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Groups with a small set of generators

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Dedicated to Professor S. Naimpally on the occasion of his 70th birthday.

Abstract. Following [22] we study the class $\mathcal{S}$ of all groups that admit a small set of generators. Here we adopt also another notion of smallness (P-small) introduced by Prodanov in the case of abelian groups. We push further some results obtained in [22] (by adding some new members of $\mathcal{S}$) and partially resolve an open question posed in [22]. We show that in most cases the groups in $\mathcal{S}$ admit a P-small set of generators.

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1. Introduction.

The question of measuring the size of a set of generators of a group is certainly a relevant one. In the case of topological group one puts *topological* restrictions on the set of generators to ensure smallness (see [19, 20] for the so called "suitable sets" – "small" sets of generators born in the Theory of Cohomology of infinite Galois groups in the work of Tate and Douady [8]). In the case of discrete groups the following notion of smallness was proved to be a very useful property in this respect in [22].

A subset B of a group G is *large* if $G = F \cdot B = B \cdot F$ for some finite set F of G. This property has been largely studied in the literature also under different names (big, discretely syndetic, relatively dense). A set $S \subseteq G$ is *small* if for every finite set F the sets $S \cdot F$ and $F \cdot S$ have a large complement in G [2, 3] (clearly, only infinite groups may have small sets). The role of small and large

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subsets of groups in number theory, compact representations of groups and dynamics can hardly be overestimated [7, 13, 14].

The question when an infinite group may have a small set of generators was addressed in [22]. Let us denote by $\mathcal{S}$ the class of groups with small set of generators. It was proved in [22] that $\mathcal{S}$ contains all groups that have an infinite abelian normal subgroup (in particular, all groups with infinite center) as well as all solvable groups.

Call a subset S of an abelian group G *small in the sense of Prodanov* (briefly, P-*small*) if there exist $x_1, \ldots, x_n \ldots$ in G such that the sets $\{S + x_n\}_n$ are pairwise disjoint. It is easy to see that when $S - S$ is not large, then S is P-small. This was the motivation for the introduction of P-small sets in [23]. It was noticed by Gusso [15] that P-small sets of the abelian groups are small. Their advantage is also that they are much easier to understand and construct.

The main contributions of the present paper go in two directions. In §2.1 we define approriate versions of P-smallness in non-abelian groups, as well as other versions of smallness that turn out to be stronger than smallness or P-smallness in many cases. We show that in most of the cases the small generated groups from [22] have a set of generators satisfying also a much stronger property of smallness. On the other hand, we add to the list of groups in $\mathcal{S}$ found in [22] some new classes of non-abelian groups with a small set of generators proceding in two ways. In the case of permutation groups and linear groups our arguments are purely algebric. We offer also another approach to the problem that heavily leans on topology. This allows us to produce a wealth of small and P-small sets and to provide many examples of compact-like topological groups that belong to $\mathcal{S}$. Finally, we partially answer a question from [22] by showing that $\mathcal{S}$ contains all compact groups with eventual exception of the topologically finitely generated pro-p groups and the profinite groups with trivial Frattini subgroup.

1.1. **Notation and terminology.** We denote by $\mathbb{N}$ and $\mathbb{P}$ the sets of natural and prime numbers, respectively; by $\mathbb{Z}$ the integers, by $\mathbb{Q}$ the rationals, by $\mathbb{R}$ the reals, by $\mathbb{T}$ the unit circle group $\mathbb{R}/\mathbb{Z}$, by $\mathbb{Z}(p)$ the cyclic group of order p and by $\mathbb{Z}_p$ the p-adic integers ($p \in \mathbb{P}$). The cardinality of continuum 2^ω will be denoted by $\mathfrak{c}$. If S is a set, we denote by $\mathcal{P}(S)$ its power set.

Let G be a group. We denote by 1 the neutral element of G and by $Z(G)$ the center of G. For a subset S of G we denote by $\langle S \rangle$ the subgroup generated by S. The group G is divisible if for every $n \in \mathbb{N}$ and $g \in G$ there exists $x \in G$ with $x^n = g$. The semidirect product of the groups G and K is denoted by $G \rtimes K$. Abelian groups are mostly written additively. In particular, 0 denotes the neutral element of an additively written abelian group G and for $A \subseteq G$ and $n \in \mathbb{N}$ we let $A_{(n)} = \underbrace{A + \cdots + A}_{n}$ when $n > 0$ and $A_{(n)} = \{0\}$ for completeness.

Topological groups are Hausdorff. A topological group G is *precompact* if its completion is compact, *pseudocompact* if every continuous real-valued function on G is bounded. For a topological group G we denote by $c(G)$ the connected component of the identity and by $\psi(G)$ the pseudocharacter G (the minimum cardinality of a set $\mathcal{U}$ of neighborhoods of 1 such that $\bigcap_{U\in\mathcal{U}} U = \{1\}$).

Unless explicitly stated, all groups are assumed to be infinite.

If X is a topological space and $A \subseteq X$, the closure of A is denoted by $\overline{A}$. For undefined symbols or notions see [7], [9], [12], or [17]. We denote by $\mathbb{S}(G)$ and $\mathbb{S}_P(G)$ the collections of all small and P-small sets of a group G respectively.

2. The various levels of smallness.

Let us give the following more precise form of largeness.

Definition 2.1. Let G be a group. A subset B of G is *left large* (resp. *right large*) if for some finite set F the union $F \cdot B$ (resp. $B \cdot F$) of left (resp. right) translates of B covers G.

The following trivial equalities are helpful when passing to complements:

$$\{g \in G : g \cdot F \not\subseteq A\} = (G \setminus A) \cdot F^{-1}, \qquad \{g \in G : g \cdot F \not\subseteq G \setminus A\} = A \cdot F^{-1} \quad (1)$$

Clearly, they imply that $G \setminus A$ is right large iff there exists a finite F such that A contains no left translate gF of F. Or, $G \setminus A$ is not right large iff A contains left translates gF of every finite set F.

Definition 2.2. A subset S of a group G is called:

(a) *left small* if for every finite set F the sets $S \cdot F$ and $F \cdot S$ have a left large complement; *right small* is defined analogously.
(b) *weakly left small* if for every finite set F the set $S \cdot F$ has a left large complement; *weakly right small* is defined analogously.
(c) *n-small*, for a positive $n \in \mathbb{N}$, if S is small and the sets $(S^{-1} \cdot S)^{n-1}$ and $(S \cdot S^{-1})^{n-1}$ are not large.
(d) *microscopic* if for every $n \in \mathbb{N}$ the sets $(S^{-1} \cdot S)^n$ and $(S \cdot S^{-1})^n$ are small.

A set is small iff it is left small and right small. Clearly the 1-small sets are precisely the small ones. The 2-small sets are those small sets S such that $S^{-1} \cdot S$ and $S \cdot S^{-1}$ are not large. For $n > 1$ we denote by $\mathbb{S}_n(G)$ the family of n-small sets of G. In additive notation, $S \in \mathbb{S}_{n+1}(G)$ for an abelian group G and $n > 0$ iff $(S - S)_{(n)} = S_{(n)} - S_{(n)}$ is not large, i.e., $S_{(n)} \in \mathbb{S}_2(G)$ (indeed, 2-small implies P-small which in turn implies small by [15]).

The equalities (1) give the following useful criterion mentioned in [1] and [22] in the bilateral version of smallness:

Lemma 2.3. *A set S is weakly left small (left small) iff for every finite set F there exists a finite K such that the set $S \cdot F$ contains no right translate Kg (and the set $F \cdot S$ contains no right translate Kg).*

2.1. **Left and right smallness in the sense of Prodanov.** Call a set S of a group G:

(a) *right small in the sense of Prodanov* (briefly *right P-small*) if there exist $x_1, \dots, x_n \dots$ in G such that the sets $\{S \cdot x_n\}_n$ are pairwise disjoint (or, equivalently, $x_n \cdot x_m^{-1} \notin S^{-1} \cdot S$ for $n \neq m$).

(b) *left small in the sense of Prodanov* (briefly *left P-small*) if there exist $x_1, \dots, x_n \dots$ in G such that the sets $\{x_n \cdot S\}_n$ are pairwise disjoint (or, equivalently, $x_m^{-1} \cdot x_n \notin S \cdot S^{-1}$ for $n \neq m$).

(c) *strongly right P-small* if there exist $x_1, \dots, x_n \dots$ in G such that the sets $\{S^f \cdot x_n\}_n$ are pairwise disjoint for every $f \in G$ (or, equivalently, $x_n \cdot x_m^{-1} \notin (S^{-1} \cdot S)^f$ for $n \neq m$); *strongly left P-small* is defined analogously.

(d) (*strongly*) *P-small* if it is (strongly) left and (strongly) right P-small.

It is easy to see that S is left P-small (strongly left P-small) if and only if S^{-1} is right P-small (strongly right P-small).

Here we give separately the smallnes conditions in terms of the **difference sets** $S \cdot S^{-1}$ and $S^{-1} \cdot S$.

