Functorial approach structures

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Abstract. We show that there exists at least a proper class of functorial approach structures, i.e., right inverses to the forgetful functor $T : \text{AP} \to \text{Top}$ (where $\text{AP}$ denotes the topological construct of approach spaces and contractions as introduced by R. Lowen). There is however a great difference in nature of these functorial approach structures when compared to the quasi-uniform paradigm which has been extensively studied by the first author: whereas it is well-known from [2] that a large class of epireflective subcategories of $\text{Top}_0$ can be 'parametrized' using the interaction of functorial quasi-uniformities with the quasi-uniform bicompletion, we show that using functorial approach structures together with the approach bicompletion developed in [10], only $\text{Top}_0$ itself can be retrieved in this way.


Keywords: Approach space, (approach) bicompleteness, epireflective subcategory, functorial approach structure, spanning, topological space.

1. Introduction and Preliminaries

In [2, 3, 4, 5, 7, 8, 9] so-called functorial (quasi-)uniformities were extensively studied. A functorial quasi-uniformity, is a functor $F : \text{Top} \to \text{QU}$ (where $\text{Top}$, resp. $\text{QU}$, stands for the topological construct of topological spaces and continuous maps, resp. of quasi-uniform spaces and uniformly continuous maps) which is a section for the usual forgetful functor $T_{qu} : \text{QU} \to \text{Top}$, i.e. such that $T_{qu}F = 1_{\text{Top}}$. First of all, let us recall from [3, 5] that there is a one-to-one correspondence between functorial quasi-uniformities in the above sense, and functorial quasi-uniformities $F : \text{Top}_0 \to \text{QU}_0$ in the $T_0$ case.

We refer to [11] as our blanket reference for categorical material and to [12] for all information about quasi-uniformities. Let us only mention that for the order in the fibres of a topological construct $\text{A}$, we take the opposite convention to the one taken in [11]: if $A, B$ are two objects on the same underlying set, we call $A$ finer than $B$ (or $B$ coarser than $A$), and write $A \geq B$, iff the identity map on the underlying set becomes a morphism $A \to B$. Then all the fibres
become complete lattices. One of the most important results about functorial quasi-uniformities, is their interplay with the quasi-uniform bicompletion (cf. [2]). It was shown by the first named author that functorial quasi-uniformities can e.g. be used to classify epireflective subcategories of $\text{Top}_0$, in the sense that for every (full) epireflective subcategory $E$ of $\text{Top}_0$ with

$$|\text{Sob}| \subseteq |E| \subseteq |\text{TopBicompl}_0|$$

(where $\text{Sob}$ stands for the subcategory of $\text{Top}_0$ formed by all sober objects and $\text{TopBicompl}_0$ for the one formed by all topologically bicomplete $T_0$-spaces (i.e. those $T_0$ topological spaces admitting a bicomplete quasi-uniformity)), there exists a functorial quasi-uniformity $F : \text{Top}_0 \to \text{QU}_0$ such that

$$E = \{ X \in \text{Top}_0 \mid FX \text{ bicomplete} \}.$$

We refer to [2] for a survey on this topic.

In [13], the topological construct $\text{AP}$ of approach spaces and contractions was introduced by R. Lowen as a quantified supercategory of $\text{Top}$ and it has been proved since then that approach spaces are an interesting framework for quantitative topology (see e.g. [14, 15, 16, 19] for applications of approach theory to hyperspaces or topological vector spaces). For a detailed motivation and more information about $\text{AP}$, we refer the reader to [13]. For an approach space $X$, we will write $\underline{X}$ for its underlying set and $\delta_X$ for its approach distance. In this context, $\text{Top}$ can be proved to be concretely bicoreflectively embedded into $\text{AP}$, and the corresponding concrete bicoreflector $T : \text{AP} \to \text{Top}$ plays the role of forgetful functor in this setting. Recall that it was shown in [17] that the $T_0$-objects in the sense of Maruy [18], in the setting of approach spaces, are exactly those approach spaces with $T_0$ topological coreflection. We denote by $\text{AP}_0$ the corresponding subcategory of $\text{AP}$.

Recently, in [10], the present authors derived a notion of symmetry for approach spaces, and together with it a categorically satisfactory (i.e. sub-firmly epireflective in the sense of [19]) notion of approach bicompleteness and bicompletion, which has a totally different behavior compared to the behavior in the quasi-uniform case. This different behavior again highlights the structural difference between approach spaces and quasi-uniform spaces: although both of them can be described using pseudo-quasi-metrics, approach spaces simultaneously quantify topological and (pseudo-quasi)-metric spaces, so with regard to concepts such as ‘bicompleteness’ or ‘Cauchy filters’, a very different paradigm compared to the quasi-uniform one is to be expected.

