Applied General Topology
© Universidad Politécnica de Valencia
Volume 4, No. 1, 2003
pp. 91–97

Functorial approach structures

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Abstract. We show that there exists at least a proper class of functorial approach structures, i.e., right inverses to the forgetful functor $T : \mathbf{AP} \to \mathbf{Top}$ (where **AP** denotes the topological construct of approach spaces and contractions as introduced by R. Lowen). There is however a great difference in nature of these functorial approach structures when compared to the quasi-uniform paradigm which has been extensively studied by the first author: whereas it is well-known from [2] that a large class of epireflective subcategories of $\mathbf{Top_0}$ can be 'parametrized' using the interaction of functorial quasi-uniformities with the quasi-uniform bicompletion, we show that using functorial approach structures together with the approach bicompletion developed in [10], only $\mathbf{Top_0}$ itself can be retrieved in this way.

2000 AMS Classification: 18B30, 18B99, 54B30, 54E15, 54E99.
Keywords: Approach space, (approach) bicompleteness, epireflective subcategory, functorial approach structure, spanning, topological space.

1. Introduction and Preliminaries

In [2, 3, 4, 5, 7, 8, 9] so-called functorial (quasi-)uniformities were extensively studied. A functorial quasi-uniformity, is a functor $F : \mathbf{Top} \to \mathbf{QU}$ (where **Top**, resp. **QU**, stands for the topological construct of topological spaces and continuous maps, resp. of quasi-uniform spaces and uniformly continuous maps) which is a section for the usual forgetful functor $T_{\mathrm{qu}} : \mathbf{QU} \to \mathbf{Top}$, i.e. such that $T_{\mathrm{qu}}F = \mathbf{1_{Top}}$. First of all, let us recall from [3, 9] that there is a one-to-one correspondence between functorial quasi-uniformities in the above sense, and functorial quasi-uniformities $F : \mathbf{Top_0} \to \mathbf{QU_0}$ in the T_0 case.

We refer to [1] as our blanket reference for categorical material and to [11] for all information about quasi-uniformities. Let us only mention that for the order in the fibres of a topological construct **A**, we take the opposite convention to the one taken in [1]: if A, B are two objects on the same underlying set, we call A finer than B (or B coarser than A), and write $A \geq B$, iff the identity map on the underlying set becomes a morphism $A \to B$. Then all the fibres become complete lattices. One of the most important results about functorial quasi-uniformities, is their interplay with the quasi-uniform bicompletion (cf. [2]). It was shown by the first named author that functorial quasi-uniformities can e.g. be used to classify epireflective subcategories of $\mathbf{Top_o}$, in the sense that for every (full) epireflective subcategory $\mathbf{E}$ of $\mathbf{Top_o}$ with

$$|\mathbf{Sob}| \subseteq |\mathbf{E}| \subseteq |\mathbf{TopBicompl_o}|$$

(where $\mathbf{Sob}$ stands for the subcategory of $\mathbf{Top_o}$ formed by all sober objects and $\mathbf{TopBicompl_o}$ for the one formed by all topologically bicomplete T_0-spaces (i.e. those T_0 topological spaces admitting a bicomplete quasi-uniformity)), there exists a functorial quasi-uniformity $F : \mathbf{Top_o} \to \mathbf{QU_o}$ such that

$$\mathbf{E} = \{X \in \mathbf{Top_o} \mid FX \text{ bicomplete}\}. \tag{1.1}$$

We refer to [2] for a survey on this topic.

In [13], the topological construct $\mathbf{AP}$ of approach spaces and contractions was introduced by R. Lowen as a quantified supercategory of $\mathbf{Top}$ and it has been proved since then that approach spaces are an interesting framework for quantitative topology (see e.g. [14, 15, 16, 19] for applications of approach theory to hyperspaces or topological vector spaces). For a detailed motivation and more information about $\mathbf{AP}$, we refer the reader to [13]. For an approach space X, we will write $\underline{X}$ for its underlying set and δ_X for its approach distance. In this context, $\mathbf{Top}$ can be proved to be concretely bicoreflectively embedded into $\mathbf{AP}$, and the corresponding concrete bicoreflector $T : \mathbf{AP} \to \mathbf{Top}$ plays the role of forgetful functor in this setting. Recall that it was shown in [17] that the T_0-objects in the sense of Marny [18], in the setting of approach spaces, are exactly those approach spaces with T_0 topological coreflection. We denote by $\mathbf{AP_o}$ the corresponding subcategory of $\mathbf{AP}$.

Recently, in [10], the present authors derived a notion of symmetry for approach spaces, and together with it a categorically satisfactory (i.e. sub-firmly epireflective in the sense of [6]) notion of approach bicompleteness and bicompletion, which has a totally different behavior compared to the behavior in the quasi-uniform case. This different behavior again highlights the structural difference between approach spaces and quasi-uniform spaces: although both of them can be described using pseudo-quasi-metrics, approach spaces simultaneously quantify topological and (pseudo-quasi)-metric spaces, so with regard to concepts such as 'bicompleteness' or 'Cauchy filters', a very different paradigm compared to the quasi-uniform one is to be expected.