Claim 2.4. *Let S be a subset of a group G. Then:*

(a_l) *S is left P-small iff there exists an infinite set X such that*

$$(S \cdot S^{-1}) \cap (X \cdot X^{-1}) = \{1\};$$

(a_r) *S is right P-small iff there exists an infinite set X such that*

$$(S^{-1} \cdot S) \cap (X^{-1} \cdot X) = \{1\};$$

(b) *S is strongly left P-small iff there exists an infinite set X such that*

$$(\forall f \in G)(S \cdot S^{-1})^f \cap (X \cdot X^{-1}) = \{1\}.$$

Consequently, each one of the properties weakly left (right) small, left (right) small, left (right) P-small, P-small, n-small, microscopic, is invariant under left and right translations.

It is clear that strongly P-small implies P-small (by taking $f = 1$ in the definitions (c) and (d) above). In the non-abelian case, P-small need not imply small (cf. Example 2.17). The final part of the claim shows that microscopic implies small.

Remark 2.5. (1) Assume that S is right large. Then $S \cdot F = G$ for some finite F. Then for every infinite set X one of the sets Sf ($f \in F$) contains infinitely many members X_1 of X. In particular, there exist $x, y \in X$ with $x \neq y$ and $x, y \in Sf$. This gives $xy^{-1} \in X \cdot X^{-1} \cap S \cdot S^{-1}$. Thus $X \cdot X^{-1} \cap S \cdot S^{-1} \neq \{1\}$ for every infinite X. (More precisely, for every infinite X there exists an infinite $X_1 \subseteq X$ such that $X_1 \cdot X_1^{-1} \subseteq S \cdot S^{-1}$). Hence S is not left P-small. Therefore **left P-small sets cannot be right large** (but can be left large, cf. Example 2.18). The same conclusion may be obtained by using (a) in the next Lemma 2.6.

(2) If $S^{-1} \cdot S$ contains a finite index subgroup H, then S is not right P-small. Indeed, if $X \subseteq G$ is an infinite set, then some coset aH will contain an infinite subset X_1 of X, hence $X_1^{-1} \cdot X_1 \subseteq H \subseteq S^{-1} \cdot S$, so S is nor right P-small by (a_r) of the preceding Claim.

In the next lemma we give some connection between these notions of smallnes. Note that $S^{-1} \cdot S$ is symmetric, so all three versions of "large" coincide for $S^{-1} \cdot S$.

Lemma 2.6. *Let S be a subset of a group G.*

(a) *If S is left (right) P-small, then it is also weakly left (right) small.*
(b) *If $S^{-1} \cdot S$ is not large then S is right P-small, if $S \cdot S^{-1}$ is not large, then S is left P-small,*
(c) *If S is 2-small then it is P-small.*
(d) *If S is strongly left P-small, then it is left small.*
(e) *If S is strongly P-small, then it is small.*
(f) *If S is microscopic, then it is n-small for every $n \in \mathbb{N}$. In particular, microscopic sets are 2-small. If G is abelian, then the first implication can be reversed.*

Proof. (a) There exist $x_1, \dots, x_n, \dots$ in G such that $x_m^{-1} \cdot x_n \notin S \cdot S^{-1}$ for $n \neq m$. Let F be a finite subset of G. To see that $S \cdot F$ has a left large complement choose n such that $|F| < n$ and take $K = \{x_1^{-1}, \dots, x_n^{-1}\}$. Then $S \cdot F$ contains no right translates $K \cdot g$ of K. Indeed, assume that $K \cdot g \subseteq S \cdot F$ for some $g \in G$. Then there exist $x_i \neq x_j$ in K such that for some $f \in F$ one can find $s, s_1 \in S$ with $x_i^{-1} g = sf$ and $x_j^{-1} g = s_1 f$. The second equation gives $g^{-1} \cdot x_j = f^{-1} \cdot s_1^{-1}$. Multiplying this by the first equation we get $x_i^{-1} x_j = s s_1^{-1}$. Hence $x_i^{-1} x_j \in S \cdot S^{-1}$, that leads to contradiction. Analogously one proves that right P-small implies weakly right small.

(b) Indeed, there exists $x_1, \dots, x_n \dots$ in G such that

$$x_n \notin S^{-1} \cdot S \cdot \{x_1, \dots, x_{n-1}\},$$

so that $x_n \cdot x_m^{-1} \notin S^{-1} \cdot S$ for $m < n$. Since $S^{-1} \cdot S$ is symmetric, we have also $x_m \cdot x_n^{-1} \notin S^{-1} \cdot S$. Thus S is right P-small. One can see analogously, that S is left P-small whenever $S \cdot S^{-1}$ is not large.

(c) Follows from (b).

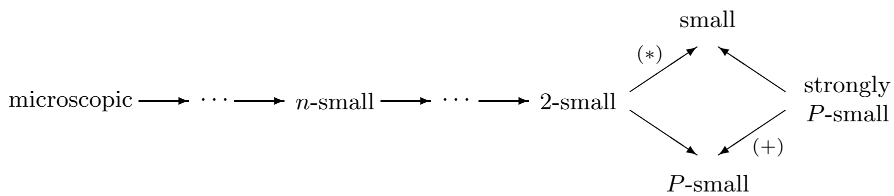
(d) To prove that S is left small take any finite set F. We can assume without loss of generality that $1 \in F$. We have to find a finite set K such that $S \cdot F$ contains no right translates $K \cdot g$ and $F \cdot S$ contains no right translates $K \cdot g$. The former property was already checked above in (a). To check the latter one note that by assumption there exist $x_1, \dots, x_n, \dots$ in G such that $x_m^{-1} \cdot x_n \notin S^f \cdot (S^{-1})^f$ for $n \neq m$ and for every $f \in F$. Take $n > |F|$ and $K = \{x_1^{-1}, \dots, x_n^{-1}\}$. Assume that $K \cdot g \subseteq F \cdot S$ for some $g \in G$. Then there exist $x_i \neq x_j$ in K such that for some $f \in F$ one can find $t, t_1 \in S$ with $x_i^{-1} g = ft$ and $x_j^{-1} g = f t_1$. The second equation gives $g^{-1} x_j = t_1^{-1} \cdot f^{-1}$.

Multiplying this with the first equation we get $x_i^{-1}x_j = ftt_1^{-1}f^{-1}$. Hence $x_i^{-1}x_j \in S^f \cdot (S^{-1})^f$, that leads to contradiction.

(e) Follows directly from (d).

(f) It suffices to observe that S is small (indeed $S^{-1} \cdot S$ contains a translated set of S). For the reverse implication in the Abelian case, it suffices to apply (a) and (b) to the set $(S-S)_{(n)}$. □

In the next diagram we give all the implications that are always valid between the various symmetric smallness properties we have introduced so far.



The arrow (+) becomes an equivalence for abelian groups, so that the diagram becomes linear with P-small=strongly P-small placed between 2-small and small. The inverse arrow of (∗) fails in two ways: a P-small set need not be small (as mentioned above) and $S \cdot S^{-1}$, $S^{-1} \cdot S$ need not be large for a P-small set S (an example for the simultaneous failure of these implications is given in 2.17). The following question remains open even in the abelian case:

Problem 2.7. *Do the notions 2-small and strongly P-small coincide?*

Now we produce a series of examples of *small sets on $\mathbb{Z}$ that are not P-small.*

Example 2.8. (a) Let P_r denote the set of all products of at most r odd primes and let $A_r = P_r \cup -P_r$. It was proved in [1] that the set P_r is small for every $r \in \mathbb{N}$, hence A_r is small as well. Brun proved in 1919 that $P_9 + P_9$ contains all sufficiently large even integers. This was improved by Rademacher in 1924 who brought r down to 7 and Selberg improved it to $r = 3$ in 1950. According to Goldbach's conjecture $P_1 + P_1$ contains all even integers n with $|n| > 4$. In these terms, the set $A_r - A_r$ contains the subgroup of all even integers, hence it is surely large for $r \geq 3$ (actually, A_r is not P-small by Remark 2.5) (2)). This shows that if Goldbach's conjecture has positive answer, then the set A_1 of all odd primes (positive and negative) is small but not P-small. Obviously, to the same result leads the positive answer to the (unsolved) conjugate Goldbach's conjecture (every even integer is a difference of two odd primes).

(b) Let $f(x) = ax^2 + bx + c$, with $a, b, c, \in Z$, $a \neq 0$. Then $S_f = \{f(n) : n \in \mathbb{Z}\}$ is small, but not P-small. Since these properties are invariant under translation, we can assume without loss of generality that $c = 0$. We shall see that $S_f - S_f$ always contains a non-zero subgroup H, so that it is not P-small by Remark 2.5. Indeed, if $b \neq 0$, then for every $n \in \mathbb{Z}$ one has $2bn = f(n) - f(-n) \in S_f - S_f$, so $H = 2b\mathbb{Z} \subseteq S_f - S_f$. If $b = 0$, then $4an = f(n+1) - f(n-1) \in S_f - S_f$, so $H = 4a\mathbb{Z} \subseteq S_f - S_f$. Furthermore S_f is small because $\lim_{n\to+\infty}(f(n+1) - f(n)) = +\infty$ [1].

