NORMAL SUBGROUPS WHOSE CONJUGACY CLASS GRAPH HAS DIAMETER THREE.

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Abstract
Let $G$ be a finite group and $N$ a normal subgroup of $G$. We determine the structure of $N$ when the graph associated to the $G$-conjugacy classes contained in $N$ has diameter three.

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1 Introduction
Let $G$ be a finite group and let $N$ be a normal subgroup of $G$ and let $x \in N$. We denote by $x^G = \{x^g \mid g \in G\}$ the $G$-conjugacy class of $x$. Let $\Gamma_G(N)$ be the graph associated to these $G$-conjugacy classes, which was defined in [2] as follows: its vertices are the $G$-conjugacy classes of $N$ of cardinality bigger than 1, that is, $G$-classes of elements in $N \setminus (\mathbb{Z}(G) \cap N)$, and two of them are joined by an edge if their sizes are not coprime. It was proved in [2] that $d(\Gamma_G(N)) \leq 3$ where $d(\Gamma_G(N))$ denotes the diameter of the graph. In this paper we analyze
the structure properties of $N$ when $d(\Gamma_G(N)) = 3$.

The above graph extends the ordinary graph, $\Gamma(G)$, which was formerly defined in [3], whose vertices are the non-central conjugacy classes of $G$ and two vertices are joined by an edge if their sizes are not coprime. The graph $\Gamma_G(N)$ can be viewed as the subgraph of $\Gamma(G)$ induced by those vertices of $\Gamma(G)$ which are vertices in $\Gamma_G(N)$. This fact does not allow to obtain directly properties of the graph of $G$-classes.

Concerning ordinary classes, L.S. Kazarin characterizes in [7] the structure of a group $G$ having two “isolated classes”. Remember that a group $G$ has isolated conjugacy classes if there exist elements $x, y \in G$ with coprime conjugacy class sizes such that every element of $G$ has conjugacy class size coprime to either $|x^G|$ or $|y^G|$. Particularly Kazarin determined the structure of the groups $G$ with $d(\Gamma(G)) = 3$. It should be noted that similar results have also been tested for other graphs. In [5], Dolfi defines the graph $\Gamma'(G)$ whose vertices are the elements of the set of all primes which occur as divisors of the lengths of the conjugacy classes of $G$, and two vertices $p, q$ are joined by an edge if there exists a conjugacy class in $G$ whose length is a multiple of $pq$. In [6] Dolfi and Casolo describe all finite groups $G$ for which $\Gamma'(G)$ is connected and has diameter three.

We have to remark that the primes dividing the $G$-conjugacy class sizes not necessarily divide $|N|$, it can occur the case when $N$ is abelian and it is non-central in $G$ and consequently we have not control on these primes. For this reason, we observe that new cases appear when we work with $G$-classes which are not contemplated in the ordinary case. The main result of this paper is the following theorem. From now on, if $H$ is a subgroup of a finite group $G$ we denote by $\pi(H)$ the set of primes dividing $|H|$.

**Theorem A.** Let $G$ be a finite group and $N \trianglelefteq G$. Suppose that $x^G$ and $y^G$ are two non-central $G$-conjugacy classes of $N$ such that any $G$-conjugacy of $G$ has size coprime with $|x^G|$ or $|y^G|$. Let $\pi_x = \pi(|x^G|)$, $\pi_y = \pi(|y^G|)$ and $\pi = \pi_x \cup \pi_y$. Then, $N = O_{\pi}(N) \times O_{\pi}(N)$ with $x, y \in O_{\pi}(N)$ which is a quasi-Frobenius group with abelian kernel and complement or $O_{\pi}(N) = P \times A$ with $A \leq Z(N)$ and $P$ a $p$-group for a prime $p$.

Notice that in the conditions of Theorem A if $d(\Gamma_G(N)) \leq 2$ it follows that the graph is disconnected and the structure of $N$ is determined in Theorem E of [2]. Consequently, $d(\Gamma_G(N)) = 3$ and we obtain the following result.

**Corollary.** Let $G$ be a finite group and $N \trianglelefteq G$. Suppose that $\Gamma_G(N)$ is connected and $d(\Gamma_G(N)) = 3$. Let us consider $x, y \in N$ such that $d(x^G, y^G) = 3$. Set $\pi = \pi(|x^G|) \cup \pi(|y^G|)$. We have that $x, y \in O_{\pi}(N)$, $N = O_{\pi}(N) \times O_{\pi}(N)$ with $O_{\pi}(N)$ a quasi-Frobenius group with abelian kernel and complement or $O_{\pi}(N) = P \times A$ with $A \leq Z(N)$ and $P$ a $p$-group for a prime $p$. 

