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Constrained extrema of two variables functions

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1. Abstract and objectives

In this paper we work with functions of two variables and introduce the concept of constrained extrema for this kind of functions. That is, we calculate the maximum and minumum value of a function of two variables under some extra conditions (constraints). These extreme values are not necessarily the free extreme values of the function, they depend on the conditions considered. This fact represents a difference between calculating free or constrained extreme values of a two variables function.

Once studied this paper the student will be able to determine the constrained extrema of a function of two variables under some extra conditions.

2. Introduction

For many problems it is important to know in which points a function reaches the biggest or smallest value. But sometimes we have more conditions to be satisfaid and we can be interested in knowing where the function has the biggest or the smallest value between the points that satisfy some equality. For example, it can be useful to know which point of an sphere is nearest to a given point in the space or which is the distance between a point and a line.

First of all, we recall that for a function the concept of maxima and minima values consists on those points for which the value of the function is bigger or smaller than any other point near them. But may be those points do not satisfy the constraint equations we are considering. Then, the idea of maxima and minima of the function under some conditions is different from the idea of relative (or free) maxima and minima of the function.

When we talk about constrained extrema of a function under one or more conditions, we are talking about those points that satisfy all the conditions and for which the value of the function is bigger or smaller than for any other point satisfying the conditions considered. So, our problem consists on determine the maximum and minimum value of a function between the points that are solution of the equality constraints considered in the domain of definition of the function. That is to obtain the **constrained extrema** of $f(x, y)$. Then, we are going to study how to obtain these extrema.

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3. Constrained extrema of a function of two variables

First of all, we recall that the maxima and minima values of a function are those points for which the value of the function is bigger or smaller than any other point near them. That is,

Definition 1 *A real function $f(x,y)$ has*

- *a* **relative maximum** *at the point (x_0, y_0) if there exists $\delta > 0$ such that $\forall(x,y) \in \mathbb{R}^2$ with $|(x,y)-(x_0,y_0)| < \delta$ it is satisfied that*

$$f(x,y) \leq f(x_0,y_0)$$

- *a* **relative minimum** *at the point (x_0, y_0) if there exists $\delta > 0$ such that $\forall(x,y) \in \mathbb{R}^2$ with $|(x,y)-(x_0,y_0)| < \delta$ it is satisfied that*

$$f(x,y) \geq f(x_0,y_0)$$

These extrema are also called **free** or **local extrema** of the function. These definitions are the same concepts of maximum and minumim value for a one variable function in $\mathbb{R}$.

However, sometimes problems of calculus of extrema of functions are subjected to some equality constraints. That is, the problem consists on determining the maximum and minimum value of a function $f(x,y)$ under some extra equality constraints in the domain of the function. In this case, we are talking about **constrained extrema** of $f(x,y)$. For example, the problem of finding the closest point to the origin on the line $2x+3y=6$ is to minimize the function $f(x,y)=\sqrt{x^2+y^2}$ where (x,y) must satisfy the line equation.

This idea of constrained extrema in one variable functions is reduced to calculate the value of the function at a point (if the constraint is like $x=a$) or to calculate the intersection points between the constraints and the function and to compare which has the biggest or smallest value of the y coordinate (if the constraints are of the form $g(x,y)=0$).

For a function of two variables, the technique of working will be different depending on the condition. We are going to work with constraints given in the form $g(x,y)=0$.

3.1. Lagrange multipliers

As we have said, the problem consists on determining the maximum and minimum value of a function $f(x, y)$ under an extra equality constraint in the domain of definition of the function given in the form $g(x, y) = 0$.

In this case we have two methods of working:

- Reduce the problem to a one variable problem of relative extrema or
- Use Lagrange multipliers

The first method can be applied if the constraint allows to express one of the variables as a function of the other variable as we show in the following example.

Example 1 *Find the closest point to the origin on the line $2x + 3y = 6$.*

Solution: *Since the distance from (x, y) to the origin is given by $\sqrt{x^2 + y^2}$, the problem consists on minimizing this function, subject to $2x + 3y - 6 = 0$. Equivalently, the problem is to minimize the function $f(x, y) = x^2 + y^2$ subject to $2x + 3y - 6 = 0$.*

In this case, from the constraint, $y = \dfrac{6 - 2x}{3}$ and define

$$g(x) = f\left(x, \frac{6 - 2x}{3}\right) = x^2 + \left(\frac{6 - 2x}{3}\right)^2 = \frac{13x^2 - 24x + 36}{9}.$$

The minimum of $g(x)$ will be the minimum searched. The problem has been reduced to a one variable calculus. The nullity of the first derivative of $g(x)$ give the equation

$$g'(x) = \frac{26x - 24}{9} = 0 \quad \Rightarrow \quad x = \frac{12}{13} \qquad \left(y = \frac{18}{13}\right).$$

And, from the second derivative criterium, $g''(x) = \dfrac{26}{9} > 0$, this point is a relative minimum.