2.2. **Smallness vs topology, measure, asymptotical density and growth.** It was proved in [2, Proposition 3.7] that every compact subset of a non-compact topological group is small. In fact, one can prove the following much stronger property.

Lemma 2.9. *A compact set K of a non-compact topological group is always microscopic.*

Proof. Indeed, K^{-1} is compact as well. Hence $K^{-1}\cdot K$ and $K\cdot K^{-1}$ are compact as continuous images under multiplication of the compact spaces $K^{-1}\times K$ and $K\times K^{-1}$. For every $n\in\mathbb{N}$ and for every finite set F the sets $F\cdot(K\cdot K^{-1})^n$ and $F\cdot(K^{-1}\cdot K)^n$ are still compact, hence small by [2, Proposition 3.7]. □

We shall apply this lemma very often when G is countable, so surely non-compact. Also we shall have often K a converging sequence. This is why we make use of the following modified version of a notion introduced by Protasov and Zelenyuk [29] (the original version was for G abelian and H replaced by G).

Definition 2.10. A sequence $a_1, a_2, \dots, a_n, \dots$ in an infinite group G is a *T-sequence* if the subgroup H generated by the set $\{a_1, a_2, \dots, a_n, \dots\}$ admits a Hausdorff group topology such that $a_n \to 1$.

Since the underlying set of every convergent sequence, along with its limit, is a compact set, we get

Corollary 2.11. *The underlying set of a T-sequence $\{a_n\}_{n=1}^{\infty}$ of an infinite group G is a microscopic set.*

Recall, that a *Banach measure* on a group G is a finitely additive invariant measure on G such that *every* subset of G is measurable and $\mu(G)=1$ (see [10, 11] for the existence of Banach measure on *all* abelian groups and the existence of groups that admit no Banach measure). Iv. Prodanov noted that Banach measures are a very convenient tool when dealing with large sets. Indeed, if μ is a right invariant Banach measure on the group G, then obviously every right large set has a positive Banach measure, while every right P-small set has measure zero (hence a right P-small set is never right large in such a group). Therefore, an infinite group G that admits a right invariant Banach measure is never a finite union of right P-small sets (see [27] or [7, p. 29] for an elementary proof of this fact in the Abelian case that makes no recourse to Banach measures). It is proved in [1, Theorem 3.4] that every compact group G admits closed small sets A of positive Haar measure arbitrarily close to 1 hence small sets need not be neither left nor right P-small. (In this case one can apply also Steinhouse-Weil theorem to conclude that $A^{-1}\cdot A$ has non-empty interior, so $A^{-1}\cdot A$ is large.)

Example 2.12. Here we give examples of small sets in $\mathbb{Z}$ with various levels of smallness.

(1) Let $f(x) \in \mathbb{Z}[x]$ be polynomial of degree $d > 0$. Then the set $S_f = \{f(n) : n \in \mathbb{Z}\}$ is large if $d = 1$. It follows from the criterion given in [1] that S_f is small if $d > 1$. According to Example 2.8 $S_f - S_f$ is large if $d \leq 2$. As $S_f - S_f \supseteq S_g$, where $g(x) = f(x+1) - f(x)$ (so deg $g = d-1$), it is easy to see that in general $(S_f - S_f)_{(m)} = (S_f)_{(m)} - (S_f)_{(m)}$ is large if $m \geq 2^{d-2}$. Hence, S_f is never microscopic. The above implications cannot be inverted for arbitrary polynomials f (for $f(x) = x^{2d} - x$ the set $S_f - S_f$ contains all even integers, hence it is large), but seemingly this can be done for *monomials* $f(x) = ax^d$ (e.g., S_f is 2-small iff $d > 2$, etc.). In this way one can construct n-small sets of $\mathbb{Z}$ that are not $n+1$-small for every $n \in \mathbb{N}$.

(2) For every $n > 1$ let r_n be the rest of n^2 modulo 2^k, where 2^k is the greatest power of 2 with $2^k \leq n$. Then $S = \pm\{n^2 - r_n : n > 1\}$ is microscopic by the above corollary (as $n^2 - r_n$ converges to 0 in the 2-adic topology of $\mathbb{Z}$) even if the function defining S grows slower than the polynomial function $n \mapsto n^2$.

Remark 2.13. It will be desireable to understand better the various kinds of small sets of $\mathbb{Z}$. Here we propose two more points of view.

(a) One can connect smallness or largeness of subsets of $\mathbb{Z}$ with density (for an infinite set $A \subseteq \mathbb{Z}$ let the density of A be the limit $\lim_n \frac{|A \cap [-n,n]|}{2n}$, whenever it exists). Clearly, every large set has positive density (cf. [1, Proposition 1.13]). One can build examples of small sets of positive density ([1, Proposition 1.14]). On the other hand, by Schnilermann's approach, for every set A with positive density $A_{(n)} = \mathbb{Z}$ for n sufficiently large. Hence a set of positive density cannot be microscopic. We do not know whether such a set can be n-small for some $n > 1$.

(b) Another approach to the smallness of a set $A = \{a_n\}$ of positive integers is to study the asymptotic behavior of the ratio $\frac{a_{n+1}}{a_n}$. There exist T-sequences $\{a_n\}$ in $\mathbb{Z}$ such that $\lim_n \frac{a_{n+1}}{a_n} = 1$ (cf. Example 2.12 (2), note that $n^2 \geq a_n = n^2 - r_n \geq n^2 - n$ by the choice $r_n < n$, so $\lim_n \frac{a_{n+1}}{a_n} = 1$ holds true). Hence there exists a microscopic set $A = \{a_n\}$ in $\mathbb{Z}$ such that the ratio $\frac{a_{n+1}}{a_n}$ converges to 1. It is known that when $\lim_n \frac{a_{n+1}}{a_n} = \infty$ or the limit is a transcendental real number, then $\{a_n\}$ is a T-sequence [29], while every algebraic number $\alpha \geq 1$ admits a sequence $\{a_n\}$ with $\lim_n \frac{a_{n+1}}{a_n} = \alpha$ that is not a T-sequence [29]. We do not know whether the condition $\lim_n \frac{a_{n+1}}{a_n} = \alpha > 1$ may imply that A is microscopic, strongly small or P-small (it yields A is small as $a_{n+1} - a_n \to \infty$ when $\alpha > 1$, so that [1, Proposition 1.2] applies). If yes, then we obtain new examples of small (microscopic) sets that are not the underlying set of a T-sequence.

2.3. **Smallness of subgroups and transversals.** Let us recall that a *left transversal* of a subgroup H of a group G is a subset T of G such that

$$G = T \cdot H \text{ and } (T^{-1} \cdot T) \cap H = \{1\}. \tag{2}$$

Lemma 2.14. *Let H be a subgroup of G and let T be a left transversal of H. Then:*

(a) *if H is infinite, then T is right P-small (or, equivalently, T^{-1} is left P-small);*
(b) *if T is infinite, then H is P-small; moreover, if H is small, then H is microscopic.*
(c) *if $(T^{-1} \cdot T) \cap H^f = \{1\}$ for every $f \in G$, then H is microscopic if T is infinite (and T is strongly right P-small if H is infinite).*

Proof. (a) If H is infinite, then (2) implies that T is right P-small since every infinite subset $\{x_n\}$ of H will witness right P-smallness. This yields T^{-1} is left P-small.

(b) The part $(T^{-1} \cdot T) \cap H = \{1\}$ of (2), in view of $H = H^{-1} \cdot H$, witnesses H is right P-small. If T_1 is a right transversal of H, then analogous argument proves that H is left P-small. Since $H^{-1} = H$, in both cases H is P-small. Clearly, H is not large when T is infinite. Since H is a subgroup, this implies that H is microscopic, if H is small.

(c) For $t \neq t_1$ in T and every $f \in G$ the cosets tH^f and t_1H^f are disjoint by hypothesis. Then H is strongly left P-small by definition. Since H is a subgroup, this implies that H is strongly P-small, hence small (by Lemma 2.6). As T is infinite, H must be microscopic, according to item (b). □

Item (a) cannot be inverted (cf. Example 2.18).

Corollary 2.15. *Let G be a group and H, D be infinite subgroups of G.*

(a) *if $D \cap H = \{1\}$, then both H and D are P-small.*
(b) *if $D \cap H^f = \{1\}$, for every $f \in G$ then both H and D are strongly P-small and microscopic.*

Corollary 2.16. *Let H be a normal subgroup of a group G.*

(a) If H has infinite index, then H is strongly P-small and microscopic.

(b) If H is infinite, then any transversal T of H is strongly P-small (hence, small).

(c) If H is infinite and has infinite index, then $H \cup T$ is small for any transversal T of H.