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Proof. It follows immediately by Theorem A. □

Proofs of these results are based on the techniques appeared in [7] although we do not use them in ours. When \( N = G \) we obtain the result of Kazarin.

2 Proof of Theorem A

First, we show three elementary results necessary to prove the main theorem.

Lemma 1. Let \( G \) a \( \pi \)-separable group. Then the conjugacy class length of every \( \pi \)-element of \( G \) is a \( \pi \)-number if and only if \( G = H \times K \), where \( H \) and \( K \) are a Hall \( \pi \)-subgroup and a \( \pi \)-complement of \( G \), respectively.

Proof. This is Lemma 8 of [1].

In the particular case in which \( \pi = p' \), the complement of some prime \( p \), the above Lemma is true without assuming \( p \)-separability (which is equivalent to \( p \)-solvability).

Lemma 2. If, for some prime \( p \), every \( p' \)-element of a group \( G \) has index prime to \( p \), then the Sylow \( p \)-subgroup of \( G \) is a direct factor of \( G \).

Proof. This is Lemma 1 of [4].

Lemma 3. Let \( G \) be a finite group and \( N \leq G \). Let \( B = b^G \) and \( C = c^G \) be two non-central \( G \)-conjugacy classes of \( N \). If \( (|B|,|C|) = 1 \). Then

a. \( C_G(b)C_G(c) = G \).
b. \( BC = CB \) is a non-central \( G \)-class of \( N \) and \( |BC| \) divides \( |B||C| \).
c. Suppose that \( d(B,C) \geq 3 \) and \( |B| < |C| \). Then \( |BC| = |C| \) and \( CBB^{-1} = C \). Furthermore, \( C\langle BB^{-1} \rangle = C \langle BB^{-1} \rangle \subseteq \langle CC^{-1} \rangle \) and \( |\langle BB^{-1} \rangle| \) divides \( |C| \).

Proof. This is Lemma 2.1 of [2].

Proof of Theorem A. We proceed by induction on \( |N| \). Notice that the hypotheses are inherited by every normal subgroup in \( G \) which is contained in \( N \) and contains \( x \) and \( y \). By using the primary decomposition we can assume that \( x \) and \( y \) have order a power of two primes, say \( p \) and \( q \), respectively.

Step 1. \( q = p \) if and only if \( xy = yx \).

Suppose that \( xy = yx \) and that \( p \neq q \). Observe that \( C_G(xy) = C_G(x) \cap C_G(y) \) and consequently, \( |x^G| \) divides \( |(xy)^G| \) and \( |y^G| \) divides \( |(xy)^G| \). Thus, we
obtain a $G$-conjugacy class connected with $x^G$ and $y^G$, which is a contradiction by hypotheses. Conversely, suppose that $p = q$. We know that $p$ cannot divide either $|x^G|$ or $|y^G|$. Furthermore, the hypotheses imply that $(|x^G|, |y^G|) = 1$, so we have $G = C_G(x)C_G(y)$ and $|x^G| = |G : C_G(x)| = |C_G(y) : C_G(x) \cap C_G(y)|$. Now, since $y$ is a $p$-element in $\mathbb{Z}(C_G(y))$, we deduce that $y \in C_G(x) \cap C_G(y)$ and hence $xy = yx$.

Step 2. $p, q \in \pi$.

We define $K = C_G(x) \cap C_G(y)$. First, we assume that $p \neq q$ and $xy \neq yx$. We have $|G : K| = |G : C_G(x)||C_G(x) : C_G(x) \cap C_G(y)| = |x^G||y^G|$, which is a $\pi$-number. Since $x \in \mathbb{Z}(C_G(x))$ and $x$ is a $p$-element but $x \notin K$, we know that $p$ divides $|C_G(x) : K| = |y^G|$. This means that $p \in \pi_y$. Similarly we obtain that $q$ divides $|x^G|$, that is, $q \in \pi_x$. Consequently, $p, q \in \pi$.

Suppose now that $p = q$ and $xy = yx$. Let us see that $p \in \pi$. We denote $X = x^G$ and $Y = y^G$ and we assume for instance that $|X| > |Y|$. By hypothesis, the distance between $X$ and $Y$ in $\Gamma_G(N)$ is 3. We can apply Lemma 3(c) and we obtain $X(YY^{-1}) = X$, $YY^{-1} \subseteq XX^{-1}$ and $|\langle YY^{-1} \rangle|$ divides $|X|$. On the other hand, since $G = C_G(x)C_G(y)$ we have $X \subseteq C_G(y)$. As a result, $YY^{-1} \subseteq XX^{-1} \subseteq C_G(y)$. In particular, if we take $z = y^g \neq y$, for some $g \in G$, we have $w = zy^{-1} \in YY^{-1} \subseteq C_G(y)$, so $[z, y] = 1$. We obtain that $w$ is a non-trivial $p$-element and, since $p$ divides $|YY^{-1}|$, which divides $|X|$, we conclude that $p \in \pi_x$. If $|Y| > |X|$ we can argue similarly to get $p \in \pi_y$.

