition, which is proved in [6].

Proposition 3.7. *The locally topological groupoid (G,W) is extendible to a topological groupoid structure on G if and only if the following condition holds:*

- **(1)** *if $x \in O_G$, and s is a product $s_n \cdots s_1$ of local sections about x such that each s_i lies in $\Gamma^c(W)$ and $s(x) = 1_x$, then there is a restriction s' of s to a neighbourhood of x such that s' has its image in W and is continuous, i.e. $s' \in \Gamma^c(W)$.*

Proof. The canonical morphism $\phi : H \to G$ is an isomorphism if and only if $\ker\psi \cap J^c(W) = \ker\psi$. This is equivalent to $\ker\psi \subseteq J^c(W)$. We now show that $\ker\psi \subseteq J^c(W)$ if and only if the condition (1) is satisfied.

Suppose $\ker\psi \subseteq J^c(W)$. Let $s = s_n \cdots s_1$ be a product of admissible local sections about $x \in O_G$ with $s_i \in \Gamma^c(W)$ and $x \in \mathcal{D}_s$ such that $s(x) = 1_x$. Then $[s]_x \in J^c(G,W)$ and $\psi([s]_x) = s(x) = 1_x$. So $[s]_x \in \ker\psi$, so that $[s]_x \in J^c(W)$. So there is a neighbourhood U of x such that the restriction $s \mid U \in \Gamma^c(W)$.

Suppose the condition (1) is satisfied. Let $[s]_x \in \ker\psi$. Since $[s]_x \in J^c(G,W)$, then $[s]_x = [s_n]_{x_n} \cdots [s_1]_{x_1}$ where $s = s_n \cdots s_1$ and $[s_i]_{x_i} \in J^c(W)$, $x_{i+1} = \beta s_i x_i, i = 1,\ldots,n$ and $x_1 = x \in \mathcal{D}(s)$. Since $s(x) = 1_x$, then by (1), $[s]_x \in J^c(W)$. □

In effect, Proposition 3.7 states that the non-extendibility of (G,W) arises from the *holonomically non trivial* elements of $J^c(G,W)$. Intuitively, such an element h is an iteration of local procedures (i.e. of elements of $J^c(W)$) such that h returns to the starting point (i.e. $\alpha h = \beta h$) but h does not return to the starting value (i.e. $\psi h \neq 1$).

The following result, which is given as Corollary 4.6 in [6], gives a circumstance in which this extendibility condition is easily seen to apply.

Corollary 3.8. *Let Q be a topological groupoid and let $p : M \to Q$ be a morphism of groupoids such that $p : O_M \to O_Q$ is the identity. Let W be an open subset of Q such that*

1. $O_Q \subseteq W$*;*
2. $W = W^{-1}$*;*
3. W *generates* Q*;*
4. (α_W, β_W, W) *is continuously locally sectionable;*

and suppose that $\tilde{\imath} : W \to M$ is given such that $p\tilde{\imath} = i : W \to Q$ is the inclusion and $W' = \tilde{\imath}(W)$ generates M.

Then M admits a unique structure of topological groupoid such that W' is an open subset and $p : M \to Q$ is a morphism of topological groupoids mapping W' homeomorphically to W.

Proof. It is easy to check that (M, W') is a locally topological groupoid. We prove that condition (1) in Proposition 3.7 is satisfied (with (G, W) replaced by (M, W')).

Suppose we are given the data of (1). Clearly, $ps = ps_n \cdots ps_1$, and so ps is continuous, since G is a topological groupoid. Since $s(x) = 1_x$, there is a restriction s' of s to a neighbourhood of x such that $Im(ps) \subseteq W$. Since p maps W' homeomorphically to W, then s' is continuous and has its image contained in W. So (1) holds, and by Proposition 3.7, the topology on W' is extendible to make M a topological groupoid. □

Remark 3.9. It may seem unnecessary to construct the holonomy groupoid in order to verify extendibility under condition (1) of Proposition 3.7. However the construction of the continuous structure on M in the last corollary, and the proof that this yields a topological groupoid, would have to follow more or less the steps given in Aof and Brown [1] as sketched above. Thus it is more sensible to rely on the general result. As Corollary 3.8 shows, the utility of (1) is that it is a checkable condition, both positively or negatively, and so gives clear proof of the non-existence or existence of non-trivial holonomy.