Then, the point $\left(\dfrac{12}{13}, \dfrac{18}{13}\right)$ is the point of the line closest to the origin.

But, sometimes it is not possible to reduce the problem to calculation of maxima and minima in one variable functions. In those cases we need to use the second method: Lagrange multipliers method. This method is also valid even when the reduction is possible.

In general, one has to maximize or minimize the function

$$z = f(x, y),$$

under the constraint $g(x, y) = 0$.

The **Lagrange multipliers method** consists on:

Step 1 Construct the Lagrangian function:

$$L(x, y, \lambda) = f(x, y) + \lambda g(x, y)$$

where λ is called **Lagrange multiplier**.

Step 2 Obtain the critical points of the Lagrangian function. The constrained maxima and minima of $f(x, y)$ are some of these critical points.

Step 3 To determine which of the critical points obtained in Step 2 is the constrained maximum and minimum of $f(x, y)$ under the condition $g(x, y) = 0$. To do this we can use geometrical methods (that is, plotting the function and the constraint) or one of the following methods:

- **Second differential method**: Given the critical point (x_0, y_0, λ_0), define the function $L_0(x, y) = f(x, y) + \lambda_0 g(x, y)$ and calculate

  $$d^2L_0(x_0, y_0) = \frac{\partial^2 L_0}{\partial x^2}(x_0, y_0)dx^2 + 2\frac{\partial^2 L_0}{\partial x \partial y}(x_0, y_0)dx\, dy + \frac{\partial^2 L_0}{\partial y^2}(x_0, y_0)dy^2$$

  under the consdition $\dfrac{\partial g}{\partial x}dx + \dfrac{\partial g}{\partial y}dy = 0$.

  If $d^2L_0(x_0, y_0) > 0$ the point $(x_0, y_0, f(x_0, y_0))$ is a constraint minimum and if $d^2L_0(x_0, y_0) < 0$ the point $(x_0, y_0, f(x_0, y_0))$ is a constraint maximum. In other cases we have to use another method.

- **Hessian method**: Given the critical point (x_0, y_0, λ_0), define the function $L_0(x, y) = f(x, y) + \lambda_0 g(x, y)$ and calculate the Hessian

$$H_{L_0}(x_0, y_0) = \begin{vmatrix} \dfrac{\partial^2 L_0}{\partial x^2}(x_0, y_0) & \dfrac{\partial^2 L_0}{\partial x \partial y}(x_0, y_0) \\ \dfrac{\partial^2 L_0}{\partial y \partial x}(x_0, y_0) & \dfrac{\partial^2 L_0}{\partial y^2}(x_0, y_0) \end{vmatrix}$$

  If $H_{L_0}(x_0, y_0) > 0$ and $\dfrac{\partial^2 L_0}{\partial x^2}(x_0, y_0) > 0$, we have that $(x_0, y_0, f(x_0, y_0))$ is a constrained minimum of $f(x, y)$.
  If $H_{L_0}(x_0, y_0) > 0$ and $\dfrac{\partial^2 L_0}{\partial x^2}(x_0, y_0) < 0$, we have that $(x_0, y_0, f(x_0, y_0))$ is a constrained maximum of $f(x, y)$.
  In othe cases we will have to use other methods.

In the following examples we illustrate the method.

Example 2 *Find the closest point to the origin on the line $2x + 3y = 6$.*

Solution: *Since the distance from (x, y) to the origin is given by $\sqrt{x^2 + y^2}$, the problem consists on minimizing this function, or equivalently the function $f(x, y) = x^2 + y^2$, subject to $g(x, y) = 2x + 3y - 6 = 0$.*
Using the Lagrange multipliers method, one has:

Step 1 *The Lagrangian function is*

$$L(x, y, \lambda) = f(x, y) + \lambda g(x, y) = x^2 + y^2 + \lambda(2x + 3y - 6).$$

Step 2 *Obtain the critical points of $L(x, y, \lambda)$. Solve the system:*

$$\begin{cases} \dfrac{\partial L}{\partial x} = 2x + 2\lambda = 0 \\ \dfrac{\partial L}{\partial y} = 2y + 3\lambda = 0 \\ \dfrac{\partial L}{\partial \lambda} = 2x + 3y - 6 = 0 \end{cases}$$

The unique solution of this system is $(x_0, y_0, \lambda_0) = \left(\dfrac{12}{13}, \dfrac{18}{13}, \dfrac{-6}{\sqrt{468}}\right)$.