Step 3. We can assume that $N/\mathbb{Z}(N)$ is neither a $p$-group nor a $q$-group (particularly, we can assume that $N$ is not abelian).

As we have said at the beginning, $x$ is a $p$-element and $y$ is a $q$-element. Suppose that $N/\mathbb{Z}(N)$ is a $p$-group (the reasoning is analogous if we suppose that it is a $q$-group). Hence we can write $N = P \times A$ where $A \leq \mathbb{Z}(N)$ and $A$ is a $p'$-group. If $p \neq q$, it follows that $x \in P$ and $y \in A$, which leads to a contradiction with Step 1. Thus, $p = q$ and $x, y \in P$, so the theorem is proved.

Step 4. We can suppose that $N$ is not a $\pi$-group.

Let us see that if $N$ is a $\pi$-group, then $N$ is a quasi-Frobenius group with abelian kernel and complement or $N = P \times A$ with $A \leq \mathbb{Z}(N)$ and $A$ a $p'$-group. Assume that $N$ is a $\pi$-group. As $N$ is non-abelian by Step 3, there exists a conjugacy class $z^N$ such that $|z^N| \neq 1$. Since $|z^N|$ divides $|z^G|$, then either $(|z^N|, |x^G|) = 1$ or $(|z^N|, |y^G|) = 1$. Thus, $|z^N|$ is either a $\pi_x$-number or a $\pi_y$-number. If $\Gamma(N)$ is disconnected, we know by Theorem 2 of [3] that $N$ is quasi-Frobenius group with abelian kernel and complement. Moreover, $\Gamma(N)$ cannot be empty since by Step 3, we can assume that $N$ is not abelian. Consequently, we can assume that $\Gamma(N)$ is connected and this forces to either $|x^N| = 1$ or $|y^N| = 1$. Suppose for instance that $|x^N| = 1$, that is, $x \in \mathbb{Z}(N)$.
By Step 3 we can take $w$ an $s$-element of $N \setminus Z(N)$ with $s \neq p$. Observe that $|w^N|$ must be a $p$-number, so $w^G$ is connected to $y^G$ in $\Gamma_G(N)$. Since $x$ and $w$ have coprime orders and $x \in Z(N)$ we have that $|w^G|$ and $|x^G|$ both divide $|(wx)^G|$. As a consequence, we have a contradiction because $|(wx)^G|$ has primes in $\pi_x$ and $\pi_y$. Then we can suppose that $N$ is not a $\pi$-group.

Step 5. Conclusion in case $p \neq q$.

Let $z$ be a $\pi'$-element of $K \cap N$ and let us prove that $|z^G|$ is a $\pi'$-number. Suppose that $s \in \pi$ is a prime divisor of $|z^G|$. We can assume for instance that $s \in \pi_x$, otherwise we proceed analogously. Since $|z^G|$ divides $|(zx)^G|$ we obtain that $s$ divides $|(zx)^G|$. On the other hand, we know by the proof of Step 2 that $q \in \pi_x$. Therefore, $|(zx)^G|$ is divisible by primes in $\pi_x$ and $\pi_y$, a contradiction. Consequently, $s \notin \pi$ and $|z^G|$ is a $\pi'$-number, as wanted.

Let $M$ be the subgroup generated by all $\pi'$-elements of $K \cap N$. Note that $M \neq 1$, otherwise $K \cap N$ would be a $\pi$-group and, since $|N : K \cap N| = |KN : K|$ divides $|G : K|$, which is a $\pi$-number too, then $N$ would be a $\pi$-group, a contradiction with Step 2. Let us prove that $M \leq G$. Let $\alpha$ be a generator of $M$, so $|\alpha^G|$ is $\pi'$-number. Since $|(G : K|, |\alpha^G|) = 1$ we have $G = K\langle \alpha \rangle$ and hence, $\alpha^G = \alpha^K \subseteq K \cap N$. Therefore, $\alpha^G \subseteq M$, as wanted.

