On the exponent of mutually permutable products of two abelian groups

A. Ballester-Bolinches, John Cossey and M.C. Pedraza-Aguilera

Abstract

In this paper we obtain some bounds for the exponent of a finite group, and its derived subgroup, which is a mutually permutable product of two abelian subgroups. They improve the ones known for products of finite abelian groups, and they are used to derive some interesting structural properties of such products.
Mathematics Subject Classification (2010): 20D10, 20D20
Keywords: finite group, abelian group, exponent, factorisations, p-supersolubility, p-length.

1 Introduction

Throughout this paper all groups are finite and p denotes a fixed prime.

We recall that two subgroups A and B of a group G are said to permute if AB is a subgroup of G. A and B are called *mutually permutable* if every subgroup of A permutes with B and every subgroup of B permutes with A. If every subgroup of A permutes with every subgroup of B we say that A and B are *totally permutable*. Obviously totally permutable subgroups are mutually permutable but the converse does not hold in general.

Products of mutually and totally permutable subgroups have been widely studied in the last twenty five years and receive a full discussion in [1]. The emphasis was on how the structure of the factors A and B affects the one of the factorised group $G = AB$ and viceversa. The class of all p-supersoluble groups, or p-soluble groups in which every p-chief factor is cyclic, is one of the most useful classical classes to highlight the differences between totally and mutually permutable products. Every totally permutable product of two p-supersoluble subgroups is p-supersoluble. In particular, G', the derived subgroup of G, is p-nilpotent. This result does not remain true for mutually permutable products (see [1, Example 4.1.32]). However, in that example, the derived subgroup is still p-supersoluble.

This then brings up the natural question of whether or not the derived subgroup of a mutually permutable product of two p-supersoluble groups is p-supersoluble. A possible minimal counterexample is a primitive p-soluble group whose core-free maximal subgroup is a mutually permutable product of two abelian subgroups of exponent dividing $p-1$. Hence, the answer seems to depend on obtaining suitable bounds for the exponent of a mutually permutable product of two abelian subgroups in terms of the exponents of its factors.

If $G = AB$ is a product of two abelian subgroups, then G is metabelian by a well-known result of Itô. Futhermore, by a result of Howlett [5], if e and f are the exponents of A and B respectively, then the exponent of G divides ef. If $G = AB$ is a metabelian group which is the product of two subgroups A and B of exponents e and f respectively, then Mann [7] proves that the exponent of G divides $(ef)^3$ and if either A or B is abelian, then the exponent of G divides $(ef)^2$.

Our main aim in this paper is to obtain better bounds on the exponent of a mutually permutable product G of two abelian subgroups in terms of the exponents of the factors. A bound on the exponent of G' that allows us to give an affirmative answer to the above question is also determined. Some results about the p-length of mutually permutable products will follow as direct consequences of our main theorems.

2 Preliminaries

It is assumed that the reader is familiar with the notation presented in [1] and [3]. In order to make our paper reasonably self-contained, we collect in the following lemma some well-known facts about divisibility of binomial coefficients.

Lemma 1. *(i) ([4, Lemma]) For all positive integers q, r and i, with $i \leq p^r q$, $\binom{p^r q}{i} p^i$ is divisible by p^{r+1} .*

(ii) If $1 < i < p^{i-1}$, then $\binom{p^{i-1}}{i}$ is divisible by p.

The following expansion formulas for metabelian groups are also needed (see [6, Lemma 3.1]).

Lemma 2. *Let G be a metabelian group and n a positive integer. Then, for all $x, y \in G$,*

$$[x, y^n] = \prod_{i=1}^{n} [x, iy]^{\binom{n}{i}}$$

and

$$(xy^{-1})^n = x^n \left(\prod_{0<i+j<n} [x, iy, jx]^{\binom{n}{i+j+1}} \right) y^{-n}$$

The following lemma gives information about groups which are products of two cyclic p-groups which is crucial in the proof of the results of the next section.