Let us now for the moment only recall that the category $\text{pqMet}_\infty$ of extended pseudo-quasi-metric (or $\infty$ pq-metric) spaces can be fully and concretely bicoreflectively embedded into $\text{AP}$, and that a $T_0$ approach space $X = (\underline{X}, \delta_X)$ is approach bicomplete iff the $\infty$ pq-metric space $(X, d_{\delta_X})$ is bicomplete in the usual sense, where

$$d_{\delta_X}(x, y) := \delta_X(x, \{y\}), \quad x, y \in \underline{X}. $$
We now want to address the question, whether or not, functorial approach structures and approach bicompleteness can be used to capture (preferably more) epireflective subcategories of $\text{Top}_0$.

2. Results

In all that follows,

$$T : \text{AP} \to \text{Top}$$

denotes the topological bicoreflector, which in approach theory serves as the underlying functor. We refer to [13] for a detailed description of $T$. A functor

$$F : \text{Top} \to \text{AP}$$

which is a section for $T$, i.e. for which

$$TF = 1_{\text{Top}},$$

will be called a functorial approach structure.

We first need an obvious lemma characterizing all $T$-sections. The notation $\text{PrTop}$ stands for the topological construct of pre-topological spaces and continuous maps. For any pre-topological space, we use $X$ for its underlying set and $\text{cl}_X$ for the corresponding closure operator on $X$.

**Lemma 2.1.** $F : \text{Top} \to \text{AP}$ is a section for $T : \text{AP} \to \text{Top}$ if and only if $F$ can be written as an approach tower

$$(F_\varepsilon : \text{Top} \to \text{PrTop})_{\varepsilon \geq 0}$$
of concrete functors with

1. $F_0$ is the embedding of $\text{Top}$ into $\text{PrTop}$, which we denote by $1_{\text{Top}}$,
2. $F_\infty$ is the indiscrete functor, which we denote by $I$ and which equips each set with the coarsest possible topology,
3. $\forall X \in |\text{Top}|, \forall \varepsilon \geq 0 : F_\varepsilon X = \bigvee_{\varepsilon < \gamma} F_\gamma X,$
4. $\forall X \in |\text{Top}|, \forall \varepsilon, \gamma \geq 0 : \text{cl}_{F_\varepsilon} X \circ \text{cl}_{F_\gamma} X \geq \text{cl}_{F_{\varepsilon + \gamma}} X.$

**Proof.** This immediately follows from the description of approach spaces in terms of towers (see [13]) and the fact that $F_0 = TF$. □

Next we show that, like in the quasi-uniform paradigm, there are “enough” functorial approach structures. The proof however becomes more intricate.

**Theorem 2.2.** The conglomerate of $T$-sections is at least a proper class.

**Proof.** Suppose that the conglomerate of all different $T$-sections would be in one-to-one correspondence with a set of cardinality $\kappa$. Then consider the cardinal number $2^\kappa > \kappa$. Note that $2^\kappa$ also is an (initial) ordinal number and that $\Gamma := \{\alpha \mid \alpha \text{ ordinal number and } \alpha < 2^\kappa\}$, equipped with the inclusion $\subseteq$ is a lattice without top element. It was proved in [12] that $(\Gamma, \subseteq)$ has a lattice-isomorphic representation within the large lattice of bireflective subcategories
of $\mathbf{Top}$ (with the natural order defined there). This entails that we can find a class $\mathcal{R}$ of mutually different bireflective subcategories of $\mathbf{Top}$ which is in one-to-one correspondence with the set $2^\kappa$. (Note that we make no distinction between a bireflective subcategory and its corresponding bireflector, and that we consider such a bireflector as an endofunctor on $\mathbf{Top}$). For every $R \in \mathcal{R}$, we define:

$$F_R^\varepsilon := \begin{cases} I & \varepsilon = \infty, \\ R & \varepsilon \in [1, \infty[ \\ 1_{\mathbf{Top}} & \varepsilon \in [0, 1[ \end{cases}$$

(viewed as a functor into $\mathbf{PrTop}$),

$$1_{\mathbf{Top}} \in [0, 1[ \ (\text{viewed as a functor into } \mathbf{PrTop}).$$

Then according to Lemma 2.1, it is clear that $F := (F_R^\varepsilon)_{\varepsilon \geq 0}$ is a $T$-section, and that $\{F^R | R \in \mathcal{R}\}$ is a class of mutually different $T$-sections, being in one-to-one correspondence with the set $2^\kappa$, yielding a contradiction with the definition of $\kappa$. $\square$

Finally, we come to proving our main theorem showing that “locally around the 0-level”, all functorial approach structures however become trivial. First note that, with exactly the same proof as in the (quasi)-uniform cases treated in [7, 8, 2], we can prove that every $T$-section can be obtained through the spanning construction as defined by the first author. This means that given a $T$-section $F$, there exists a class $A \subset |\mathbf{AP}|$ for which

$$T_A := \{TA | A \in A\}$$

is initially dense in $\mathbf{Top}$ and such that for all $X \in |\mathbf{Top}|$ the source

$$(f : FX \to A)_{A \in A, f \in \mathbf{Top}(X,T_A)}$$

is $\mathbf{AP}$-initial. To summarize this we write $F = \langle A \rangle$ and we say that “$A$ spans $F$”.