Proof. (a) To see that H is strongly left P-small fix a left transversal T of H. Then it is also a right transversal. For $t \neq t_1$ in T the cosets $Ht = tH$ and $Ht_1 = t_1H$ are disjoint. Moreover, $H^f = H$ for every $f \in F$, so that $H^f t$ and $H^f t_1$ remain disjoint for every $f \in F$ and $t \neq t_1$ in T, i.e., $(T^{-1} \cdot T) \cap H = \{1\}$ and $(T \cdot T^{-1}) \cap H = \{1\}$ for every $f \in G$. Therefore, H is strongly right P-small by Lemma 2.14. Since H is a subgroup, this implies that H is strongly P-small, hence small. Therefore, H is also microscopic.

(b) To see that T is strongly left P-small let F be a finite set of G. Take any countably infinite subset $\{x_n\}$ of H. Then for every $f \in F$ the sets $T^f x_n$ are pairwise disjoint. Indeed, if $z \in T^f x_n \cap T^f x_m$, then $zx_n^{-1} = t^f$ and $zx_m^{-1} = t_1^f$

for some $t, t_1 \in T$. Now $x_m x_n^{-1} = (t_1^f)^{-1} t^f = (t_1^{-1} t)^f$. Since $x_m x_n^{-1} \in H$ and H is normal, we conclude that $t_1^{-1} t \in H$. This yields $t = t_1$ and $n = m$.

(c) Follows from (a) and (b). □

Item (a) implies in particular that H is small – this conclusion was obtained in [2, Prop. 1.7].

2.4. **Examples distinguishing left/right largness and smallness.** It is known [2] that the union of two small sets is again small. We show in the next example (item (b) inspired by [15, Example 3.2.7]) that in general the union of two P-small sets need not be neither P-small nor left or right small.

Example 2.17. Let G be the free group of two generators a, b.

(a) Let S be the set of reduced words that start with a (non-trivial) power of a, and let S' be the set of reduced words that start with a (non-trivial) power of b. Then S and S', as well as $S'' = S' \cup \{1\}$, are left P-small, hence weakly left small too. As S and S'' are complementary, this yields that S and S'' are left large (as complements of weakly left small sets). Therefore, none of them is left small. On the other hand, $G = S \cup S''$ is large. So in the non-abelian case an infinite group can be the union of two left P-small sets. Finally, neither S nor S'' are right large, consequently, neither S nor S'' are weakly right small, neither right P-small. Hence a weakly left small (actually, left P-small) set need not be weakly right small.

(b) Let $Y_{a,a}$ be the set of all words of G starting and ending by a non-trvial power of a. Define analogously $Y_{b,b}, Y_{a,b}$ and $Y_{b,a}$. Clearly

$$Y_{a,b}^{-1} = Y_{b,a}, \quad (Y_{b,b})a \subseteq Y_{b,a} \quad (Y_{a,a})b \subseteq Y_{a,b} \quad (Y_{a,b})a \subseteq Y_{a,a} \text{ and } b(Y_{a,b}) \subseteq Y_{b,b}. \tag{3}$$

All these sets are P-small. Indeed, it suffices to check that one of them is P-small and then apply (3) and the fact that the automorphism $f : G \to G$ exchanging a and b sends $Y_{a,a}$ to $Y_{b,b}$. Analogously, one can see that either all four sets are small, or none of them is small. As G is the union of the five sets $\{1\}$, $Y_{a,a}, Y_{b,b}, Y_{a,b}$ and $Y_{b,a}$, one of them is not small. Since $\{1\}$ is small it follows from the above argument that none of the sets $Y_{a,a}, Y_{b,b}, Y_{a,b}$ and $Y_{b,a}$ is small. The union $Y_{a,a} \cup Y_{a,b}$ coincides with the set S defined in (a), so it is not even left small. Since $S^{-1} = Y_{a,a} \cup Y_{b,a}$ is not right small, we see that the union of two P-small sets need not be neither left nor right small. Moreover, as S is not right P-small, we conclude that the union of two P-small sets need not be P-small.

Since the set $Y_{b,b}$ is weakly left and right small (being P-small), its complement $G \setminus Y_{b,b}$ is large. Hence the P-small set $A = Y_{a,a}$ fails to be small and has $A^{-1}A = AA^{-1}$ large (this witnesses the strong failure of the reverse implication of the arrow $(*)$ in the diagram).

Example 2.18. (see also [27]) *A right large set which is right P-small and not left large.* Let G be the free product of a cyclic group $A = \langle a \rangle$ of order two and an infinite cyclic group $C = \langle b \rangle$. Every element of $G \setminus C$ can be uniquely written as a product

$$w = b^{n_0} \cdot a \cdot b^{n_1} \cdot a \cdot b^{n_2} \cdot a \cdot \ldots \cdot a \cdot b^{n_{k-1}} \cdot a \cdot b^{n_k}, \tag{3}$$

where $k > 0$ and $n_0, \ldots, n_k$ are integers, such that if $k > 1$, then $n_1, \ldots, n_{k-1}$ are non-zero integers, while the integers n_0 and n_k may also have value 0. Obviously, the elements $w \in C$ can be obtained in the form (3) with $k = 0$. One refers to (1) as to the *reduced form* of the element $w \in G$. The product $w_1 \cdot w_2$ of two words w_1 and w_2 can be brought to a reduced form after a finite number of cancelations.

Let $Y = \{w \in G : k > 0 \text{ and } n_k = 0\}$ be the set of all reduced words that end with a and let $X = G \setminus Y$. Then $Y = Xa$, so that $X = Ya$ too. Thus, both X and Y are right large, since $G = X \cup Xa = Y \cup Ya$. Consequently, neither X nor Y are right small. Let us see now that neither X nor Y are left large. Since $Y = Xa$, it suffices to see that X is not left large.

Let us first note that the inverse of an element w as in (3) is given by

$$w^{-1} = b^{-n_k} \cdot a \cdot b^{-n_{k-1}} \cdot a \cdot b^{-n_{k-2}} \cdot a \cdot \ldots \cdot a \cdot b^{-n_1} \cdot a \cdot b^{-n_0}.$$

Assume that $G = \bigcup_{i=1}^{s} g_i X$ for some $g_1, \ldots, g_s \in G$. There exist $n \neq m$ such that $b^n a \in g_i X$ and $b^m a \in g_i X$. Then for some $x \in X$ one has $b^n a = g_i x$. Hence $g_i = b^n a x^{-1}$. Now $x = b^{n_0} \cdot a b^{n_1} a b^{n_2} a \ldots a b^{n_{k-1}} a b^{n_k}$ with either $x = 1$ or $n_k \neq 0$. In both cases the leading term of the reduced form of the word g_i is $b^n a \ldots$. Analogous argument shows that the leading term of the reduced form of the word g_i is $b^m a \ldots$, a contradiction.

Since Y is not left large, X is not left small. Indeed, if F is any finite set that contains $1 \in G$, then the complement of $F \cdot X$ is contained in Y, hence cannot be left large. Analogous argument shows that Y is not left small. Therefore we have the following properties: X and Y are right large and are neither left large nor left small.

Let us note that the right large set Y is a left transversal of A and right P-small (as all Yb^n are paiwise disjoint), hence weakly right small too. Nevertheless, A is finite (this shows that the implication in Lemma 2.14 (a) cannot be inverted).

Clearly right large sets cannot be even weakly left small. A more careful analysis of Example 2.18 shows that the set X has the following curious property: *for every $w \in G$ there exists a finite subset F of G such that $w \cdot X \subseteq X \cup F$.* (Indeed, let w have the form (1) and assume that $n_k \neq 0$. Then $F = \{w \cdot b^{-n_k}, w \cdot b^{-n_k} \cdot a \cdot b^{-n_{k-1}}, \ldots, b^{n_0} \cdot a\}$ works.) In general, call a subset X of a group G with the above property *left absorbing*. Obviously, every left absorbing set X with infinite complement is not left large. Clearly, every cofinite set is both left absorbing and large. So it is reasonable to exclude the cofinite sets to obtain:

Lemma 2.19. *Every non-cofinite left absorbing set in an infinite group G is not left large.*

Proof. It suffices to note that for a left absorbing set X and every finite subset F of G the set $F \cdot X$ can add at most finitely many new points to X, hence cannot cover G when X is not cofinite. □

Admittedly, the worst failour of coincidence of left and right largeness is presented by the right large sets that are left absorbing.

If G admits a left invariant Banach measure, then every left large set has positive measure. The free product considered above does not allow a Banach measure as it contains a copy of the free group of two generators (see [Følner]), but still the set X has a rather strange behaviour: it has "measure 1/2 from right" (in the sense that it contains, roughly, half of the elements of the group), on the other hand it is left absorbing.

2.5. **Permanence properties of small sets.** We collect here some permanence properties of large and small sets related to homomorphisms.

According to the next lemma from [16, Proposition 1.2] images and counter-images preserve largeness of sets.