Lemma 3. *Let the p-group $G = \langle a \rangle \langle b \rangle$ be the product of two cyclic subgroups $A = \langle a \rangle$ and $B = \langle b \rangle$. Assume that the exponent of G divides p^n and fix $1 \leq i \leq n-1$. Then:*

(i) $G^{p^i} \cap A = \langle a^{p^i} \rangle$ *and* $G^{p^i} \cap B = \langle b^{p^i} \rangle$ *and* $G^{p^i} = \langle a^{p^i} \rangle \langle b^{p^i} \rangle$. *In particular,* $(G^{p^i})^p = G^{p^{i+1}}$.

(ii) $(G^{p^i})^{p^j} = G^{p^{i+j}}$ *for all* j.

Proof (i) We use induction on i. According to [1, Corollary 3.1.9], G is a totally permutable product of the subgroups A and B. Therefore $G^p = (G^p \cap A)(G^p \cap B) = \langle a^\alpha \rangle \langle b^\beta \rangle$ by [1, Theorem 4.1.48]. Suppose that p does not divide α. Then $1 = \alpha\alpha_0 + p\beta_0$ and $a = (a^\alpha)^{\alpha_0}(a^p)^{\beta_0} \in G^p \leq \Phi(G)$. Consequently $G = \langle b \rangle$, as desired. Therefore we may assume that $G^p \cap A = \langle a^p \rangle$ and, analogously, $G^p \cap B = \langle b^p \rangle$. Therefore the statement holds for $i = 1$.

Assume that $i > 1$, $G^{p^{i-1}} \cap A = \langle a^{p^{i-1}} \rangle$, $G^{p^{i-1}} \cap B = \langle b^{p^{i-1}} \rangle$ and $G^{p^{i-1}} = \langle a^{p^{i-1}} \rangle \langle b^{p^{i-1}} \rangle$. Then, by [1, Corollary 3.1.9], $G^{p^{i-1}}$ is the totally permutable product of the subgroups $\langle a^{p^{i-1}} \rangle$ and $\langle b^{p^{i-1}} \rangle$. Applying [1, Theorem 4.1.48], $(G^{p^{i-1}})^p = ((G^{p^{i-1}})^p \cap \langle a^{p^{i-1}} \rangle)((G^{p^{i-1}})^p \cap \langle b^{p^{i-1}} \rangle)$. We can apply now the above arguments to conclude that $(G^{p^{i-1}})^p \cap \langle a^{p^{i-1}} \rangle = \langle a^{p^i} \rangle$ and $(G^{p^{i-1}})^p \cap \langle b^{p^{i-1}} \rangle = \langle b^{p^i} \rangle$. Since G^{p^i} is a subgroup of $(G^{p^{i-1}})^p$, it follows that $(G^{p^{i-1}})^p = G^{p^i} = \langle a^{p^i} \rangle \langle b^{p^i} \rangle$ and the induction is complete.

(ii) We proceed by induction on j. Statement (i) tells us that the result is true for $j = 1$. Assume that $j > 1$ and $(G^{p^i})^{p^{j-1}} = G^{p^{i+j-1}}$. Then $(G^{p^i})^{p^j} = (G^{p^i})^{p^{j-1}p} \leq ((G^{p^i})^{p^{j-1}})^p = (G^{p^{i+j-1}})^p = G^{p^{i+j}} \leq (G^{p^i})^{p^j}$. Consequently $G^{p^{i+j}} = (G^{p^i})^{p^j}$, as desired.

3 Bounding the exponent of mutually permutable products of two abelian groups

If the group G is the product of two abelian groups A and B and they have finite exponent, then so does G, and moreover the exponent of G is bounded in terms of the exponents of A and B [4]. The aim of this section is to prove that this bound can be considerably improved in the case when G is either a totally or a mutually permutable product of two abelian subgroups. A bound for the exponent of the derived subgroup in terms of the exponents of the factors is also exhibited.