**Theorem 2.3.** For every $T$-section $F$, there exists $\gamma > 0$ such that

$$\forall \varepsilon \in [0, \gamma[ : F_\varepsilon = 1_{\mathbf{Top}}.$$  

**Proof.** Take an arbitrary $T$-section $F : \mathbf{Top} \to \mathbf{AP}$. According to Lemma 2.1, we can write $F$ as a tower $(F_\varepsilon)_{\varepsilon \geq 0}$ of functors subject to the conditions listed there. On the other hand we know from the general spanning construction that there exists $A \subset |\mathbf{AP}|$ such that $T_A$ is initially dense in $\mathbf{Top}$ and $F = \langle A \rangle$. In particular this means for the Sierpinski space $\emptyset$, that the source

$$(f : \emptyset \to TA)_{A \in A, f \in \mathbf{Top}(\emptyset,T_A)}$$

is initial in $\mathbf{Top}$. This clearly can only happen when there exists $B \in A$ such that $\emptyset$ can be embedded into $TB$ as a topological subspace. This means that we can find $x, y \in B$ with

$$\delta_B(x, \{y\}) = 0 \quad \text{and} \quad \gamma := \delta_B(y, \{x\}) > 0.$$  

Now fix $\varepsilon \in ]0, \gamma[$. Then obviously the previous implies that $\emptyset$ still is a pre-topological subspace of $B_\varepsilon := (B, t^B_\varepsilon)$ (here $t^B_\varepsilon$ is the pre-topological closure on $B_\varepsilon$).
the level $\varepsilon$ in the approach tower corresponding to $\delta_B$, i.e. for all $Y \subseteq B$,
\[
t^B_\varepsilon(Y) := \{b \in B \mid \delta_B(b,Y) \leq \varepsilon\}.
\]
Fix $X \in |\text{Top}|$. Because $F = \langle A \rangle$, the source
\[
(f : FX \to C)_{C \in A, f \in \text{Top}(X,TC)}
\]
is initial in $\text{AP}$. Therefore, $F_\varepsilon X$ has to be finer than the initial pre-topological structure on $X$ for the $\text{PrTop}$-source
\[
(\mathcal{f} : X \to C)_{C \in A, f \in \text{Top}(X,TC)}.
\]
The initial $\text{PrTop}$-structure being the coarsest one on $X$ making all functions in the source (2.2) above continuous, it certainly is coarser than $cl_X$. Because $\mathcal{S}$ is a pretopological subspace of $B_\varepsilon$, it is clear on the other hand that the initial structure for the source (2.2) above at the same time has to be finer than the initial $\text{PrTop}$-structure for the source
\[
(\mathcal{f} : X \to \mathcal{S})_{f \in \text{PrTop}(X,\mathcal{S})}.
\]
Because $\text{Top}$ is fully and concretely embedded as an initially closed subcategory in $\text{PrTop}$, and because $\mathcal{S}$ is initially dense in $\text{Top}$, the initial $\text{PrTop}$-structure for the latter source is $cl_X$. So the initial $\text{PrTop}$-structure for (2.2) is $cl_X$, whence $F_\varepsilon X \geq X$ and since automatically $F_\varepsilon X \leq F_0 X = X$, we are done. □

Let us now recall from [3] that, with the same argument as in the quasi-uniform case used in [9], it follows from the universality of $\text{AP}$ in the sense of [18] (meaning that $\text{AP}$ is the bireflective = initial hull of its $T_0$-objects), which was proved in [17], that there is a one-to-one correspondence between $T$-sections, and sections to the functor
\[
T|_{\text{AP}_0} : \text{AP}_0 \to \text{Top}_0.
\]
In one direction this correspondence is simply given by restriction of $T$-sections to $\text{Top}_0$. We therefore immediately have the analogue of Theorem 2.3 in this setting, providing a description of all $T|_{\text{AP}_0}$-sections. This yields that we automatically also obtain:

**Theorem 2.4.** For every $T|_{\text{AP}_0}$-section $F$, there exists $\gamma > 0$ such that
\[
\forall \varepsilon \in [0,\gamma] : F_\varepsilon = 1_{\text{Top}_0}.
\]

Now take an arbitrary $T|_{\text{AP}_0}$-section
\[
F : \text{Top}_0 \to \text{AP}_0
\]
and let $\gamma > 0$ be as in the theorem above. Then for all $X \in |\text{Top}_0|$, the $\infty$ pq-metric $d_{FX}$ only takes values in
\[
\{0\} \cup [\gamma, +\infty]
\]
and therefore obviously is a bicomplete metric on $X$, whence $FX$ is automatically approach bicomplete. This answers the question posed at the end of the first paragraph in the negative, again showing a drastically different behaviour
of the approach setting in comparison to the quasi-uniform one: the only epireflective subcategory of $\text{Top}_0$ we can retrieve via functorial approach structures is $\text{Top}_0$ itself.

**References**


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