4. The Star Universal Cover of a Topological Groupoid

Let X be a topological space and suppose that each path component of X admits a simply connected covering space. It is standard that if $\pi_1 X$ is the fundamental groupoid of X, topologised as in Brown and Danish-Naruie [3] and $x \in X$, then the target map $\beta\colon (\pi_1 X)_x \to X$ is the universal covering map of X based at x. Let G be a topological groupoid. The groupoid MG is defined as follows. As a set, MG is the union of the stars $(\pi_1 G_x)_{1_x}$. The object set is the same as that of G. The initial point map $\alpha\colon MG \to X$ maps all of $(\pi_1 G_x)_{1_x}$ to x, while the final point map $\beta\colon MG \to X$ is on $(\pi_1 G_x)_{1_x}$ the composition of the two target maps

$$(\pi_1 G_x)_{1_x} \xrightarrow{\beta} G_x \xrightarrow{\beta} X.$$

As explained in [9] there is a groupoid multiplication on MG defined by concatenation, i.e.

$$[b] \circ [a] = [ba(1) + a]$$

where the $+$ inside the bracket denotes the usual composition of the paths. Here a is assumed to be a path in G_x from 1_x to $a(1)$, where $\beta(a(1)) = y$, say, so that b is a path in G_x, and for each $t \in [0,1]$, the product $b(t)a(1)$ is defined in G, yielding a path $b(a(1))$ from $a(1)$ to $b(1)a(1)$. It is straightforward to prove that in this way MG becomes a groupoid, and that the final maps of paths induces a morphism of groupoids $p\colon MG \to G$. If each G_x admits a simply connected cover at 1_x then we may topologise each $(MG)_x$ so that it is the universal cover of G_x based at 1_x, and then MG becomes a star topological groupoid, which means each star $(MG)_x$ has a topology such that each right translation (and hence each left translation) is a homeomorphism

We call MG the *star universal cover* of G.

If X is a topological space which has a simply connected cover and $G = X \times X$, then $MG = \pi_1(X)$. If G is a topological group, then MG is a universal cover of G.

Theorem 4.1. *Let G be a locally sectionable topological groupoid in which each star G_x is path connected and has a simply connected cover. Let W be an open subset of G containing O_G such that $W = W^{-1}$ and W generates G. Suppose that each star $W_x = W \cap G_x$ is connected and simply connected. Then the groupoid MG constructed above may be given a structure of topological groupoid such that each star $(MG)_x$ is a universal cover of G_x and W is isomorphic to an open subset $\widetilde{W}$ of MG.*

Proof. To get a topology on MG as required we use Corollary 3.8. For this we first define a map $\widetilde{\imath}\colon W \to MG$ as follows: Let $u \in W(x,y)$, where $W(x,y) = W \cap G(x,y)$, then $u \in W_x$. Since W_x is path connected, there is a path a in W_x from 1_x to u. Here note that $1_x \in W_x$ since $O_G \subseteq W$. Define $\widetilde{\imath}(u)$ to be the unique homotopy class of a in W_x. Note that since W_x is simply connected, $\widetilde{\imath}$ is well defined.

The map $\widetilde{\imath}\colon W \to MG$ is injective. For if $u, v \in W$ such that $\widetilde{\imath}(u) = \widetilde{\imath}(v)$, then we have $p\widetilde{\imath}(u) = p\widetilde{\imath}(v)$ and so $u = v$.

Let $\widetilde{W}$ denote the image of W under the map $\widetilde{\imath}\colon W \to MG$. Thus $\widetilde{W}$ has a topology such that the map $\widetilde{\imath}\colon W \to \widetilde{W}$ is a homeomorphism. Note that by assumption the pair (G, W) satisfies the conditions 1-4 of Corollary 3.8. So to apply Corollary 3.8 to the pair $(MG, \widetilde{W})$, we only need to prove that the subset $\widetilde{W}$ generates MG as a groupoid. We prove this in the following Lemma.

Lemma 4.2. *The subset $\widetilde{W}$ generates MG as a groupoid.*

Proof. For this let $[a] \in MG(x,y)$. So a is a path from 1_x to $g \in G(x,y)$. Let $S \subseteq [0,1]$ be the set of $s \in [0,1]$ such that $a^s = a|_{[0,s]}$, the restriction of a to $[0,s]$, can be written $a^s = a_n \circ \cdots \circ a_1$ for some n and $\operatorname{Im} a_i \subseteq W$. Since

$S \subseteq [0,1]$, S is bounded above by 1, and so $u = \sup S$ exists. Then we prove the following:

i) $u \in S$
ii) $u = 1$.