Theorem 1. *Let the group $G = AB$ be the product of the cyclic p-groups A and B. Assume that A and B have exponent dividing p^n. Then the exponent of G divides p^n. Moreover, the nilpotency class of G is at most n.*

Proof By [1, Corollary 3.1.9], G is the totally permutable product of A and B. Suppose that $A = \langle a \rangle$ and $B = \langle b \rangle$. We prove that the exponent of G divides p^n by induction on n. If $n = 1$, then G is p-elementary abelian and the theorem holds in this case. Suppose that $n > 1$ and write $Z = G^p$. Applying Lemma 3(i), $Z = \langle a^p \rangle \langle b^p \rangle$. Then Z is a totally permutable product of two cyclic p-groups of exponent dividing p^{n-1}. By induction, the exponent of Z divides p^{n-1}. Now if $x \in G$, then $x^p \in Z$. Consequently $(x^p)^{p^{n-1}} = x^{p^n} = 1$ and hence the exponent of G divides p^n.

We prove that the nilpotency class of G is at most n by induction on n. The result clearly holds when $n = 1$. Assume that $n > 1$. If $j < n$, then G/G^{p^j} is a product of two cyclic groups of exponent dividing p^j. By induction, the nilpotency class of G/G^{p^j} is at most j. Therefore $\gamma_{j+1}(G) \leq G^{p^j}$. In particular, $[a, tb] \in G^{p^{n-1}}$ for all $t \geq n-1$.

Next we show that $G^{p^{n-1}} \leq Z(G)$. By Lemma 2, we have $[a, b^{p^{n-1}}] = \prod_{j=1}^{p^{n-1}} [a, jb]^{\binom{p^{n-1}}{j}}$. Next, we see that all of the factors of the above product belong to $G^{p^n} = 1$. If $j < n$, then $[a, jb] \in G^{p^j}$. By Lemma 1, $r = \binom{p^{n-1}}{j}$ is divisible by p^{n-j}. Then $r = p^{n-j}t$ for some integer t. Since $G^r \leq G^{p^{n-j}}$, we have that $[a, jb]^r \in (G^{p^j})^r \leq (G^{p^j})^{p^{n-j}}$. By Lemma 3(ii), $[a, jb]^{\binom{p^{n-1}}{j}} \in G^{p^n} = 1$. On the other hand, applying Lemma 3(i), $|G^{p^{n-1}}| \leq p^2$. Thus $[[G^{p^{n-1}}, G], G]$=1. In particular for every $j \geq n+1$ we have $[a, jb] = [[a, (n-1)b], b, b, mb] = 1$, $m \geq 0$. Assume that $j = n$. Then $[a, nb] \in G^{p^{n-1}}$. If $p = 2$ and $n = 2$, then G has exponent dividing 4 and the nilpotency class of G is at most 2. Assume that $p \neq 2$ or $p = 2$ and $n \neq 2$. Then $n < p^{n-1}$. Hence, by Lemma 1(ii), it follows that $\binom{p^{n-1}}{n}$ is divisible by p. Hence $[a, nb]^{\binom{p^{n-1}}{n}} \in (G^{p^{n-1}})^p = G^{p^n} = 1$ by Lemma 3(i). Thus $b^{p^{n-1}}$ is central in G. Analogously, $a^{p^{n-1}}$ is central in G. Consequently, $G^{p^{n-1}} \leq Z(G)$ and G has nilpotency class at most n, as wanted.

The general case follows from Theorem 1. Specifically, we have:

Corollary 1. *Let the group $G = AB$ be the product of the totally permutable abelian subgroups A and B. Assume that the exponents of A and B divide ${p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_n}^{\alpha_n}$ for distinct primes $p_1, p_2, \ldots, p_n$. Then the exponent of G divides ${p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_n}^{\alpha_n}$.*

Proof First at all, note that the exponent of G is equal to the product of the exponents of its Sylow subgroups. By [1, Theorem 1.1.19], there exists Sylow p_i-subgroups P_i, A_{p_i} and B_{p_i} of G, A and B, respectively, such that $P_i = A_{p_i}B_{p_i}$ is the totally permutable product of the subgroups A_{p_i} and B_{p_i}, for each $i \in \{1, 2, \ldots, n\}$. Let x_i be an element of P_i. Then there exist $a_i \in A_{p_i}$ and $b_i \in B_{p_i}$ such that $x_i = a_i b_i$ and so x_i belongs to $\langle a_i\rangle\langle b_i\rangle$, which is the totally permutable product of the cyclic subgroups $\langle a_i\rangle$ and $\langle b_i\rangle$. By Theorem 1, the exponent of $\langle a_i\rangle\langle b_i\rangle$ divides ${p_i}^{\alpha_i}$. Therefore the order of x_i divides ${p_i}^{\alpha_i}$. Consequently, the exponent of G divides ${p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_n}^{\alpha_n}$, as desired.