Let us now for the moment only recall that the category $\mathbf{pqMet^\infty}$ of extended pseudo-quasi-metric (or ∞ pq-metric) spaces can be fully and concretely bicoreflectively embedded into $\mathbf{AP}$, and that a T_0 approach space $X = (\underline{X}, \delta_X)$ is approach bicomplete iff the ∞ pq-metric space (X, d_{δ_X}) is bicomplete in the usual sense, where

$$d_{\delta_X}(x, y) := \delta_X(x, \{y\}), \quad x, y \in \underline{X}.$$

We now want to address the question, whether or not, functorial approach structures and approach bicompleteness can be used to capture (preferably more) epireflective subcategories of $\mathbf{Top_0}$.

2. Results

In all that follows,

$$T : \mathbf{AP} \to \mathbf{Top}$$

denotes the topological bicoreflector, which in approach theory serves as the underlying functor. We refer to [13] for a detailed description of T. A functor

$$F : \mathbf{Top} \to \mathbf{AP}$$

which is a section for T, i.e. for which

$$TF = \mathbf{1_{Top}},$$

will be called a *functorial approach structure.*

We first need an obvious lemma characterizing all T-sections. The notation **PrTop** stands for the topological construct of pre-topological spaces and continuous maps. For any pre-topological space, we use $\underline{X}$ for its underlying set and cl_X for the corresponding closure operator on $\underline{X}$.

Lemma 2.1. *$F : \mathbf{Top} \to \mathbf{AP}$ is a section for $T : \mathbf{AP} \to \mathbf{Top}$ if and only if F can be written as an approach tower*

$$(F_\varepsilon : \mathbf{Top} \to \mathbf{PrTop})_{\varepsilon \geq 0}$$

of concrete functors with

1. *F_0 is the embedding of **Top** into **PrTop**, which we denote by $\mathbf{1_{Top}}$,*
2. *F_∞ is the indiscrete functor, which we denote by I and which equips each set with the coarsest possible topology,*
3. $\forall\, X \in |\mathbf{Top}|,\ \forall\, \varepsilon \geq 0:$
$$F_\varepsilon X = \bigvee_{\varepsilon < \gamma} F_\gamma X,$$
4. $\forall\, X \in |\mathbf{Top}|,\ \forall\, \varepsilon, \gamma \geq 0:\ \mathrm{cl}_{F_\gamma X} \circ \mathrm{cl}_{F_\varepsilon X} \geq \mathrm{cl}_{F_{\gamma+\varepsilon} X}.$

Proof. This immediately follows from the description of approach spaces in terms of towers (see [13]) and the fact that $F_0 = TF$. □

Next we show that, like in the quasi-uniform paradigm, there are "enough" functorial approach structures. The proof however becomes more intricate.

Theorem 2.2. *The conglomerate of T-sections is at least a proper class.*

Proof. Suppose that the conglomerate of all different T-sections would be in one-to-one correspondence with a set of cardinality κ. Then consider the cardinal number $2^\kappa > \kappa$. Note that 2^κ also is an (initial) ordinal number and that $\Gamma := \{\alpha \mid \alpha \text{ ordinal number and } \alpha < 2^\kappa\}$, equipped with the inclusion $\subseteq$ is a lattice without top element. It was proved in [12] that $(\Gamma, \subseteq)$ has a lattice-isomorphic representation within the large lattice of bireflective subcategories of **Top** (with the natural order defined there). This entails that we can find a class $\mathcal{R}$ of mutually different bireflective subcategories of **Top** which is in one-to-one correspondence with the set 2^κ. (Note that we make no distinction between a bireflective subcategory and its corresponding bireflector, and that we consider such a bireflector as an endofunctor on **Top**). For every $R \in \mathcal{R}$, we define:

$$F^R_\varepsilon := \begin{cases} I & \varepsilon = \infty, \\ R & \varepsilon \in [1, \infty[\text{ (viewed as a functor into } \mathbf{PrTop}), \\ \mathbf{1_{Top}} & \varepsilon \in [0, 1[\text{ (viewed as a functor into } \mathbf{PrTop}). \end{cases}$$

Then according to Lemma 2.1, it is clear that $F^R := (F^R_\varepsilon)_{\varepsilon \geq 0}$ is a T-section, and that $\{F^R \mid R \in \mathcal{R}\}$ is a class of mutually different T-sections, being in one-to-one correspondence with the set 2^κ, yielding a contradiction with the definition of κ. □

Finally, we come to proving our main theorem showing that "locally around the 0-level", all functorial approach structures however become trivial. First note that, with exactly the same proof as in the (quasi)-uniform cases treated in [7, 8, 2], we can prove that every T-section can be obtained through the spanning construction as defined by the first author. This means that given a T-section F, there exists a class $\mathcal{A} \subset |\mathbf{AP}|$ for which

$$T\mathcal{A} := \{\, TA \mid A \in \mathcal{A}\,\}$$

is initially dense in **Top** and such that for all $X \in |\mathbf{Top}|$ the source

$$(f : FX \to A)_{A \in \mathcal{A},\, f \in \mathbf{Top}(X, TA)}$$

is **AP**-initial. To summarize this we write $F = \langle \mathcal{A}\,\rangle$ and we say that "$\mathcal{A}$ spans F".