To prove (i), let $a(u) \in G(x, x_u)$. Then the map $f\colon [0,1] \to G_{x_u}$ defined by $f(t) = a(t)(a(u))^{-1}$ is continuous and $f(u) = 1_{x_u} \in W$. Hence there is an $\epsilon > 0$ such that $f([u-\epsilon, u+\epsilon]) \subseteq W$. Hence the composition map

$$\delta_W \circ (f \times f)\colon [u-\epsilon, u+\epsilon] \times [u-\epsilon, u+\epsilon] \to W \times_\alpha W \to G$$

$$(t_1, t_2) \mapsto a(t_1)(a(t_2))^{-1}$$

is continuous, where δ_W is the restriction to $W \times_\alpha W \to G$ of the groupoid difference map $\delta\colon G \times_\alpha G \to G, (b,a) \mapsto ba^{-1}$. Hence there is an $\epsilon' > 0$ such that $\epsilon' < \epsilon$ and

$$\delta_W(f \times f)([u-\epsilon', u+\epsilon'] \times [u-\epsilon', u+\epsilon']) \subseteq W \tag{$\star$}$$

Since $u = \sup S$, there is an element $s \in S$ such that $u - \epsilon' < s$. Hence a^s can be written as $a_n \circ \cdots \circ a_1$ for n with $\operatorname{Im} a_i \subseteq W$ and so we have that $a_u = a_{n+1} \circ \cdots \circ a_1$ where $a_{n+1}(t) = a(t)(a(s))^{-1}$ for $t \in [s,u]$. By ($\star$) we have that $\operatorname{Im} a_{n+1} \subseteq W$. Hence $u \in S$.

To prove (ii) suppose that $u < 1$. Since $u \in S$, we have $a^u = a_n \circ \cdots \circ a_1$ for some n such that $\operatorname{Im} a_i \subseteq W$. Let $a_i(1) = g_i \in G(x_{i-1}, x)$ with $x_0 = x$ and $x_n = y$. Hence we have $a(u) = g_n \circ \cdots \circ g_1$ and the path a can be divided into small paths as

$$a = a(u+\epsilon) + a(u) + (a_n \circ \cdots \circ a_1)$$

where $\operatorname{Im} a_i \subseteq W$. Since the map

$$[u,1] \to G_{x_n}, t \mapsto a(t)(a(u))^{-1}$$

is continuous there is an $\epsilon > 0$ such that $a(t)(a(u))^{-1} \in W$ for $t \in [u, u+\epsilon]$. Hence $a^{u+\epsilon} = a_{n+1} \circ (a_n \cdots a_1)$ with $a_{n+1}(t) = a(t)((a(u))^{-1}$ for $t \in [u, u+\epsilon]$. Hence we have that $a^{u+\epsilon} \in S$, which is a contradiction. This proves that $u = 1$. □

Hence by Corollary 3.8, the groupoid MG has a unique structure of topological groupoid such that $\widetilde{W}$ is open in MG and $p\colon MG \to G$ is a morphism of the topological groupoids. □

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References

[1] M. E.-S. A.-F. Aof and R. Brown, *The holonomy groupoid of a locally topological groupoid*, Topology Appl. **47** (1992), 97–113.

[2] R. Brown, *Topology; A Geometric account of General Topology, Homotopy Types and the Fundamental Groupoid* (Ellis Horwood, Chichester, 1988).

[3] R. Brown and G. Danesh-Naruie, *The fundamental groupoid as a topological groupoid*, Proc. Edinburgh Math. Soc. **19 (2)** (1975), 237–244.

[4] R. Brown and İ. İçen, *Lie local subgroupoids and their Lie holonomy and monodromy groupoids*, Topology Appl. **115** (2001), 125–138.

[5] R. Brown, İ. İçen and O. Mucuk, *Local subgroupoids II: Examples and properties*, Topology Appl. **127** (2003), 393–408.

[6] R. Brown and O. Mucuk, *The monodromy groupoid of a Lie groupoid*, Cah. Top. Géom. Diff. Cat. **36** (1995), 345–369.

[7] R. Brown and O. Mucuk, *Foliations, locally Lie groupoids and holonomy*, Cah. Top. Géom. Diff. Cat. **37** (1996), 61–71.

[8] C. Ehresmann, *Catégories topologiques et catégories différentiables*, Coll. Géom. Diff. Glob. Bruxelles (1959), 137–150.

[9] K. C. H. Mackenzie, *Lie groupoids and Lie algebroids in differential geometry*, London Math. Soc. Lecture Note Series **124**, (Cambridge University Press, 1987).

[10] O. Mucuk, *Covering groups of non-connected topological groups and the monodromy groupoid of a topological groupoid*, PhD Thesis, University of Wales, 1993.

[11] O. Mucuk, *Locally topological groupoid*, Turkish Journal of Mathematics **21-2** (1997), 235–243.

[12] J. Pradines, *Théorie de Lie pour les Groupoides différentiables, relation entre propriétés locales et globales*, Comptes Rendus Acad. Sci. Paris **263** (1966), 907–910.

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O. Mucuk
Erciyes University,
Faculty of Science and Art,
Department of Mathematics,
Kayseri,
Turkey.

E-mail address: mucuk@erciyes.edu.tr

İ. İçen
İnönü University,
Faculty of Science and Art,
Department of Mathematics,
Malatya,
Turkey.

E-mail address: iicen@inonu.edu.tr