Step 3 *The constrained minima of $f(x, y)$ is exactly the point found at Step 2. Then, the point $\left(\frac{12}{13}, \frac{18}{13}\right)$ is the closest point of the line to the origin.*

We can check that it is a minimum using, for example, the Hessian method. The function

$$L_0\left(x, y, \frac{-6}{\sqrt{468}}\right) = x^2 + y^2 - \frac{6}{\sqrt{468}}(2x + 3y - 6)$$

has a Hessian of the form

$$H_{L_0}(x, y) = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0.$$

Moreover $\frac{\partial^2 L_0}{\partial x^2} = 2 > 0$. *Then, the point is a minimum.*

Example 3 *Find the maximum and minimum value of $f(x, y) = 2x + 5y$ on the elipse $9x^2 + 16y^2 = 144$.*

Solution: *Using the Lagrange multipliers method, one has:*

Step 1 *The Lagrangian function is*

$$L(x, y, \lambda) = 2x + 5y + \lambda(9x^2 + 16y^2 - 144).$$

Step 2 *Obtain the critical points of $L(x, y, \lambda)$. Solve the system:*

$$\begin{cases} \dfrac{\partial L}{\partial x} = 2 + 18\lambda x = 0 \\ \dfrac{\partial L}{\partial y} = 5 + 32\lambda y = 0 \\ \dfrac{\partial L}{\partial \lambda} = 9x^2 + 16y^2 - 144 = 0 \end{cases}$$

The solution of this system are the points $(x_0, y_0, \lambda_0) = \left(\frac{32}{17}, \frac{45}{17}, -\frac{17}{288}\right)$ and $(x_1, y_1, \lambda_1) = \left(-\frac{32}{17}, -\frac{45}{17}, \frac{17}{288}\right)$.

Step 3 *The constrained maximum and minimum of $f(x,y)$ will be some of the points found at Step 2. Using the second differential method, we define*

$$L_0(x,y) = 2x+5y+\lambda_0(9x^2+16y^2-144) = 2x+5y-\frac{17}{288}(9x^2+16y^2-144)$$

and calculate

$$d^2L_0(x_0,y_0) = 18\lambda_0 dx^2 + 32\lambda_0 dy^2 = 2\lambda_0(9dx^2+16dy^2).$$

Clearly $d^2L_0(x_0,y_0) < 0$ and then (x_0,y_0) represents a constrained maximum.

Analogously, for (x_1,y_1,λ_1) we have $L_1(x,y) = 2x+5y+\lambda_1(9x^2+16y^2-144) = 2x+5y+\frac{17}{288}(9x^2+16y^2-144)$ and

$$d^2L_1(x_1,y_1) = 18\lambda_1 dx^2 + 32\lambda_1 dy^2 = 2\lambda_1(9dx^2+16dy^2) > 0.$$

Then (x_1,y_1) represents a constrained minimum.

Therefore, calculating

$$f\left(\frac{32}{17},\frac{45}{17}\right) = 2\ \frac{32}{17} + 5\ \frac{45}{17} = 17$$

and

$$f\left(-\frac{32}{17},-\frac{45}{17}\right) = -2\ \frac{32}{17} - 5\ \frac{45}{17} = -17,$$

we have that the maximum value of $f(x,y)$ on the elipse is 17 and it occurs at the point $\left(\frac{32}{17},\frac{45}{17}\right)$; and the minimum value of $f(x,y)$ on the elipse is -17 and it occurs at the point $\left(-\frac{32}{17},-\frac{45}{17}\right)$.

Remark: Note that, in the case we have more than one critical point for the Lagrangian function and the constraint represents a bounded curve (like an elipse, circumference, ...), then we can assure that the continuous function $f(x,y)$ reaches a maximum and a minimum value on the curve. Therefore, Step 3 can be reduced to compare the value of the function $f(x, y)$ for the critical points obtained in Step 2 in order to obtain the maximum and the minimum points. In the previous example, clearly the point $\left(\frac{32}{17}, \frac{45}{17}\right)$ was a maximum and the point $\left(-\frac{32}{17}, -\frac{45}{17}\right)$ was a minimum because $f\left(\frac{32}{17}, \frac{45}{17}\right) = 17$ is bigger than $f\left(-\frac{32}{17}, -\frac{45}{17}\right) = -17$.