In [4, Theorem 1], it is proved that if $G = AB$, where A and B are abelian groups with exponent dividing e, where $e = {p_1}^{r_1}{p_2}^{r_2}\cdots{p_n}^{r_n}$, and $p_1, p_2, \ldots, p_n$ are distinct primes, then the exponent of G' divides f, where $f = ep_1 \ldots p_n$. In particular, the exponent of G divides ef.

Our next results show that in the case when A and B are mutually permutable subgroups of $G = AB$, then the exponent of G divides f and the exponent of G' divides e.

Theorem 2. *Let the group $G = \langle a\rangle Z\langle b\rangle Z$ be the product of the mutually permutable abelian p-subgroups $\langle a\rangle Z$ and $\langle b\rangle Z$. Assume that $\langle a\rangle$ and $\langle b\rangle$ have exponent dividing p^n and Z has exponent dividing p^m. Then the exponent of G divides p^t, where $t = max(n + 1,\ m)$.*

Proof Let $T = \langle a\rangle Z \cap \langle b\rangle Z$. By [1, Proposition 4.1.16], we have that $G/T = (\langle a\rangle T/T)(\langle b\rangle T/T)$ is the product of the totally permutable subgroups $\langle a\rangle T/T$ and $\langle b\rangle T/T$. By Theorem 1, the exponent of G/T divides

p^n and the nilpotency class of G/T is at most n. Let x be an element of G. Then $x = shz$ for some $s \in \langle a \rangle$, $h \in \langle b \rangle$, and $z \in Z$. Next we show that $(sh)^{p^{n+1}} = 1$. Write $t = h^{-1}$. By Lemma 2, we have

$$(st^{-1})^{p^{n+1}} = s^{p^{n+1}} \left(\prod_{0<i+j<p^{n+1}} [s, it, js]^{\binom{p^{n+1}}{i+j+1}} \right) t^{-p^{n+1}} = \prod_{0<i+j<p^{n+1}} [s, it, js]^{\binom{p^{n+1}}{i+j+1}}$$

Moreover, $[s, it, js] = [[s, it, (j-1)s], s] = [s, [s, it, (j-1)s]]^{-1}$. Denote $u = [s, it, (j-1)s]$. Since $G/G^{p^k}T$ is the totally permutable product of the subgroups $\langle a \rangle G^{p^k}T/G^{p^k}T$ and $\langle b \rangle G^{p^k}T/G^{p^k}T$ by [1, Proposition 4.1.16], we can apply Theorem 1 and conclude that the nilpotency class of $(G/T)/(G^{p^k}T/T)$ is at most k, for each $k \leq n$. This means that $\gamma_{k+1}(G/T) \leq G^{p^k}T/T$ for all $k \leq n$. Hence if $i + j > n$, we have that $u \in T \leq Z(G)$ and so $[s, it, js] = [s, u]^{-1} = 1$. Assume that $i + j \leq n$. Then $uT \in G^{p^{i+j-1}}T/T$. By Lemma 3(ii), we know that $u^{p^{n-(i+j-1)}}T \in ((G/T)^{p^{i+j-1}})^{p^{n-(i+j-1)}} = T$ and $T \leq Z(G)$. Then $1 = [s, u^{p^{n-(i+j-1)}}]$. Use of Lemma 2, allows us to conclude that

$$1 = [s, u^{p^{n-(i+j-1)}}] = \prod_{r=1}^{p^{n-(i+j-1)}} [s, ru]^{\binom{p^{n-(i+j-1)}}{r}} = [s, u]^{p^{n-(i+j-1)}}$$

Note that $[s, ru] = 1$ for all $r \geq 2$ since G is metabelian. Consequently $[s, it, js]^{p^{n-(i+j-1)}} = ([s, u]^{-1})^{p^{n-(i+j-1)}} = 1$. On the other hand, by Lemma 1(i), $\binom{p^{n+1}}{i+j+1} p^{i+j+1}$ is divisible by p^{n+2}. Thus $\binom{p^{n+1}}{i+j+1}$ is divisible by $p^{n-(i+j-1)}$. Consequently, $(sh)^{p^{n+1}} = 1$, as wanted.

