Applied General Topology
© Universidad Politécnica de Valencia
Volume 5, No. 1, 2004
pp. 103- 113

Continuous functions with compact support

S. K. Acharyya, K. C. Chattopadhyaya and Partha Pratim Ghosh

Abstract. The main aim of this paper is to investigate a subring of the ring of continuous functions on a topological space X with values in a linearly ordered field F equipped with its order topology, namely the ring of continuous functions with compact support. Unless X is compact, these rings are commutative rings without unity. However, unlike many other commutative rings without unity, these rings turn out to have some nice properties, essentially in determining the property of X being locally compact non-compact or the property of X being nowhere locally compact. Also, one can associate with these rings a topological space resembling the structure space of a commutative ring with unity, such that the classical Banach Stone Theorem can be generalized to the case when the range field is that of the reals.

2000 AMS Classification: Primary 54D80, 54D35, 54G99; Secondary 12J15, 13J25, 06F25

Keywords: Ordered Fields, Zero Dimensional Spaces, Strongly Zero Dimensional Spaces, Compactifications

1. Introduction

The rings of continuous functions on a topological space X with values in a linearly ordered field, equipped with its order topology, was initiated in the paper [4]. The purpose of this paper is to investigate a lattice ordered commutative subring of $\mathfrak{C}^*(X, F)$, see Definition 2.1, which consists of precisely those continuous functions defined on a topological space X and taking values in a linearly ordered field equipped with its order topology that have a compact support. Similar study for the case of real valued continuous functions have been done elsewhere as well as in [5], many of whose results are generalised in the present paper.

It is seen that unless X is compact, these rings are commutative rings without the identity. Commutative rings without identity fail to have many nice properties which hold in commutative rings with identity. For instance in a commutative ring K without identity it need not be true that all maximal ideals are prime, a fact that is crucially required in proving that the sets $K_a = \{M : M \text{ is a maximal ideal and } a \in M\}$, as one varies $a \in M$ make a base for the closed sets of a topology on the set of all maximal ideals of K, see the last part of §2. However, it turns out that these rings do provide much information about the space X in the following sense :

(1) we can provide characterizations for completely F regular locally compact non-compact topological spaces (see Theorem 3.4) as well as completely F regular nowhere locally compact topological spaces (see Theorem 3.5),
(2) develop an analogue of the idea of structure spaces so as to determine the class of locally compact non-compact Tychonoff topological spaces in terms of these function rings (see Theorem 4.9), thereby generalizing the Banach Stone Theorem, see [3, Theorem 4.9, page 57].

The paper is organized as follows : §2 introduces the basic notions and results from [4] that are required to make the paper self-contained. §3 develops the basic properties of the function ring $\mathfrak{C}_{\mathfrak{k}}(X, F)$ culminating in the characterizations stated above. §4 develops the analogue of a structure space for these rings and culminates in generalizing Banach Stone's theorem as noted above.

We fix some conventions for this paper. The term "order" refers to a "linear order" and all the topological spaces considered are at least Hausdorff topological spaces. Any ordered field F is always equipped with its order topology. The symbol $f : A \xrightarrow{\simeq} B$ refers to an isomorphism in the category of which f is a map. For instance the statement "$f : A \xrightarrow{\simeq} B$ as topological spaces" will mean that f is a homeomorphism between the topological spaces A and B. The symbol $A \xrightarrow{\simeq} B$ will mean that in the concerned category the objects A and B are isomorphic.

2. Preliminaries

Definition 2.1. *Let X be a topological space and F be an ordered field.*

(1) $\mathfrak{C}(X, F) = \{f \in F^X : f \text{ is continuous on } X\}$.
(2) $\mathfrak{B}(X, F) = \{f \in \mathfrak{C}(X, F) : (\exists t \in F)(\forall x \in X)(-t \leq f(x) \leq t)\}$.
(3) $\mathfrak{C}^*(X, F) = \{f \in \mathfrak{C}(X, F) : \mathrm{cl}_F(f(X)) \text{ is compact}\}$.
(4) *The set* $\mathrm{cl}_X(\mathrm{coz}(f))$ *is said to be the* support *of the function* $f \in \mathfrak{C}(X, F)$ *and will henceforth be denoted by the symbol* $\mathrm{supp}_X(f)$.
(5) $\mathfrak{C}_{\mathfrak{k}}(X, F) = \{f \in \mathfrak{C}(X, F) : f \text{ has compact support}\}$.
(6) $\mathcal{Z}_{X,F}(f) = \{x \in X : f(x) = 0\}$, *for* $f \in \mathfrak{C}(X, F)$.
(7) $\mathfrak{Z}(X, F) = \{\mathcal{Z}_{X,F}(f) : f \in \mathfrak{C}(X, F)\}$.