However, if the constraint is not a bounded curve (a straight line, parabole, ...) then the extreme values of the function $f(x, y)$ may not exist (no critical points for the Lagrangian function) or in case of existing they will represent local maxima or minima. In this case we cannot reduce the work to be done in Step 3 and we have to use some criterium to determine the character of the point.

3.2. Several constraints and functions of three variables

Lagrange multipliers method can be generalized for functions with three or more variables and for two or more constraints taking into account that the more conditions we have, the more Lagrange multipliers we have to add.

Example 4 *Find the maximum and minimum value of $f(x, y, z) = x$ under the constraints $x + 2z = 0$ and $x^2 + y^2 + z^2 = 1$.*

Solution: *Using the Lagrange multipliers method, ones has:*

Step 1 *The Lagrangian function is*

$$L(x, y, \lambda, \mu) = x + \lambda(x + 2z) + \mu(x^2 + y^2 + z^2 - 1).$$

Step 2 *Obtain the critical points of $L(x,y,z,\lambda,\mu)$. Solve the system:*

$$\begin{cases} \dfrac{\partial L}{\partial x} = 1+\lambda+2x\mu = 0 \\ \dfrac{\partial L}{\partial y} = 2y\mu = 0 \\ \dfrac{\partial L}{\partial z} = 2\lambda+2z\mu = 0 \\ \dfrac{\partial L}{\partial \lambda} = x+2z = 0 \\ \dfrac{\partial L}{\partial \mu} = x^2+y^2+z^2-1 = 0 \end{cases}$$

From the second equation $\mu = 0$ or $y = 0$. If $\mu = 0$ then, the first equation tells that $\lambda = -1$ and the third equation leads to $\lambda = 0$, which is a contradiction. So, the case $\mu = 0$ is not possible and necessarily $y = 0$. For $y = 0$ one has

$$\begin{cases} \lambda = -1-2x\mu \\ \lambda = -z\mu \\ x = -2z \\ x^2+z^2 = 1 \end{cases}$$

Then, $(-2z)^2+z^2 = 1$, which implies $z = \pm\dfrac{1}{\sqrt{5}}$. So, the critical points are:

$$\left(\frac{-2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, \frac{\sqrt{5}}{5}\right) \quad \text{and} \quad \left(\frac{2}{\sqrt{5}}, 0, \frac{-1}{\sqrt{5}}, -\frac{1}{3}, -\frac{\sqrt{5}}{3}\right).$$

Step 3 *The constrained maximum and minimum of $f(x,y,z)$ are some of the points found at Step 2. Since the constraints represent the intersection of an sphere an a plane, that is, a circumference, the curve is bounded and we can reduce this step to evaluate $f(x,y,z)$ for the critical points and compare the results. Then, from*

$$f\left(\frac{-2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right) = \frac{-2}{\sqrt{5}} \quad \text{and} \quad f\left(\frac{2}{\sqrt{5}}, 0, \frac{-1}{\sqrt{5}}\right) = \frac{2}{\sqrt{5}},$$

one has that the maximum value of $f(x,y,z)$ *under both constraints is* $\dfrac{2}{\sqrt{5}}$ *and it occurs at the point* $\left(\dfrac{2}{\sqrt{5}}, 0, \dfrac{-1}{\sqrt{5}}\right)$*; and the minimum value of* $f(x,y,z)$ *under both constraints is* $\dfrac{-2}{\sqrt{5}}$ *and it occurs at the point* $\left(\dfrac{-2}{\sqrt{5}}, 0, \dfrac{1}{\sqrt{5}}\right)$.

4. Closing

We have studied how to calculate the maximum and minimum values of a function $f(x,y)$ under a constraint of the form $g(x,y) = 0$. We can assure that these extreme values exist for a continuous function if the constraint is a bounded curve. In other cases, the extreme values will be local extrema or even they may not exist.

The constrained extrema can be obtained using the Lagrange multipliers method and they are the critical points of the Lagrangian function, which is defined using the function and the constraints. In the case of a bounded constraint the comparison between the calue of the function on the critical points is enough to choose the maximum and the minimum values. When the constraint is not a bounded curve we have to use other criterion (second differential method or Hessian method) to choose the relative constrained maximum and minimum of the function. We have presented some examples to clarify the method.

The Lagrange multipliers method can also be generalized to functions with three or more variables and for two or more constraints. An example illustrates the method in this situation.

5. Bibliography

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