Finally, as $z \in Z \in Z(G)$, if $t = max(n+1, m)$, we obtain that $x^t = (sh)^{p^t} z^{p^t} = 1$. This completes the proof of the theorem.

We shall draw two conclusions from the preceding theorem.

Corollary 2. *Let the group $G = AB$ be the product of the mutually permutable abelian subgroups A and B. Assume that the exponents of A and B divide ${p_1}^{\alpha_1} {p_2}^{\alpha_2} \cdots {p_n}^{\alpha_n}$ for distinct primes $p_1, p_2, \ldots, p_n$. Then the exponent of G divides ${p_1}^{\alpha_1+1} {p_2}^{\alpha_2+1} \cdots {p_n}^{\alpha_n+1}$.*

Proof The result will follow if we bound the exponent of the Sylow subgroups of G. By [1, Theorem 1.1.19], for each prime p_i dividing $|G|$, there exists a Sylow p_i-subgroup of G, P_i say, such that P_i is prefactorised, that is, $P_i = (P_i \cap A)(P_i \cap B)$. Moreover $P_i \cap A = A_{p_i}$ and $P_i \cap B = B_{p_i}$ are the Sylow p_i-subgroups of A and B, respectively. By [1, Corollary 4.1.22], P_i is the mutually permutable product of A_{p_i} and B_{p_i}. Let x denote an element of P_i of maximum order. Then $x = ab$ with $a \in A_{p_i}$ and $b \in B_{p_i}$. Consider the subgroup $T = \langle a\rangle Z\langle b\rangle Z$ of P_i, where $Z = A_{p_i} \cap B_{p_i}$. Applying [1, Proposition 4.1.16], T is the mutually permutable product of the subgroups $\langle a\rangle Z$ and $\langle b\rangle Z$. By Theorem 2, the exponent of T divides p_i^r, where $r = max(\alpha_i+1, \alpha_i)$. Consequently the order of x divides $p_i^{\alpha_i+1}$, as desired.

Theorem 3. *Let the group $G = AB$ be the product of the mutually permutable abelian subgroups A and B. Assume that the exponents of A and B divide ${p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots {p_n}^{\alpha_n}$ for distinct primes $p_1, p_2, \ldots, p_n$. Then the exponent of G' divides ${p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots {p_n}^{\alpha_n}$.*

Proof By [1, Lemma 3.1.5], $G' = [A, B]$. It is enough to show that the exponent of a Sylow p_i-subgroup of G' divides ${p_i}^{\alpha_i}$ for each $i = 1, 2, \ldots, n$. Fix an index $i \in \{1, 2, \ldots n\}$ and denote $p = p_i$ and $\alpha = \alpha_i$. Let A_p and B_p be the Sylow p-subgroups of A and B respectively. Then, as in the above theorem, $P = A_pB_p$ is a Sylow p-subgroup of G which is the mutually permutable product of the subgroups A_p and B_p. Let $a \in A_p$, and $b \in B_p$. Write $T = \langle a\rangle Z\langle b\rangle Z$, where $Z = A_p \cap B_p$. By [1, Proposition 4.1.16], T is the mutually permutable product of $\langle a\rangle Z$ and $\langle b\rangle Z$. Since $(T/Z)/(T/Z)^p$ is abelian, it follows that $T'Z \leq T^pZ$. In particular, $[a, b] \in T^pZ$. On the other hand, by Lemma 3(i), $T^pZ = \langle a^p\rangle Z\langle b^p\rangle Z$. Moreover the exponent of $\langle a^p\rangle$ and $\langle b^p\rangle$ divides $p^{\alpha-1}$ and the exponent of Z divides p^α. Now the application of Theorem 2 yields the exponent of $T^pZ = \langle a^p\rangle Z\langle b^p\rangle Z$ divides p^α. Consequently the order of $[a, b]$ divides p^α.