Let $D = \langle x^G, y^G \rangle$. Notice that $D \leq G$ and $D \subseteq N$. Let $\alpha$ be a generator of $M$. As we have proved that $|\alpha^G|$ is $\pi'$-number, then $|(\alpha^G, |x^G|) = 1$, so $G = C_G(x)C_G(\alpha)$. Thus, $x^G = x^{C_G(\alpha)} \subseteq C_G(\alpha)$ because $\alpha \in K$. The same happens for $y$, that is, $y^G \subseteq C_G(\alpha)$, so we conclude that $[M, D] = 1$.

We define $L = MD$ and we distinguish two cases. Assume first that $L < N$. Note that $x, y \in L \leq G$ and $L$ trivially satisfies the hypotheses of the theorem. By applying induction to $L$ we have in particular $L = O_\pi(L) \times O_{\pi'}(L)$. Observe that the fact that $M \neq 1$ implies that $O_{\pi'}(L) > 1$. Now, by definition of $M$, we have that $|K \cap N : M|$ is a $\pi$-number. As $|N : K \cap N|$ is also a $\pi$-number, it follows that $|N : O_{\pi'}(L)|$ is a $\pi$-number too. Then, $O_{\pi'}(L) = O_{\pi'}(N)$ is a Hall $\pi'$-subgroup of $N$. We can apply Lemma 1 so as to conclude that $N = O_\pi(N) \times O_{\pi'}(N)$ with $x, y \in O_\pi(N)$. Since $O_{\pi'}(N) > 1$, we apply the inductive hypotheses to $O_\pi(N) \leq N$ and we deduce that $O_\pi(N)$ is a quasi-Frobenius group with abelian kernel and complement or $O_\pi(N) = P \times A$ with $A \leq Z(N)$ and $P$ is a $p$-group so the theorem is finished.

From now on, we assume that $L = N$ and let us see that $Z(N) = 1$. Otherwise, we take $\overline{N} = N/Z(N)$ and $\overline{G} = G/Z(N)$. If $|\overline{x}| = 1$, then $|\overline{x}, \overline{y}| = 1$, and thus $[x, y] \in Z(N)$. Since $(o(x), o(y)) = 1$, it is easy to prove that $[x, y] = 1$, a contradiction. Analogously, we have $|\overline{y}| \neq 1$. Consequently, $\overline{N}$ satisfies the assumptions of the theorem. By induction, we have $\overline{N} = O_{\pi'}(\overline{N}) \times O_{\pi}(\overline{N})$ with $\overline{x}, \overline{y} \in O_{\pi}(\overline{N})$ and $O_{\pi}(\overline{N})$ is either a quasi-Frobenius group with abelian kernel
and complement or $\overline{N} = \overline{P} \times \overline{A}$ with $\overline{A} \leq Z(\overline{N})$ and $\overline{P}$ a $p$-group. In the latter case, $[\overline{y}, \overline{x}] = 1$ which leads to a contradiction as we have seen before. So we are in the former case. It follows that $N = O_{\pi}(N) \times O_\pi(N)$ with $x, y \in O_\pi(N)$ and by applying induction to $O_\pi(N) < N$, we have the result. Therefore, $Z(N) = 1$. On the other hand, we have proved that $[M, D] = 1$. Thus $M \cap D \subseteq Z(N) = 1$ and $N = M \times D$ with $x, y \in D$. Since $M \neq 1$, we can apply induction to $D$ and we get $D = O_{\pi}(D) \times O_\pi(D)$ with $x, y \in O_\pi(D)$ and $O_\pi(D)$ is a Frobenius group with abelian kernel and complement (notice that $Z(O_\pi(D)) = 1$ because $Z(N) = 1$). The $p$-group case cannot occur because $x$ and $y$ do not commute. Notice that if $M$ is a $\pi'$-group then the theorem is proved. Assume then that $M$ is not a $\pi'$-group and we will obtain a contradiction. Let $s \in \pi$ such that $s$ divides $|M|$. We can assume that $s \in \pi_x$ (we proceed analogously if $s \in \pi_y$). Suppose that there exists an $s'$-element $z \in M$ such that $|z^M|$ is divisible by $s$. Since $N$ is the direct product of $M$ and $D$, we have that $(zy)^N = z^Ny^N$ is a non-trivial class of $N$ whose size is divisible by $s$ and by some prime of $|y^N| \neq 1$. This is not possible because $|(zy)^G|$ would have primes in $\pi_x$ and $\pi_y$. Thus, the class size of every $s'$-element of $M$ is a $s'$-number. It is known that $M = M_1 \times S$ with $S \in Syl_\pi(M)$. In this case, $Z(S) \subseteq Z(N) = 1$, a contradiction.