Lemma 2.20. *Let $f : G \to G_1$ be a surjective homomorphism and X be a (left, right) large subset of G. Then $f(X)$ is (left, right) large in G_1. A subset Y of G_1 is (left, right) large iff $f^{-1}(X)$ is (left, right) large in G.*

According to the above lemma the image of a large set cannot be small. On the other hand, trivial examples show that the image of a small set can be large. Now we show that inverse images preserve smallness.

Lemma 2.21. *Let $f : G \to G_1$ be a surjective homomorphism and X be a subset of G_1. Then X has one of the properties (left, right) small, weakly left (right) small, (left, right) P-small, strongly (left, right) P-small, n-small, microscopic, iff $f^{-1}(X)$ has the same property.*

Proof. Assume that $S = f^{-1}(X)$ is left small. It suffices to show that for every finite set F of G the sets $f(F) \cdot X$ and $X \cdot f(F)$ have left large complements in G_1. By assumption, $F \cdot S$ has left large complement. Thus $f(G \setminus F \cdot S)$ is large in G_1 by Lemma 2.20. Since $f(G \setminus F \cdot S) \subseteq G_1 \setminus f(F) \cdot X$ this proves that $G_1 \setminus f(F) \cdot X$ is left large. Analogously we show that $X \cdot f(F)$ has a left large complement in G_1. Thus X is left small. If X is left small, then $G \setminus F \cdot f^{-1}(X)$ is left large for every finite $F \subseteq G$, containing the left large set $f^{-1}(G_1 \setminus f(F) \cdot X)$. Analogous proof works for "right small". The conjunction of these two properties gives the counterpart for "small".

Assume that for the infinite set $Y = \{y_n\}$ of G_1 the sets Xy_n are pairwise disjoint for $y_n \in G_1$. Let $z_n \in G$ satisfy $f(z_n) = y_n$ for every n. Then the sets $f^{-1}(X)y_n$ are pairwise disjoint. Hence $f^{-1}(X)$ is right P-small whenever X is right P-small. Similar proof works for "right P-small".

The preservation of remaining versions of smallness may be proved in a similar way. □

Remark 2.22. Gusso [16] showed that "smallness" has the following "transitivity" property: if H is a subgroup of G and S is a small subset of H, then S is small in G. Obviously, left and right P-smallness, as well as n-smallness (hence microscopic), have this property. Finally, if $S \subseteq H$ is not left (right) large in H, then it is not left (right) large in G.

3. The class $\mathcal{S}$ of small generated groups.

We start with some direct consequences of Corollary 2.16 including for readers convenience also some results from [22, Corollary 5]:

Lemma 3.1. *An infinite group has a small set of generators in each of the following cases:*

(a) *if G admits an infinite normal subgroup of infinite index;*
(b) *if G is a semi-direct product of two infinite groups;*
(c) *if G is a restricted direct product of infinitely many non-trivial groups;*
(d) *if G is generated by a finite family of subgroups $H_1, \dots, H_n \in \mathcal{S}$.*

Proof. (a) follows from item (c) of Corollary 2.16, (b) follows from (a), (c) follows from (b). For (d) Remark 2.22 should be applied. □

Remark 3.2. If N and H are infinite groups, then the set $S = H \cup N$ in $G = N \times H$ is small as a union of two small sets, but it is not P-small. Indeed, $S^{-1} = S$ and $S^2 = G$, thus S is neither left nor right P-small (this was proved in the more general case of a normal subgroup N of infinite index of G in [15]). Nevetheless, we show in Lemma 3.3 (a) that the group G has a P-small set of generators whenever either H or N have this property.

Let us introduce the classes:

(a) $\mathcal{S}_P$ of all groups that admit a strongly P-small set of generators;
(b) $\mathcal{S}^P$ of all groups that admit a set of generators that is both small and P-small;
(c) $\mathcal{S}_n$ of all groups that admit a n-small set of generators;
(d) $\mathcal{M}$ of all groups that admit a microscopic set of generators.

Clearly, $\mathcal{S}_P \subseteq \mathcal{S}$ and $\mathcal{S}^P \subseteq \mathcal{S}$. As 2-small sets are both small and P-small, we have the following inclusions

$$\mathcal{M} \subseteq \ldots \subseteq \mathcal{S}_n \subseteq \ldots \subseteq \mathcal{S}_2 \subseteq \mathcal{S}^P \subseteq \mathcal{S} \supseteq \mathcal{S}_P.$$

3.1. Some general properties of $\mathcal{S}$, $\mathcal{S}^P$, $\mathcal{S}_n$, $\mathcal{S}_P$ and $\mathcal{M}$.

Lemma 3.3. *Let G be a group.*

(a) *If G admits a quotient $G/N \in \mathcal{S}$ (resp. $\mathcal{S}_P$, $\mathcal{S}^P$, $\mathcal{S}_n$, $\mathcal{M}$), then G has the same property;*
(b) *if there exists is a normal subgroup N of G with $N \in \mathcal{S}$ (resp. $N \in \mathcal{S}^P$), then G has the same property.*

Proof. (a) By Lemma 2.21, $G/N \in \mathcal{S}$ implies $G \in \mathcal{S}$ since the inverse image of a small set of generators is a small set of generators. Analogous argument shows that if some quotient $G/N \in \mathcal{S}_P$, then $G \in \mathcal{S}_P$, etc.

(b) Assume that S is a small set of generators of the normal subgroup N of G. This yields that N is infinite. Then by Lemma 2.16 any transversal T to H is small (see also [22, Proposition 3]). Then $S \cup T$ is a small set of G. Clearly this is a set of generators of G.

We prove now that $N \in \mathcal{S}^P$ implies $G \in \mathcal{S}^P$. Let S be a P-small set of generators of N. Choose any transversal T_0 of N and put $T = T_0 \setminus N$. Then the set of generators $S_1 = S \cup T$ of G is P-small. Indeed, if $X \subseteq N$ witnesses left P-smallness of S, then for $n \neq m$

$$\begin{aligned} x_n S_1 \cap x_m S_1 &= (x_n S \cup x_n T) \cap (x_m S \cup x_m T) \\ &= (x_n S \cap x_m T) \cup (x_n T \cap x_m S) \cup (x_n T \cap x_m T) = \varnothing, \end{aligned}$$

as $x_n T \cap x_m T = \varnothing$ since T is a transversal of N, $x_m S \cap x_n T = \varnothing$ and $x_n S \cap x_m T = \varnothing$ as $x_m S \subseteq N$ and $x_n S \subseteq N$, while $x_n T \cap N = \varnothing$ and $x_m T \cap N = \varnothing$. This proves that S_1 is left P-small. Right P-smallness of S_1 can be established in the same way. If in addition we assume that S is small in N, then by Remark 2.22 it is also small in G, so that the union S_1 is small too. □

Here we give a topological proof of a result similar to another one (regarding small sets) already known in the case of infinite abstract groups (cf. [2, 3]).

Theorem 3.4. *Every non-discrete topological group G admits an infinite microscopic set.*

Proof. If G is uncountable, just take any countable subgroup of G. Assume now that G is countable. By a theorem of Arhangel′ski (cf. [7, Theorem 7.5.3]) the group G admits coarser metrizable group topology τ that in the given case must be non-discrete. Let $s_n \to 1$ be a non-trivial converging sequence in (G, τ). Then the set $K = \{1\} \cup \{s_n : n \in \mathbb{N}\}$ is compact, so by Lemma 2.9 K is microscopic. □

There exist infinite groups that admit no Hausdorff group topology beyond the discrete one; let us call them *Markov groups*. The question of whether Markov groups exist was raised by Markov in the early forties, and answered positively by Shelah [25]. He constructed (under CH) a Markov group G of size ω_1 that is also a Kurosch group (i.e., all proper subgroups of G are countable; an example of a countable Markov group was given in the same year by Ol′shankii without any additional set-theoretic assumptions.). Call a Markov group *hereditary Markov group*, if every infinite subgroup of G is a Markov group. It is not known whether such groups exist. Clearly, the above theorem works for all abstract groups G that have a countable subgroup admitting a non-discrete Hausdorff group topology, i.e., for non-hereditarily-Markov groups.

We prove a new version of the theorem from [22] that establishes the existence of small sets of generators for some groups. Here we claim less (since left P-small sets need not be small), but we gain in generality since no restrictions are needed on the group. In the case of abelian groups the results obviously coincide. In this case we get also a sharper result (cf. Corollary 3.6).

Theorem 3.5. *Let G be an infinite group. Then*

(a) G has a set of generators X that is a union of a small set and a left (right) P-small set;

(b) if G is not hereditarily Markov, then X can be chosen to be the union of a microscopic set and a left (right) P-small set.

Proof. Let S be an infinite small subset of G (it exists by [2]). Fix a right transversal set T_r of $\langle S\rangle$ and a left transversal set T_l of $\langle S\rangle$. By Lemma 2.14 T_r is left P-small (hence weakly left small by Lemma 2.6). Analogously, T_l is right P-small. Clearly, $X = S \cup T_r$ as well as $S \cup T_l$ generate G.