Let x be an element of $P \cap G'$ of maximum order and suppose that the order of x is p^t for some $t > \alpha$. Since G' is abelian, we have that $x = \prod_{j=1}^{m}[a_j, b_j]^{t_j}$ where $[a_j, b_j]^{t_j}$ is a p-element for each $j \in \{1, 2, \ldots, m\}$ (note that for each $[s, t] \in G'$, $[s, t] \in \langle[s, t]\rangle = \langle[s, t]^\alpha\rangle\langle[s, t]^\beta\rangle$ where $\langle[s, t]^\alpha\rangle$ and $\langle[s, t]^\beta\rangle$ are the Sylow p-subgroup and Hall p'-subgroup of $\langle[s, t]\rangle$, respectively). Moreover there exists $r \in \{1, 2, \ldots, m\}$ such that the order of $[a_r, b_r]^{t_r}$ is p^t. Denote $[a_r, b_r] = [a, b]$. Let $A_{p'}$ and $B_{p'}$ be the Hall p'-subgroups of A and B respectively. By [1, Theorem 1.1.19], $A_{p'}B_{p'}$ is a Hall p'-subgroup of G. Then $a = uv$ and $b = wz$ for some $u \in A_{p'}$, $v \in A_p$, $w \in B_{p'}$ and $z \in B_p$. Thus $[a, b] = [u, z]^v[u, w]^{zv}[v, z][v, w]^z$ and

$[a,b]^{t_r} = ([u,z]^v)^{t_r}([u,w]^{zv})^{t_r}[v,z]^{t_r}([v,w]^z)^{t_r}$. It is clear that $[u,w]^{zv}$ is a p'-element. Suppose that $([u,z]^v)^{t_r}$ and $([v,w]^z)^{t_r}$ are not p-elements. Then we can express $[a,b]^{t_r}$ as a product of three p-elements in the following way:

$$[a,b]^{t_r} = ([u,z]^v)^{t_r}([u,w]^{zv})^{t_r}[v,z]^{t_r}([v,w]^z)^{t_r} = ([u,z]^v)^f[v,z]^{t_r}([v,w]^z)^l$$

for some non-negative integers f and l. Since $([u,z])^f$ belongs to a Sylow p-subgroup of $\langle u\rangle Z\langle z\rangle Z$, where $Z = A\cap B$, we have that the order of $([u,z]^v)^f$ divides p^α. Analogously the order of $([v,w]^z)^l$ divides p^α. Furthermore, v and z are both p-elements. Applying the above argument, it follows that the order of $[v,z]^{t_r}$ divides p^α. Hence the order of $[a,b]^{t_r}$ and x divides p^α, contrary to our supposition. Consequently, the exponent of $P\cap G'$ divides p^α. The proof of the theorem is now complete.

4 Applications to the structure of mutually permutable products

One well-known feature of the saturated formation of all p-supersoluble groups is that it is closed under the formation of totally permutable products ([1, Theorem 4.1.31]). Unfortunately, this result does not remain true for mutually permutable products, even if the factors are normal (see [1, Example 4.1.32]). However, a mutually permutable product G of p-supersoluble groups is p-supersoluble if and only if G' is p-nilpotent.

These results show the central role played by the derived subgroup in the structure of a mutually permutable product of p-supersoluble groups (see [2]). In particular, the question of whether or not the derived subgroup of a mutually permutable product of p-supersoluble groups is p-supersoluble is of interest.

Our first result of this section provides an affirmative answer to that question and it is a consequence of Theorem 3.

Theorem 4. *Let the group $G = AB$ be the mutually permutable product of the p-supersoluble subgroups A and B. Then G' is p-supersoluble.*

Proof We suppose that the theorem is false and derive a contradiction. Let G be a counterexample of minimal order. Since the class of all p-supersoluble groups is a saturated formation, it follows that G has a unique minimal normal subgroup, N say. Applying [1, Theorem 4.1.15], G is p-soluble and so N is an elementary abelian p-group of rank greater than 1. Assume that $\Phi(G) \neq 1$. Then $G'\Phi(G)/\Phi(G) \simeq G'/(G' \cap \Phi(G))$ is p-supersoluble. Consequently, G' is p-supersoluble. This contradiction shows that $\Phi(G) = 1$, and so there exists a core-free maximal subgroup M of G such that $G = NM$ and $M \cap N = 1$. Moreover, $N = C_G(N) = F(G) = O_p(G) = O_{p',p}(G)$.

