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Free extrema of two variables functions

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1. Abstract and objectives

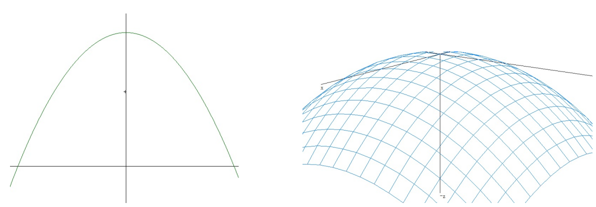
In this paper we work with functions of two variables and introduce the concept of relative (free) extrema for this kind of functions. That is, we calculate the maximum and minumum value of a function of two variables in its domain of definition. This represents a generalization of the same concept as in one variable functions.

Once studied this paper the student will be able to obtain the relative extrema of a function of two variables and to classify them.

2. Introduction

In many situations it is important to know where a function reaches a local maximum or minimum value. For instance, if we consider the temperature function, it can be interesting to know in which points the temperature is higher or lower than at any point near them. This is what we are going to consider in this work.

It is known that a one variable function has a local maximum or minimum when the growing behavior changes from increasing to decreasing (maximum) or from decreasing to increasing (minimum). These situations can be characterized using the first derivative of the function. In the same way we generalize the concept of maximum and minimum value of a function to functions of several variables.



Then, we are going to study what a local (relative) extreme of a two variables function is, how to calculate the relative extrema and how to classify them to know if they are maximum or minimum points of the function.

1. Free extreme points

Firstly, we introduce the concept of relative or free maxima and minima values of a function of two variables pointing out the difference between local and absolute extrema as happens in one variable functions.

Definition 1 *A real function* $f(x,y)$ *has*

- *a* **relative maximum** *at the point* (x_0, y_0) *if there exists* $\delta > 0$ *such that* $\forall (x,y) \in D$ *with* $|(x,y) - (x_0 - y_0)| < \delta$ *it is satisfied that*

$$f(x,y) \leq f(x_0, y_0)$$

- *a* **relative minimum** *at the point* (x_0, y_0) *if there exists* $\delta > 0$ *such that* $\forall (x,y) \in D$ *with* $|(x,y) - (x_0 - y_0)| < \delta$ *it is satisfied that*

$$f(x,y) \geq f(x_0, y_0)$$

- *an* **absolute or global maximum** *at the point* (x_0, y_0) *if* $f(x,y) \leq f(x_0, y_0)$ *for any point* (x,y) *of its domain of definition.*

- *an* **absolute or global minimum** *at the point* (x_0, y_0) *if* $f(x,y) \geq f(x_0, y_0)$ *for any point* (x,y) *of its domain of definition.*

In this work we are interested in the obtaining of the relative maxima and minima of a function of two variables.

Remark 1 *The relative extreme points of a function are the points where the function has a relative maximum or minimum. They are also called free extreme points.*

Remark 2 *Note that the above definitions are also valid for functions of several variables in general.*

As in one variable functions, derivatives play an important role to study the relative extrema of a function $z = f(x,y)$. So, we can define the critical points as follows.

Definition 2 *A point* (x_0, y_0) *is called a* **critical point** *of* $z = f(x,y)$ *if the first order partial derivatives of* $f(x,y)$ *at that point are equal to zero or don't exist:*

- $\dfrac{\partial f}{\partial x}(x_0, y_0)$ *don't exist or equals* 0 *and*

- $\dfrac{\partial f}{\partial y}(x_0, y_0)$ *don't exist or equals* 0.

This concept can also be generalized to functions of more than two variables. In fact, a point $\vec{x_0} \in \mathbb{R}^n$ is a critical point of a function $f(\vec{x})$, with $\vec{x} \in \mathbb{R}^n$, if all the first partial derivatives of the function at that point are null or one of them doesn't exist.

Theorem 1 *Necessary condition*
If the function $f(x, y)$ *has a relative maximum or minimum ar the point* (x_0, y_0) *then* (x_0, y_0) *is a critical point of* $f(x, y)$.

Note that, as in one variable functions, the first order partial derivatives can vanish for points that are no extrema. That is, not all critical points are extrema of $f(x, y)$.

Example 1 *Obtain the critical points of* $f(x, y) = x^3 - 3x^2 + y^2 - 2xy$
Solution*: Calculate the first partial derivatives of* $f(x, y)$ *and equal them to zero.*

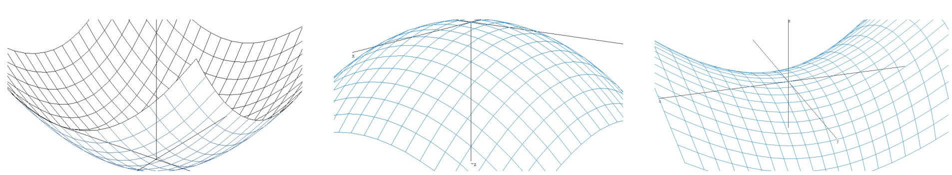
$$\begin{cases} \dfrac{\partial f}{\partial x} = 3x^2 - 6x - 2y = 0 \\ \dfrac{\partial f}{\partial y} = 2y - 2x = 0 \end{cases}$$

From second equation, $x = y$ *and substituting at the firs equation*

$$3x^2 - 6x - 2x = 3x^2 - 8x = x(3x - 8) = 0$$

So, $x = 0$ *or* $x = 8/3$. *Then, the points* $(0, 0)$ *and* $(8/3, 8/3)$ *are critical points of* $f(x, y)$.

In one variable functions, critical points can be a relative maximum, a relative minimum or an inflection point. Something similar occurs for functions of two variables. Look at the following graphs, where $(0, 0)$ represents a relative minimum, a relative maximum and a saddle point, respectively.