A subset $A \subseteq X$ is said to be a zero set in X with respect to F, if and only if, there exists an $f \in \mathfrak{C}(X, F)$ such that $A = \mathcal{Z}_{X,F}(f)$; the complement of a zero set in X with respect to F is called cozero set in X with respect to F. The complement of the set $\mathcal{Z}_{X,F}(f)$ shall be denoted by $\mathrm{coz}(f)$, i.e., $\mathrm{coz}(f) = X \setminus \mathcal{Z}_{X,F}(f)$.

If one identifies the element $t \in F$ with the constant function $X \xrightarrow{\mathbf{t}} F$, defined by $\mathbf{t} : x \mapsto t$, then one easily gets $F \subseteq \mathfrak{C}^*(X,F) \subseteq \mathfrak{B}(X,F) \subseteq \mathfrak{C}(X,F)$. Also, if one assigns the operations of addition, multiplication, maximum and minimum pointwise on the functions, then one easily gets $\mathfrak{C}(X,F)$ to be a lattice ordered commutative ring with unity, and $\mathfrak{B}(X,F)$, $\mathfrak{C}^*(X,F)$ and $\mathfrak{C}_{\mathfrak{k}}(X,F)$ as subsystems of it.

An element of a commutative ring with identity is said to be a *unit*, if and only if, there exists an inverse of it. It is easily seen that the units of $\mathfrak{C}(X,F)$ are precisely those continuous functions $X \xrightarrow{f} F$ that do not vanish anywhere, i.e., with $\mathcal{Z}_{X,F}(f) = \varnothing$. However, this condition is not enough to ensure that the inverse will be bounded. It follows from this observation that $f \in \mathfrak{B}(X,F)$ is a unit of the ring $\mathfrak{B}(X,F)$, if and only if, it does not vanish anywhere and that it is *bounded away from zero*, in the sense that there exists an $\epsilon \in F_{>0} = \{x \in F : x > 0\}$ such that for all $x \in X$, $|f(x)| > \epsilon$. Since $f \in \mathfrak{C}^*(X,F)$, if and only if, f has a pre-compact co-domain, and $\cdot^{-1} : F_{\neq 0} \xrightarrow{\simeq} F_{\neq 0}$ as topological spaces, where $F_{\neq 0} = \{x \in F : x \neq 0\}$, it follows that $f \in \mathfrak{C}^*(X,F)$ is a unit of $\mathfrak{C}^*(X,F)$, if and only if, f does not vanish anywhere on X and is bounded away from 0.

Remark 2.2. Note that a subset U of a topological space is said to be *pre-compact*, if and only if, $\mathrm{cl}_X(U)$ is a compact subset of X.

The following result will be required in the sequel :

Theorem 2.3. *If $f, g \in \mathfrak{C}(X,F)$ such that $\mathcal{Z}_{X,F}(f) \subseteq \mathrm{int}_X(\mathcal{Z}_{X,F}(g)) \subseteq \mathcal{Z}_{X,F}(g)$ then f divides g.*

Proof. Consider the function $h(x) = \begin{cases} \dfrac{g(x)}{f(x)}, & \text{if } x \notin \mathrm{int}_X(\mathcal{Z}_{X,F}(g)) \\ 0, & \text{otherwise} \end{cases}$. Clearly $h \in \mathfrak{C}(X,F)$ and consequently, $g = hf$, proving the theorem. □

We shall require some regularity properties for the topological spaces under consideration, see [4] for details.

Definition 2.4.