In case G is not hereditarily Markov, then G has a countable subgroup H admitting a non-discrete Hausdorff group topology. Now H has a microscopic set S by Theorem 3.4 and S is microscopic also as a subset of G. Now the argument goes as before. □

Theorem 3.4 implies that every infinite Abelian group admits an infinite microscopic set. Now the compact sets can be obtained also by embeddings in $\mathbb{T}^\omega$. The next theorem offers a more precise result and improves [22, Theorem 9].

Theorem 3.6. *Every infinite Abelian group has an infinite microscopic set of generators.*

Proof. First we prove the assertion for the infinite abelian groups G which admit a surjective homomorphism onto some Prüfer group $\mathbb{Z}(p^\infty)$. By Lemma 2.21, it suffices to check that $\mathbb{Z}(p^\infty) \in \mathcal{M}$. For this purpose, notice that the sequence $S = \{1/p^n : n \in \mathbb{N}\}$ converges to 0 in $\mathbb{Z}(p^\infty)$ equipped with the topology induced by the circle group $\mathbb{T}$. Hence S is a microscopic set in $\mathbb{Z}(p^\infty)$ (by Corollary 2.11) and S is a set of generators of $\mathbb{Z}(p^\infty)$.

For the remaining groups, according to a result from [6], a group G that admits no surjective homomorphism $G \to \mathbb{Z}(p^\infty)$ has finite rank n and for every copy $N \cong \mathbb{Z}^n$ in G one has $G/N \cong \bigoplus_p (F_p \times B_p)$, where F_p is a finite p-group and B_p is a bounded p-group. It is clear now that a group G with this property is either finitely generated or admits a surjective homomorphism onto a group of the form $H = \bigoplus_{n=1}^\infty C_i$, where all groups C_i are finite cyclic. To see that H has a microscopic set of generators let c_i be a generator of C_i and let $S = \{c_i\}_{i\in I}$. It suffices to observe that $c_i \to 0$ in the Tychonov topology of H, when each group C_i is equipped with the discrete topology. Now Corollary 2.11 and Lemma 2.21 apply.

It remains only to note that if G is finitely generated it suffices to consider the case $G = \mathbb{Z}$ (see (a) in Lemma 3.3). Now one takes the set $S = \{2^n\}_{n=0}^\infty$. This set is microscopic as $2^n \to 0$ in the 2-adic topology of $\mathbb{Z}$ (cf. Corollary 2.11). □

Recall that the *derived series* $G^{(n)}$ of a group G is defined by: $G^{(0)} = G$ and $G^{(n+1)}$ is the commutator group of $G^{(n)}$. A group G is *solvable* if $G^{(n)} = \{1\}$

for some integer n, G is perfect if $G = G'$. The group G is *hyperabelian* if G contains no perfect subgroups beyond the trivial one.

Proposition 3.7. (a) *If G/G' is infinite then G has a set of generators that is microscopic and strongly P-small;*

(b) *All free groups belong to $\mathcal{S}_P \cap \mathcal{M}$.*

(c) *$G \in \mathcal{S}_2$ if G is an infinite solvable group.*

(d) *$G \in \mathcal{S}_2$ when $G/G^{(n)}$ is infinite for some n.*

(e) *$G \in \mathcal{S}^P$ if G has an infinite abelian normal subgroup (in particular, if G has infinite center).*

Proof. (a) Follows from Lemma 3.3 and Theorem 3.6 (for abelian groups, microscopic implies strongly P-small).

(b) Follows from (a) and Lemma 3.3 since F/F' is infinite for a free group F.

(c) Pick the smallest k such that the factor $G^{(k)}/G^{(k+1)}$ is infinite (since G is infinite such a k must exist). Then by (a) the (infinite) subgroup $G^{(k)}$ has a microscopic and strongly P-small set of generators (recall that microscopic implies 2-small).

To see that this yields $G \in \mathcal{S}_2$ we need the following property. If N is a normal subgroup of a group G with $N \in \mathcal{S}_2$, then $|G/N| < \infty$ implies $G \in \mathcal{S}_2$. Indeed, let S be a 2-small set of generators of N. Chose a finite transversal F of N. Then the set $S_1 = S \cup F$ is 2-small. Indeed, it suffices to see that if S is 2-small, i.e., $S \in \mathbb{S}(N)$ and $S^{-1} \cdot S$, $S \cdot S^{-1}$ are not large in N then also for any $x \in G$ the union $S_1 = S \cup \{x\}$ is 2-small in G. Now $S_1 \cdot S_1^{-1} = (S \cdot S^{-1}) \cup Sx^{-1} \cup xS^{-1} \cup \{1\}$. Now $Sx^{-1} \cup xS^{-1} \cup \{1\}$ is a small set, and $S \cdot S^{-1}$ is not large (by Remark 2.22), therefore their union $S_1 \cdot S_1^{-1}$ cannot be large (as the difference between a large set and a small one is a large set [2, Theorem 1.4]).

(d) The quotient $G/G^{(n)}$ is an infinite solvable group, so (c) and Lemma 3.3 apply.

(e) Follows from (a) of Lemma 3.3 and Theorem 3.6. □

Remark 3.8. (i) For the class $\mathcal{S}$ (b) and (e) were proved in [22, Proposition 6] and [22, Prop. 10] respectively.

(ii) In the proof of (c) above, it is proved implicitly that the union of a 2-small set and a finite set is still 2-small. We do not know if this property holds for microscopic sets as well (this holds true in abelian groups, but the above proof needs a non-abelian version of the property).

(iii) A more careful analysis of the proof shows that every solvable group admits a *skinny* set of generators, i.e., a finite union of strongly left (or right) P-small sets, that are surely small. Note that there are small sets that are not skinny, since skinny sets have Banach measure zero, while small sets need not have this property.

It follows from (e) that $A \rtimes Aut(A) \in \mathcal{S}$ when A is an infinite abelian group (since A is a normal subgroup of $A \rtimes Aut(A)$).

3.2. **Linear groups and permutation groups.** According to Proposition 3.7 (e) the linear group $GL_n(K)$ has a small set of generators for every infinite field K and $n \geq 1$ (its center is infinite). On the other hand, smaller linear groups (as the one of all upper (resp., lower) triangular matrices in $GL_n(K)$) are solvable, hence again have small sets of generators (by Proposition 3.7). The following two instances cannot be obtained directly from that proposition.

Theorem 3.9. *The group $G = SL_n(A)$ has a small set of generators for every infinite euclidean domain A and $n > 1$.*

Proof. Let T_+ (T_-) denote the subgroup of all upper (resp., lower) triangular matrices in $G = SL_n(K)$ and let D denote the subgroup of all diagonal matrices in G. Then one can easily check that D, T_- and T_+, along with the finite set $\{\pi_1, \ldots, \pi_s\}$ of all matrices in G, having a single non-zero entry equal to ± 1 on each row and column, generate the group G. Now the subgroups T_- and T_+ are solvable, hence small generated. The same holds for the abelian group D. Hence Lemma 3.1 applies to conclude $G \in \mathcal{S}$. □

Example 3.10. For some euclidean domains A (e.g., $A = \mathbb{Z}$) the groups $SL_n(A)$ are actually finitely generated. This holds true when the additive group $(A, +)$ is torsion-free and finitely generated (so isomorphic to $\mathbb{Z}^m$ for some $m \in \mathbb{N}$). Indeed, for every $i \neq j$ and for every generator b of the additive group $(A, +)$ consider the matrix $\alpha_{ij}^b = I_n + bE_{ij}$, where E_{ij} is the matrix with only one non-zero entry 1 placed at position ij. Then the matrices α_{ij}^b along with the matrices π_i generate $SL_n(A)$. For a short proof by induction note that starting with an arbitrary matrix $\xi \in SL_n(A)$ after appropriate permutation of the rows and columns (achieved by multiplication by various π_i) and multiplication by matrices α_{ij}^b one can arrange to obtain from ξ a matrix (a_{ij}) having the entry $a_{11} \neq 0$ with minimal $\delta(a_{11})$, where δ denotes the Euclidean norm in A, all other entries on the first row or column are zero (as $\xi \in SL_n(A)$, the entry a_{11} is necessarily invertible). Now the inductive hypothesis applies to the minor obtained by removing the first row and the first column.

Theorem 3.11. *Let X be an infinite set, let $S(X)$ be the group of all permutations of X and let $S_\omega(X)$ be the subgroup of $S(X)$ of all permutations of finite support. Then $S(X) \in \mathcal{S}^P$ and $S_\omega(X)$ has a set of generators that is microscopic and strongly P-small.*

Proof. First we prove that the group $G = S_\omega(X)$ has a microscopic strongly P-small set of generators. Obviously the set $\mathcal{T}$ of all transpositions of G generates G. We show that this set is microscopic and strongly P-small. Indeed, for any subset $A \subseteq X$ denote by G_A the subgroup of all permutations with support contained in A. Since the set $\mathcal{T} = \mathcal{T}^{-1}$ is symmetric and invariant under conjugation, to check that $\mathcal{T}$ is strongly P-small it suffices to check that it is right P-small. Indeed, $S = \mathcal{T} \cdot \mathcal{T} = \mathcal{T}^{-1} \cdot \mathcal{T} = \mathcal{T} \cdot \mathcal{T}^{-1}$ is not large, as every element of S is either a 2-cycle, or a 3-cycle or a product of two disjoint

2-cycles. Hence $G \neq S \cdot G_F$ for any finite $F \subseteq X$. Now we can conclude with Remark 2.5 that $\mathcal{T}$ is strongly left P-small and strongly right P-small, hence $\mathcal{T}$ is small. Since an analogous argument shows that S^n is not large for every $n \in \mathbb{N}$ we conclude that $\mathcal{T}$ is also microscopic.