Since A is p-supersoluble, A' is p-nilpotent by [1, Lemma 2.1.6], and subnormal in G by [1, Corollary 4.1.26]. Hence $A' \leq N$. This implies that A is supersoluble and so is B. Consequently they are Sylow tower groups with respect to the reverse natural ordering of the prime numbers. By [1, Corollary 4.1.30], the same is true for G. In particular, p is the largest prime dividing $|G|$ and N is the Sylow p-subgroup of G.

Suppose that $N \leq A$ and $N \cap B = 1$ (or $N \leq B$ and $N \cap A = 1$). By [1, Lemma 4.3.3(5)], we have that $N \leq C_G(B)$. Hence $B \leq N \leq A$ and $G' = A'$ is p-supersoluble, contrary to assumption. Therefore $N \leq A \cap B$ by [1, Lemma 4.3.9 and Lemma 4.3.3(4)]. Since A and B are both supersoluble and N is self-centralising in G, it follows that the Hall p'-subgroups of A and B are abelian of exponent dividing $p-1$. Let $A_{p'}$ and $B_{p'}$ be Hall p'-subgroups of A and B such that $A_{p'}B_{p'}$ is a Hall p'-subgroup of G. Without loss of generality, we may assume that $M = A_{p'}B_{p'}$. By [1, Corollary 4.1.22], M is the mutually permutable product of $A_{p'}$ and $B_{p'}$. By Theorem 3, the exponent of M' divides $p-1$. Applying [3, B; Lemma 7.1] and [3, B; Theorem 9.8], N is a sum of irreducible modules for M' of dimension 1. Consequently G' is p-supersoluble, the final contradiction.

The class of all p-soluble groups of p-length at most 1 is a saturated formation which is not closed under taking mutually permutable products: the symmetric group of order 4 has 2-length 2 and it is a mutually permutable product of a Sylow 2-subgroup and the alternating group of degree 4, both of 2-length 1.

Our next result establishes that the class of all p-soluble groups of p-length at most 1 is closed under the formation of mutually permutable products with p-supersoluble derived subgroup.

Theorem 5. *Let the group $G = AB$ be the mutually permutable product of the subgroups A and B. If A and B are p-soluble with p-length at most 1 and G' is p-supersoluble, then the p-length of G is at most 1.*

Proof Assume that the result is false and consider a counterexample G with $|G|+|A|+|B|$ minimal. Then, by a routine argument, G has a unique minimal normal subgroup, N say. Applying [1, Theorem 4.1.15], G is p-soluble and so N is an elementary abelian p-group. Moreover $N = Soc(G) = C_G(N) = F(G) = O_p(G)$. If N is cyclic, then $G/N = G/C_G(N)$ is abelian of exponent dividing $p-1$ and G has p-length at most 1, which is a contradiction. Thus we may assume that N has rank greater than 1. Suppose that $G'' = 1$. Then $G' = N$ and G has a normal Sylow p-subgroup. This contradiction yields $G'' \neq 1$. Since G' is p-supersoluble, it follows that G'' is p-nilpotent by [1, Lemma 2.1.6]. Hence $G'' = N$ and N is the Sylow p-subgroup of G'. In particular, G is soluble.

Suppose that $N \leq A$ and $N \cap B = 1$. Then, by [1, Theorem 4.3.3(5)], $N \leq C_G(B)$. Hence $B \leq C_G(N) = N \leq A$, contrary to supposition. Suppose that $N \cap A = N \cap B = 1$. Then N is cyclic by [1, Theorem 4.3.9]. In particular, G has p-length at most 1, contrary to the choice of G. Therefore we may assume that $N \leq A \cap B$ by [1, Lemma 4.3.3]. Since $O_{p'}(A) \leq C_G(N) = N$, and A has p-length at most 1, we have that a Sylow p-subgroup A_p of A is normal in A. Analogously, B has a normal Sylow p-subgroup, B_p say. By [1, Theorem 1.1.19], $G_p = A_pB_p$ is a Sylow p-subgroup of G.