(1) *$A, B \subseteq X$ are said to be completely F separated, if and only if, there exists some $f \in \mathfrak{C}(X,F)$ such that $f(x) = 0$ on A and $f(x) = 1$ on B.*
(2) *X is said to be completely F regular, if and only if, for every closed subset A of X and every $x \in X \setminus A$, the sets $\{x\}$ and A are completely F separated.*

The property of being completely F regular is clearly the analogue of the Tychonoff property when $F = \mathbb{R}$, and the following statements strengthen the similarity.

Theorem 2.5. *For any topological space X the following are equivalent :*

(1) *X is completely F regular;*

(2) *$\mathfrak{Z}(X,F)$ is a base for the closed subsets of X;*
(3) *X has the weak topology induced by $\mathfrak{C}(X,F)$;*
(4) *X has the weak topology induced by a subset of $\mathfrak{C}(X,F)$;*
(5) *$\mathfrak{B}(X,F)$ separates points and closed subsets of X;*
(6) *$\mathfrak{C}^*(X,F)$ separates points and closed subsets of X.*

The following provide examples of completely F regular topological spaces :

Theorem 2.6. (1) *F is completely F regular.*
(2) *The property of complete F regularity is productive and hereditary.*
(3) *For $F \neq \mathbb{R}$, a topological space X is completely F regular, if and only if, it is zero dimensional.*
(4) *A topological space X is completely F regular, if and only if, it is homeomorphic to a subspace of a product of F.*

Theorem 2.6(4) equates the class of completely F regular topological spaces to the F-completely regular spaces of Mrowka, see [2]. Theorem 2.6(3) is an immediate consequence of the equivalent formulations in Theorem 2.5 and the fact that an ordered field F is either connected, in which case it is just isomorphic to $\mathbb{R}$, or else is zero dimensional, see [1].

The next theorem settles our choice of spaces, and is the analogue of the celebrated theorem of Stone :

Theorem 2.7. *For any topological space X there exists a completely F regular topological space Y and an isomorphism $\sigma : \mathfrak{C}(X,F) \xrightarrow{\simeq} \mathfrak{C}(Y,F)$ of lattice ordered commutative rings with unity, such that σ restricted to $\mathfrak{B}(X,F)$ and $\mathfrak{C}^*(X,F)$ also produce isomorphisms.*

In other words, $\sigma \upharpoonright_{\mathfrak{B}(X,F)} : \mathfrak{B}(X,F) \xrightarrow{\simeq} \mathfrak{B}(Y,F)$ as well as the restriction to $\mathfrak{C}^*(X,F)$, $\sigma \upharpoonright_{\mathfrak{C}^*(X,F)} : \mathfrak{C}^*(X,F) \xrightarrow{\simeq} \mathfrak{C}^*(Y,F)$.

We will often deal with ideals of the rings. An ideal I of $\mathfrak{C}(X,F)$ is called a *fixed ideal*, if and only if, the intersection of the zero sets in X with respect to F of the members of I be non-empty, i.e., the formula $(\exists x \in X)(\forall f \in I)(f(x) = 0)$ is true. An ideal is said to be *free*, if and only if, it is not a fixed ideal.

In §4 we shall develop an entity like the structure space of a commutative ring with unity, and for ease of reference when we compare it with our construction we include the required definitions and results.

Given a commutative ring K with unity, let the set of all its maximal ideals be denoted by $\mathfrak{M}$, and let for any $a \in K$, $\mathfrak{M}_a = \{M \in \mathfrak{M} : a \in M\}$. It is easy to see that $\{\mathfrak{M}_a : a \in K\}$ makes a base for the closed subsets of some unique topology on $\mathfrak{M}$, often called the *structure space* of the commutative ring K, and this topology is sometimes referred to as the *Stone topology* or as the *hull kernel topology* on $\mathfrak{M}$. It is easy to see that :

(1) For any $\mathfrak{x} \subseteq \mathfrak{M}$, $\mathrm{cl}_{\mathfrak{M}}(\mathfrak{x}) = \{M \in \mathfrak{M} : M \supseteq \bigcap \mathfrak{x}\}$. Indeed, it is this fact from which the name *hull kernel topology* is derived.
(2) For any $\mathfrak{x} \subseteq \mathfrak{M}$, $\mathfrak{x}$ is dense in $\mathfrak{M}$, if and only if, $\bigcap \mathfrak{x} = \bigcap \mathfrak{M}$.
(3) $\mathfrak{M}$ is a compact T_1 space.