In order to classify the critical points of a function of two variables we can use a geometrical method (plot the function) or the second derivative test to determine what kind of extrema is each critical point. The difference with one variable function is that we don't have a second derivative of the function but four second partial derivatives. To consider all of them we construct a square matrix with all of them and calculate its determinant.

Definition 3 *The* **Hessian** *of the function $f(x,y)$ at the point (x_0,y_0) is the determinant*

$$H_f(x_0,y_0) = \begin{vmatrix} \dfrac{\partial^2 f}{\partial x^2}(x_0,y_0) & \dfrac{\partial^2 f}{\partial x\, \partial y}(x_0,y_0) \\ \dfrac{\partial^2 f}{\partial x\, \partial y}(x_0,y_0) & \dfrac{\partial^2 f}{\partial y^2}(x_0,y_0) \end{vmatrix}.$$

Using the second partial derivatives and this determinant we obtain a way of classifying the critical points to determine if they are minima, maxima or saddle points.

Theorem 2 *Sufficient condition*

Let (x_0,y_0) be a critical point of a function $f(x,y)$ whose second partial derivatives are continuous at (x_0,y_0).

- *If $H_f(x_0,y_0) > 0$ and $\dfrac{\partial^2 f}{\partial x^2}(x_0,y_0) > 0$ then $f(x_0,y_0)$ is a relative minimum.*
- *If $H_f(x_0,y_0) > 0$ and $\dfrac{\partial^2 f}{\partial x^2}(x_0,y_0) < 0$ then $f(x_0,y_0)$ is a relative maximum.*
- *If $H(x_0,y_0) < 0$ then $f(x,y)$ has a saddle point at (x_0,y_0).*
- *If $H(x_0,y_0) = 0$ then this test is inconclusive.*

There are other equivalent versions of the previous test. For example, some texts may use $\dfrac{\partial^2 f}{\partial y^2}$ instead of the corresponding derivative of $f(x,y)$ with respect to x twice.

Example 2 *Classify the critical points of* $f(x, y) = x^3 - 3x^2 + y^2 - 2xy$

Solution*: The first partial derivatives of* $f(x, y)$ *are*

$$\frac{\partial f}{\partial x} = 3x^2 - 6x - 2y \quad \text{and} \quad \frac{\partial f}{\partial y} = 2y - 2x$$

Considering their nullity, the points $(0, 0)$ *and* $(8/3, 8/3)$ *are critical points of* $f(x, y)$.

Now, the hessian of the function at any point (x, y) *is*

$$H_f(x, y) = \begin{vmatrix} 6x - 6 & -2 \\ -2 & 2 \end{vmatrix} = 12x - 8.$$

For the first critical point one has $H_f(0, 0) = -8 < 0$, *then the function has a saddle point at* $(0, 0)$.

For the second critical point one has

$$H_f\left(\frac{8}{3}, \frac{8}{3}\right) = 24 > 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2}\left(\frac{8}{3}, \frac{8}{3}\right) = 10 > 0,$$

then, the function has a relative minimum at the point $\left(\frac{8}{3}, \frac{8}{3}\right)$ *and its value is* $f\left(\frac{8}{3}, \frac{8}{3}\right) = -\frac{256}{27}$.

Example 3 *Classify the critical points of* $f(x, y) = x^3 + y^2 - 6xy - 39x + 18y + 20$

Solution*: Calculate the first partial derivatives of* $f(x, y)$,

$$\frac{\partial f}{\partial x} = 3x^2 - 6y - 39 \quad \text{and} \quad \frac{\partial f}{\partial y} = 2y - 6x + 18,$$

and equal them to zero. From the second equation $y = 3x - 9$ *and substituting at the first equation*

$$3x^2 - 6(3x - 9) - 39 = 3x^2 - 18x + 15 = 3(x^2 - 6x + 5) = 0$$

So $x = 5$ *or* $x = 1$. *Then, the points* $(5, 6)$ *and* $(1, -6)$ *are critical points of* $f(x, y)$.

Now, the hessian of the function at any point (x, y) *is*

$$H_f(x,y) = \begin{vmatrix} 6x & -6 \\ -6 & 2 \end{vmatrix} = 12x - 36.$$

For the first critical point one has

$$H_f(5,6) > 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2}(5,6) = 30 > 0,$$

then, the function has a relative minimum at the point $(5, 6)$ *and its value is* $f(5, 6) = -86$.

For the second critical point one has $H_f(1, -6) < 0$ *and , then the function has a saddle point at* $(1, -6)$.

3. Closing

We have studied relative extrema of a two variables function, how to calculate and classify them. Extrema of a function are necessarily critical points, so we can use the first partial derivatives of the function to find them. Critical points are those points satisfying that the first partial derivatives of the function are both null or at least one of them doesn't exist. Once we have the critical points we can use the second derivative test to determine if they are maxima, minima or saddle points of the function. That is, we calculate the hessian of the function at each critical point. If the hessian is negative then the point represents a saddle point, if the hessian is positive then the point is a maximum or a minimum. In some cases (when the hessian equals zero) this test can not conclude what kind of extrema is the critical point. In such cases we would have to use other methods (for example, geometrical methods).

Definitions of relative maxima and minima can be extended to functions with more than two variables. In this cases extrema are also critical points, which can be obtained from the nullity of all the first partial derivatives of the function or if some of them do not exist. Then, the Hessian is a determinant of a square matrix with all the possible second derivatives (that is, of size n, with n equal to the number of variables of the function). However the second derivative test is different and it is not studied in this work.

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4. Bibliography

There are many books studying several variables functions. Some of them are:

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