Step 6. Conclusion in case $p = q$.

Let $K = C_G(x) \cap C_G(y)$ as in Step 2. Let $z$ be a $p'$-element of $K \cap N$ and let us prove that $|z^G|$ is a $\pi'$-number. Suppose that $s \in \pi$ is a prime divisor of $|z^G|$. We can assume for instance that $s \in \pi_y$, otherwise we proceed analogously. Since $|z^G|$ divides $|(zx)^G|$ we obtain that $s$ divides $|(zx)^G|$. On the other hand, we know by the proof of Step 2 that $q \in \pi_x$. Therefore, $|(zx)^G|$ is divisible by primes in $\pi_x$ and $\pi_y$, a contradiction. Consequently, $s \not\in \pi$ and $|z^G|$ is a $\pi'$-number, as wanted.

Let $T$ be the subgroup generated by all $p'$-elements of $K \cap N$. We have that $T \neq 1$ because otherwise $K \cap N$ would be a $\pi$-group and this implies that $N$ is a $\pi$-group as in Step 5, a contradiction. Let us prove that $T \subseteq G$. If $\alpha$ is a generator of $T$, we know that $|\alpha^G|$ is a $\pi'$-number. Then $|\alpha^G| = 1$, so we have $G = K C_G(\alpha)$ and $\alpha^G = \alpha^K \subseteq K \cap N$. Therefore, $\alpha^G \subseteq T$ as wanted.

Since the class size of every $p'$-element of $T$ is a $p'$-number then, by Lemma 2, $T = O_p(T) \times O_{p'}(T)$. However, by definition of $T$, we have $O_p(T) = 1$, or equivalently $M = O_{p'}(T)$. Now, notice that if $s \in \pi$ and $s \neq p$, then the class size of every element of $T$ is an $s'$-number so, it is well known that $T$ has a Sylow $s$-subgroup central and we can write $T = O_{\pi}(T) \times O_\pi(T)$. On the other hand, $|N : T| = |N : K \cap N||K \cap N : T|$ where $|N : K \cap N| = |KN : K|$ is a $\pi$-number and $|K \cap N : T|$ is a power of $p \in \pi$. Therefore $O_{\pi}(N) = O_{\pi}(T)$ and $O_\pi(N)$ is a Hall $\pi'$-subgroup of $N$. We have proved that the class size of every $p'$-element of $N$ is a $\pi'$-number, so by Lemma 1, we have $N = O_{\pi'}(N) \times O_\pi(N)$. We apply induction to $O_\pi(N) < N$ and the proof is finished. □
We give an example showing that the converse of Theorem A is not true.

**Example 1.** We take the Special Linear group $H = \text{SL}(2, 5)$ which is a group of order 120 that acts Frobeniusly on $K = \mathbb{Z}_{11} \times \mathbb{Z}_{11}$. Let $P \in \text{Syl}_5(H)$ and we consider $N_H(P)$. Then, we define $N = KP$, which is trivially a normal subgroup of $G = KN_H(P)$. We have that the set of the $G$-conjugacy class sizes of $N$ is $\{1, 20, 242\}$. Consequently, there are not two non-central $G$-classes of $N$ such that any non-central $G$-class of $N$ has size coprime with one of both.

Let us look at several examples illustrating Theorem A.

**Example 2.** We take the following groups from the library *SmallGroups* of GAP $G_1 = \text{Id}(324, 8)$ and $G_2 = \text{Id}(168, 44)$ that have the normal subgroups exposed now. The abelian 3-subgroup $P = \mathbb{Z}_3 \times \mathbb{Z}_3$ and $A = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, respectively. It follows that the set of conjugacy class sizes of $P$ is $\{1, 2, 3, 3\}$ and the set of conjugacy class sizes of $A$ is $\{1, 7\}$. We construct $N = P \times A$ and $G = G_1 \times G_2$. We have that $N$ is a normal subgroup of $G$ and the set of $G$-conjugacy class sizes of $N$ is $\{1, 2, 3, 7, 14, 21\}$ so $d(\Gamma_G(N)) = 3$ and $N$ satisfies that it is the direct product of a 3-group and $A \leq \mathbb{Z}(N)$. Note that in this example it follows that $O_{\pi'}(N) = 1$ and $\pi = \{2, 3\}$.

**Example 3.** In order to illustrate the quasi-Frobenius case it is enough to consider any group $G$ and a normal subgroup $N = G$ such that $\Gamma(N)$ has two connected components. Thus, by applying Theorem of [3] we know that $N$ is a quasi-Frobenius group with abelian kernel and complement.

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