The normal subgroup $S_\omega(X)$ of $S(X)$ belongs to $\mathcal{S}^P$, so Lemma 3.3 implies $S(X) \in \mathcal{S}^P$. For a direct alternative proof of $S(X) \in \mathcal{S}$, one can apply Lemma 3.1 since G is an infinite normal subgroup of $S(X)$ of infinite index (any permutation of infinite order of X will witness it). □

The famous theorem of Graham Higman, Bernhard Neumann and Hanna Neumann [18] says that every countable group is isomorphic to a subgroup of a 2-generated group. In this spirit we have:

Theorem 3.12. *Every group is a subgroup of a small generated group and a quotient of an $\mathcal{M}$-generated group.*

Proof. Let G be an infinite group. It suffices to embed G in the group $S(G)$ of all permutations of the set G.

For the second assertion it suffices to write G as a quotient of a free group and apply Lemma 3.1 □

It is still an open question whether $\mathcal{S}$ coincides with the class $\mathcal{G}$ of *all* groups [22]. It follows from the above theorem that $\mathcal{S} = \mathcal{G}$ is equivalent to either of the following invariance properties of $\mathcal{S}$: (i) stability under taking subgroups; (ii) stability under taking quotients.

Another open question from [22] is whether every uncountable group G admits a small set of size $|G|$. If this is true for all groups of size ω_1, then Shelah's group G ([25], let us recall that all proper subgroups of G are countable) admits a small uncountable set S, then S will generate G, so that G will be small generated. In the next section we study the question when $G \in \mathcal{S}$ for groups G that admit a non-discrete group topology (close to being compact).

3.3. **Some (compact-like) topological groups have small set of generators.** It was shown in [22] that all non-metrizable compact groups belong to $\mathcal{S}$ and it was asked whether the class $\mathcal{S}$ contains all compact groups. Here we obtain further progress in this direction.

Theorem 3.13. *$\mathcal{S}$ contains all compact groups that are not totally disconnected.*

Proof. Let G be a compact group that is not totally disconnected. Then $c(G) \neq \{1\}$, hence it is an infinite normal subgroup of G. If it has infinite index, then $G \in \mathcal{S}$ by Lemma 3.1. Hence it remains to consider the case when $c(G)$ has finite index. In this case it suffices to prove $c(G) \in \mathcal{S}$, this will imply $G \in \mathcal{S}$. Hence it suffices to prove that every compact connected group belongs to $\mathcal{S}$.

Let us see first that every connected compact Lie group G belongs to $\mathcal{S}$. Indeed, let T be a maximal torus of G. Then G has a finite number of Borel subgroups $B_1, \ldots, B_n$ containing T and they generate G ([21, §25.2, Corollary

B]). Since the subgroups B_i are solvable, we can apply Lemma 3.1 and conclude $G \in \mathcal{S}$.

We prove now that every compact connected group G belongs to $\mathcal{S}$. If G is abelian then we apply Theorem 3.6. Otherwise, the quotient $G/Z(G)$ is a product of simple connected Lie groups [26], [19, Th. 9.24]. In particular, some non-trivial quotient of G is a compact connected Lie group, so that we can apply the first part of the argument and Lemma 3.3. □

Following the above proof one can expect that the above theorem extends to all *locally* compact groups. Indeed, it suffices to prove that every connected locally compact group belongs to $\mathcal{S}$. Since such groups are projectively Lie groups (i.e., inverse limits of Lie groups), it suffices to prove that every simple Lie group belongs to $\mathcal{S}$. This makes us believe that this extension to the locally compact case is possible.

As far as compact groups are concerned, the above theorem reduces the problem to totally disconnected compact groups, i.e., *profinite groups*. Let us recall, that profinite (pro-p) groups are inverse limits of finite (finite p-torsion) groups. A topological group G is said to be *topologically finitely generated* if it has a dense finitely generated subgroup.

We have no proof at hand for the following conjecture (see Remark 2.5 for a detailed comment).

Conjecture 3.14. *Every topologically finitely generated pro-p group belongs to $\mathcal{S}$ for every prime number p.*

The *Frattini subgroup* $\Phi(G)$ of a profinite group G is the intersection of all maximal open subgroups of G. It is easy to see that $\Phi(G)$ is a closed normal subgroup of G.

Conjecture 3.15. *Every profinite group with trivial Frattini subgroup belongs to $\mathcal{S}$.*

Since non-metrizable profinite groups already belong to $\mathcal{S}$, to verify this conjecture one should consider only metrizable profinite groups G with $\Phi(G) = \{1\}$. Then there will be countably many maximal open subgroups M_n. For every maximal open subgroup M the largest normal subgroup M_G of G contained in M is still open in G. Thus we obtain a countable family N_n of open normal subgroups, such that each one is an intersection of maximal open subgroups, and $\bigcap_n N_n = \{1\}$. Since G is compact, this implies that the open normal subgroups $U_n = \bigcap_{k=1}^{n} N_k$ form a local base at 1.

Let us see now that a positive answer to these two conjecture would imply that every compact group is small generated.

Theorem 3.16. *If $\mathcal{S}$ contains all topologically finitely generated pro-p groups for every prime p and all profinite group with trivial Frattini subgroup, then all compact groups belong to $\mathcal{S}$.*

Proof. Let G be a compact group. If G is not totally disconnected Theorem 3.13 applies. Hence from now on we suppose that G is totally disconnected, i.e., profinite. Then its Frattini subgroup $\Phi(G)$ is pronilpotent ([24, Corollary 2.8.4]), hence isomorphic to the direct produt $\prod_p G_p$ of its p-Sylow subgroups G_p ([24, Proposition 2.3.8]). If the subgroup $\Phi(G)$ of G has infinite index, then consider the quotient $G_1 = G/\Phi(G)$ that has $\Phi(G_1) = \{1\}$ (cf. [24, Proposition 2.8.2 (a)]), so $G_1 \in \mathcal{S}$ by Conjecture 3.15 and $G \in \mathcal{S}$ by Lemma 3.1. Therefore, from now on we assume that the subgroup $\Phi(G)$ has finite index, i.e., $\Phi(G)$ is open. Hence it suffices to prove that $\Phi(G) \in \mathcal{S}$. If at least two of the groups G_p are infinite, then we are done by Lemma 3.1. If infinitely many groups G_p are non-zero then again Lemma 3.1 works. So it remains the case when precisely one of the groups G_p is infinite and there exists finitely many non-trivial groups G_q (with $q \neq p$) and all of them are finite. In other words, G_p is open in G. Hence we can assume without loss of generality that G is a pro-p-group. Then by [24, Lemma 2.8.7 (b)] the quotient group $G/\Phi(G)$ is an abelian group, hence $G/\Phi(G) \in \mathcal{S}$ in case it is infinite. Therefore, we can assume from now on that $G/\Phi(G)$ is finite, i.e., $\Phi(G)$ is open in G. Then G is topologically finitely generated by [24, Proposition 2.8.10], so that Conjecture 3.14 applies now. □

According to this theorem, if there exists a compact group without a small set of generators, then there exists also a profinite metrizable group G with this property, that is either a topologically finitely generated pro-p group, or has $\Phi(G) = \{1\}$.

Remark 3.17. Let us discuss here our grounds to believe that Conjecture 3.14 is true. When G is a topologically finitely generated pro-p group, the topology of G coincides with its pro-finite topology (that has as typical neighborhoods of 1 *all* subgroups of finite index of G), since by Serre's theorem [28, §4.3] every finite index subgroup of G is open. Moreover, the Frattini series $\{\Phi^n(G)\}_{n\in\mathbb{N}}$ (defined inductively by $\Phi^1(G) = \Phi(G)$ and $\Phi^{n+1}(G) = \Phi(\Phi^n(G))$ for $n \geq 1$) forms a fundamental system of neighborhoods of 1 in G for the pro-finite topology of G. Taking into account that $\Phi(G) = \overline{G^pG'}$, where $G^p = \{x^p : x \in G\}$ (cf. [24, Lemma 2.8.7 (c)]), this describes completely the topological group G in purely algebraic terms. It was proved by E. Zel′manov [30] (see also [24, Theorem 4.8.5c]) that the torsion finitely generated pro-p groups are finite. So our conjecture holds true for *torsion* groups. On the other hand, it holds true also for all profinite groups of *finite rank* (a profinite group G is said to be of finite rank d if every closed subgroup of G has at most d topological generators, [28, Chap. 8]). Indeed, it is known that every profinite groups of finite rank G admits closed normal subgroups $C \leq N$ such that N has finite index in G, C is nilpotent and N/C is solvable. Now it is clear that $G \in \mathcal{S}$ when C is infinite (as then $C \in \mathcal{S}$ as a solvable group so that Lemma 3.1 applies). When C is finite, then N and N/C are necessarily infinite, so that $N/C \in \mathcal{S}$ (by Proposition 3.7) and consequently $G/C \in \mathcal{S}$. Now Lemma 3.1 applies again. Hence the conjecture remains to be proved only for the topologically finitely generated pro-p groups of infinite rank.