Theorem 2.3. *For every T-section F, there exists $\gamma > 0$ such that*

$$\forall\, \varepsilon \in [0, \gamma\,[\,: F_\varepsilon = \mathbf{1_{Top}}.$$

Proof. Take an arbitrary T-section $F : \mathbf{Top} \to \mathbf{AP}$. According to Lemma 2.1, we can write F as a tower $(F_\varepsilon)_{\varepsilon \geq 0}$ of functors subject to the conditions listed there. On the other hand we know from the general spanning construction that there exists $\mathcal{A} \subset |\mathbf{AP}|$ such that $T\mathcal{A}$ is initially dense in **Top** and $F = \langle \mathcal{A}\,\rangle$. In particular this means for the Sierpinski space \$, that the source

$$(f : \$ \to TA)_{A \in \mathcal{A},\ f \in \mathbf{Top}(\$, TA)}$$

is initial in **Top**. This clearly can only happen when there exists $B \in \mathcal{A}$ such that \$ can be embedded into TB as a topological subspace. This means that we can find $x, y \in \underline{B}$ with

$$\delta_B(x, \{y\}) = 0 \text{ and } \gamma := \delta_B(y, \{x\}) > 0.$$

Now fix $\varepsilon \in]\,0, \gamma\,[$. Then obviously the previous implies that \$ still is a pretopological subspace of $B_\varepsilon := (\underline{B},\, t^B_\varepsilon)$ (here t^B_ε is the pretopological closure on the level ε in the approach tower corresponding to δ_B, i.e. for all $Y \subset \underline{B}$,

$$t^B_\varepsilon(Y) := \{b \in \underline{B} \mid \delta_B(b, Y) \leq \varepsilon\}).$$

Fix $X \in |\mathbf{Top}|$. Because $F = \langle \mathcal{A} \rangle$, the source

$$(f : FX \to C)_{C \in \mathcal{A},\, f \in \mathbf{Top}(X, TC)}$$

is initial in **AP**. Therefore, $F_\varepsilon X$ has to be finer than the initial pre-topological structure on $\underline{X}$ for the **PrTop**-source

$$(f : \underline{X} \to C_\varepsilon)_{C \in \mathcal{A},\ f \in \mathbf{Top}(X, TC)}. \tag{2.2}$$

The initial **PrTop**-structure being the coarsest one on $\underline{X}$ making all functions in the source (2.2) above continuous, it certainly is coarser than cl_X. Because \$ is a pretopological subspace of B_ε, it is clear on the other hand that the initial structure for the source (2.2) above at the same time has to be finer than the initial **PrTop**-structure for the source

$$(f : \underline{X} \to \$)_{f \in \mathbf{PrTop}(X, \$)}.$$

Because **Top** is fully and concretely embedded as an initially closed subcategory in **PrTop**, and because \$ is initially dense in **Top**, the initial **PrTop**-structure for the latter source is cl_X. So the initial **PrTop**-structure for (2.2) is cl_X, whence $F_\varepsilon X \geq X$ and since automatically $F_\varepsilon X \leq F_0 X = X$, we are done. □

Let us now recall from [3] that, with the same argument as in the quasi-uniform case used in [9], it follows from the universality of **AP** in the sense of [18] (meaning that **AP** is the bireflective = initial hull of its T_0-objects), which was proved in [17], that there is a one-to-one correspondence between T-sections, and sections to the functor

$$T|_{\mathbf{AP_0}} : \mathbf{AP_0} \to \mathbf{Top_0}.$$

In one direction this correspondence is simply given by restriction of T-sections to $\mathbf{Top_0}$. We therefore immediately have the analogue of Theorem 2.3 in this setting, providing a description of all $T|_{\mathbf{AP_0}}$-sections. This yields that we automatically also obtain:

Theorem 2.4. *For every $T|_{\mathbf{AP_0}}$-section F, there exists $\gamma > 0$ such that*

$$\forall \varepsilon \in [0, \gamma[\ : F_\varepsilon = \mathbf{1}_{\mathbf{Top_0}}.$$

Now take an arbitrary $T|_{\mathbf{AP_0}}$-section

$$F : \mathbf{Top_0} \to \mathbf{AP_0}$$

and let $\gamma > 0$ be as in the theorem above. Then for all $X \in |\mathbf{Top_0}|$, the ∞ pq-metric d_{FX} only takes values in

$$\{0\} \cup [\gamma, +\infty]$$

and therefore obviously is a bicomplete metric on $\underline{X}$, whence FX is automatically approach bicomplete. This answers the question posed at the end of the first paragraph in the negative, again showing a drastically different behaviour of the approach setting in comparison to the quasi-uniform one: the only epireflective subcategory of $\mathbf{Top_0}$ we can retrieve via functorial approach structures is $\mathbf{Top_0}$ itself.

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Received January 2002
Revised December 2002

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