Suppose that $G = A_pB$. Then $A = A_p(A \cap B)$. Let $(A \cap B)_{p'}$ and $B_{p'}$ be Hall p'-subgroups of $A \cap B$ and B respectively such that $(A \cap B)_{p'} \leq B_{p'}$. Then $B_{p'}$ is a Hall p'-subgroup of G, $G = A_p(A \cap B)B_{p'}B_p = A_pB_{p'}B_p$ and so $1 \neq B_{p'} \neq B$. Note that G' is a subgroup of B and so N is also a Sylow p-subgroup of B. Note that AG' is a mutually permutable product of A and G'. If G' is a proper subgroup of B, then the assumption about G implies that the p-length of AG' is at most 1. Since $O_{p'}(AG') = 1$, it follows that A_p is a normal subgroup of AG'. Therefore A_p is a subnormal subgroup of G and then $A_p = N$. We conclude that N is a Sylow p-subgroup of G, which is a contradiction. Consequently $B = G'$ is a p-supersoluble subgroup of G. This means that $A = A_p(A \cap B)$ is a mutually permutable product of a nilpotent group and a p-supersoluble group. Applying [1, Theorem 4.1.35], A is p-supersoluble. By [1, Theorem 4.1.40], G has p-length at most 1. This contradiction implies that A_pB and AB_p are proper subgroups of G. Note that $A_pB = A_p(A \cap B)B$ and $AB_p = A(A \cap B)B_p$ are mutually permutable products by [1, Theorem 4.1.16]. The choice of (G, A, B) implies that the p-length of A_pB and B_pA is at most 1. Since $N \leq A_pB \cap B_pA$, we obtain that $O_{p'}(A_pB) = O_{p'}(B_pA) = 1$. Therefore $G_p = A_pB_p$ is a normal subgroup of G. This is the final contradiction.

Acknowledgements

The first author has been supported by the grant MTM2014-54707-C3-1-P from the Ministerio de Economía y Competitividad, Spain, and FEDER, European Union. He has been also supported by a project from the National Natural Science Foundation of China (NSFC, No. 11271085) and a project of Natural Science Foundation of Guangdong Province (No. 2015A030313791).

References

[1] A. Ballester-Bolinches, R. Esteban-Romero and M. Asaad, *Products of finite groups,* vol. 53 of the Gruyter Expositions in Mathematics. Walter de Gruyter, Berlin 2010.

[2] James C. Beidleman and Hermann Heineken, *Mutually permutable subgroups and group classes,* Arch. Math., 85, 18-30, 2005.

[3] K. Doerk and T. Hawkes, *Finite Soluble Groups,* Walter De Gruyter, Berlin/New York (1992).

[4] D. F. Holt and Robert B. Howlett, *On groups which are the product of two abelian groups,* J. London Math. Soc., (2)29, 453-461, 1984.

[5] Robert B. Howlett, *On the exponent of certain factorizable groups,* J. London Math. Soc., (2)31, 265-271, 1985.

[6] Luise-Charlotte Kappe and Robert Fitzgerald Morse, *On commutators in p-groups,* J. of Group Theory, 8, 415-429, 2005.

[7] Avinoam Mann, *On the exponent of the product of two groups,* Rend. Sem. Mat. Univ. Padova, 115, 205-207, 2006.

A. Ballester-Bolinches
Departament d'Àlgebra, Universitat de València
Dr. Moliner 50, 46100 Burjassot, València (Spain)
e-mail: Adolfo.Ballester@uv.es

John. Cossey
Department of Mathematics, Mathematical Sciences Institute

Australian National University,
ACT 2601, Australia
e-mail: john.cossey@anu.edu.au

M.C. Pedraza-Aguilera
Instituto Universitario de Matemática Pura y Aplicada
Universidad Politécnica de Valencia,
Camino de Vera, 46022, Valencia (Spain)
e-mail: mpedraza@mat.upv.es