It should be noted that unlike the finite p-groups, the pro-p groups may have a trivial center even in the case of very nice groups. For example, if K_1

denotes the subgroup $\begin{pmatrix} 1+p\mathbb{Z}_p & p\mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix}$ of the linear group $GL_2(\mathbb{Z}_p)$, then the profinite group $G = SL_2(\mathbb{Z}_p) \cap K_1$ has trivial center. Nevertheless, one can prove explicitly that this group belongs to $\mathcal{S}$. Indeed, one has to observe that the subgroups T^+ and T^- of upper and lower triangular matrices, respectively, in G and the finite set of all permutation matrices generate G. Then one notes also that both groups T^+ and T^- are soluble, so that Lemma 3.1 applies.

The topologically finitely generated free pro-p groups belong to $\mathcal{M} \subseteq \mathcal{S}$ as they have infinite abelian quotients.

A subset A of a topological group is said to be (*totally*) *bounded* if for every non-empty open set U of G there exists a finite set F such that $A \subseteq F \cdot U$, A is said to be *σ-bounded* if A is a countable union of bounded subsets of G.

Proposition 3.18. *If G is a σ-bounded topological group, then G admits a closed normal subgroup N of infinite index with $\psi(G/N) \leq \omega$. If $\psi(G) > \omega$, then N must be infinite and G is small generated.*

Proof. If G is discrete, then the conclusion is obvious. Otherwise, since the union of finitely many bounded sets is again a bounded set, there exists an increasing chain $K_1 \subseteq K_2 \subseteq \ldots \subseteq K_n \subseteq \ldots$ of bounded sets of G such that $G = \bigcup_{n=1}^{\infty} K_n$. We shall use in the sequel the following property of a bounded set K: for every neighborhood V of 1 there exists a symmetric neighborhood U of 1 such that $U^x \subseteq V$ for every $x \in K$. Let $U_0 \neq G$ be a symmetric open neighborhood of 1. Define inductively a decreasing chain

$$U_0 \supseteq U_1 \supseteq \ldots \supseteq U_n \supseteq \ldots \tag{1}$$

of symmetric neighborhoods of 1 such that for each n

(a) $U_n^2 \subset U_{n-1}$ and the inclusion is proper;
(b) $U_n^x \subseteq U_{n-1}$ for every $x \in K_n$.

Note that a) yields $\overline{U}_n \subseteq U_{n-1}$. Therefore the set $N = \bigcap_n U_n$ is a closed subgroup of G. Let $x \in G$. Then there exists $n \in \mathbb{N}$ such that $x \in K_n$. Since $N = \bigcap_{m \geq n} U_m$, one can easily see that $N^x \leq N$. Therefore N is a normal subgroup of G. Let us show now that the subgroup N has infinite index. Indeed, let $f : G \to G/N$ be the canonical quotient map. Since all inclusions in (a) are chosen to be proper, then also all inclusions $f(U_n) \supset f(U_{n+1})$ are proper as $f(U_n) = f(U_{n+1})$ yields $U_n \subseteq N \cdot U_{n+1} \subseteq U_{n+1}^2$, a contradiction. Therefore G/N is infinite and obviously $\psi(G/N) \leq \omega$.

Finally, N finite would immediately imply $\psi(G) \leq \omega$. Consequently, N is infinite and G is small generated by Lemma 3.1 (a). □

Remark 3.19. For an alternative argument in the case of a non-metrizable locally compact σ-compact group G see [22]. Note that every group with those properties satisfies also the hypothesis of our proposition as locally compact non-metrizable groups necessarily have uncountable pseudocharacter. The local compactness was exploited in [22] to get metrizability of the quotient G/N that in general has countable pseudocharacter, as the above proposition shows.

A topological group G is *totally minimal* if every continous surjective homomorphism with domain G is open.

Theorem 3.20. *A topological group $G \in \mathcal{S}$ in each of the following eight cases:*

(a) *G is σ-bounded with $\psi(G) > \omega$;*
(b) *G is σ-compact with $\psi(G) > \omega$;*
(c) *G is SIN-group with $\psi(G) > \omega$;*
(d) *G is precompact with $\psi(G) > \omega$;*
(e) ([22, Theorem 14]) *G is compact and non-metrizable;*
(f) *G is pseudocompact and non-compact;*
(g) *G is totally minimal, precompact and non-metrizable;*
(h) *G is not totally disconnected and $c(G)$ has infinite index (in particular, if it is not open in G).*

Proof. (a) By Proposition 3.18 we get an infinite normal subgroup N of infinite index. Now Lemma 3.1 gives $G \in \mathcal{S}$.

(b) Every compact set is bounded, thus (a) applies.

(c) Let us recall that a SIN-group has (by definition) a base of *invariant* neighborhoods of 1 (so that SIN stands for small invariant neighborhoods). Now the proof of Proposition 3.18 can be carried out in this case to get a normal subgroup N of infinite index with $\psi(G/N) \leq \omega$. Now the assumption $\psi(G) > \omega$ yields N is infinite.

(d) Since every precompact group is a SIN-group, we can apply (c).

(e) Since every non-metrizable compact group has uncountable pseudocharacter, we can apply (d).

(f) Every pseudocompact group is precompact ([5]) and every pseudocompact group of countable pseudocharacter is compact [4]. Therefore G is precompact of uncountable pseudocharacter, so that (d) applies.

(g) Indeed, if G is compact then this follows from (e). Now assume that G is not compact. Then the completion K of G is a compact non-metrizable group. Then K has a closed normal subgroup N of infinite index with $\psi(K/N) = \omega$ by Proposition 3.18. Since K/N is compact, we get $\chi(K/N) = \psi(K/N) = \omega$, so that K/N is metrizable. Therefore, N is infinite as K is not metrizable. Moreover, $N \cap G$ is dense in N by the total minimality of G ([7, Theorem 4.3.3]), hence $N \cap G$ is infinite. Moreover, $N \cap G$ has infinite index in G, since otherwise it would be an open subgroup of G and consequently $N = \overline{G \cap N}$ would be an open subgroup of K, a contradiction. Therefore, $G \in \mathcal{S}$ by Lemma 3.1 (a).

(h) $c(G)$ is an infinite closed normal subgroup of G. Since it has infinite index in G, Lemma 3.1 (a) applies. □

It follows from (h) that if there exists an infinite topological group failing to have a small set of generators then there exists also such a group G that is either connected or totally diconnected. In the former case the arc-component of G is either trivial or has finite index (note that the the arc-component need not be closed even if G is compact, hence this need not immediately imply, by connectendness of G, that G is also arc-wise connected).

4. Questions.

(i) Does $\mathcal{S}$ contain all perfect groups?
(ii) Does $\mathcal{S}$ contain all hyperabelian groups of class ω with finite factors (i.e., all groups G such that $\bigcap_{n=1}^{\infty} G^{(n)} = \{1\}$ and all quotients $G^{(n)}/G^{(n+1)}$ are finite)?
(iii) Does every countable group have a small set of generators?
(iv) For which commutative rings K the group $SL_n(K)$ has a small set of generators?

Positive answers to (i) and (ii) yield that $\mathcal{S}$ contains all groups. Indeed, if the derived series of G stops after a finite number of steps, then $G^{(n)}$ is perfect for some n. If $G/G^{(n)}$ is infinite, then Lemma 3.3 and Proposition 3.7 apply to give $G \in \mathcal{S}$. Otherwise, $G^{(n)}$ is infinite and $G \in \mathcal{S}$ iff $G^{(n)} \in \mathcal{S}$ (i.e., (i) applies). The alternative is to have all quotients $G^{(n)}/G^{(n+1)}$ non-trivial finite. Then to the infinite quotient $G/\bigcap G^{(n)}$ (ii) applies.

Note 4.1. Added in proof, July 2003: Recently, I. Protasov and T. Banakh resolved the main problem of this paper by establishing that every group has a small set of generators (see Theorem 13.1 in the monograph "Ball Structures and Colorings of Graphs and Groups". Mathematical Studies Monograph Series, 11. VNTL Publishers, Lviv, 1999, 147 pp. ISBN 966-7148-99-8). In particular, this answers positively also our conjectures 3.14 and 3.15 and questions (i)–(iv) above.

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