(4) $\mathfrak{M}$ is Hausdorff, if and only if, for every pair of distinct maximal ideals M and N of K there exist points $a, b \in K$ such that $a \notin M$, $b \notin N$ and $ab \in \bigcap \mathfrak{M}$. Thus, the structure space of $\mathbb{Z}$ is not Hausdorff, while if X is completely F regular then the structure space of $\mathfrak{C}(X, F)$ is Hausdorff.

For a much more detailed account of structure spaces see [3, Ex. 7A (page 108), Ex. M and Ex. N (page 111)].

We will fix some notations for the rest of the paper. Henceforth, in this paper F shall always refer to a fixed ordered field equipped with its order topology; topological spaces shall always be completely F regular, unless mentioned to the contrary, and for any topological space X and $x \in X$, $\mathcal{N}_x^X$ will refer to the neighborhood filter at the point x.

3. Functions with Compact Support

We first clarify the position of $\mathfrak{C}_\mathfrak{k}(X, F)$ in the hierarchy of subsystems of $\mathfrak{C}(X, F)$.

Theorem 3.1. *For any topological space X, $\mathfrak{C}_\mathfrak{k}(X, F) \subseteq \mathfrak{C}^*(X, F)$, and if X is non-compact then $\mathfrak{C}_\mathfrak{k}(X, F)$ is a proper ideal of both $\mathfrak{C}^*(X, F)$ and $\mathfrak{C}(X, F)$.*

Proof. If X is a compact topological space, then clearly $\mathfrak{C}_\mathfrak{k}(X, F) = \mathfrak{C}(X, F)$.

Let X be not compact. Then $\mathfrak{C}(X, F) \supset \mathfrak{C}_\mathfrak{k}(X, F)$ and $\mathbf{1} \notin \mathfrak{C}_\mathfrak{k}(X, F)$; indeed no unit of any of the function rings $\mathfrak{C}(X, F)$, $\mathfrak{B}(X, F)$ or $\mathfrak{C}^*(X, F)$ then belongs to $\mathfrak{C}_\mathfrak{k}(X, F)$. Finally, for any $f \in \mathfrak{C}_\mathfrak{k}(X, F)$, as $\mathrm{supp}_X(f)$ is compact and $X = \mathcal{Z}_{X,F}(f) \cup \mathrm{supp}_X(f) \Rightarrow f(X) = \{0\} \cup f(\mathrm{supp}_X(f))$, so that $f(X)$ is a compact subset of F, implying thereby that $f \in \mathfrak{C}^*(X, F)$. Thus we have $\mathfrak{C}_\mathfrak{k}(X, F) \subseteq \mathfrak{C}^*(X, F)$.

We shall show that $\mathfrak{C}_\mathfrak{k}(X, F)$ is a proper ideal of $\mathfrak{C}^*(X, F)$. Obvious modifications of this argument will show that $\mathfrak{C}_\mathfrak{k}(X, F)$ is a proper ideal of $\mathfrak{C}(X, F)$, too.

For $f, g \in \mathfrak{C}_\mathfrak{k}(X, F)$, since both $\mathrm{supp}_X(f)$ and $\mathrm{supp}_X(g)$ are compact, it follows from $\mathrm{supp}_X(f + g) \subseteq \mathrm{supp}_X(f) \cup \mathrm{supp}_X(g)$, that $f + g \in \mathfrak{C}_\mathfrak{k}(X, F)$. Similarly, if $h \in \mathfrak{C}^*(X, F)$, then since $\mathrm{supp}_X(fh) \subseteq \mathrm{supp}_X(f) \cap \mathrm{supp}_X(h)$, it follows that $fh \in \mathfrak{C}_\mathfrak{k}(X, F)$, too. Consequently $\mathfrak{C}_\mathfrak{k}(X, F)$ is a proper ideal of $\mathfrak{C}^*(X, F)$. □

Indeed, not only $\mathfrak{C}_\mathfrak{k}(X, F)$ is a proper ideal of $\mathfrak{C}^*(X, F)$ or $\mathfrak{C}(X, F)$, but more strongly we have :

Theorem 3.2. *For any completely F regular topological space X, the subring $\mathfrak{C}_\mathfrak{k}(X, F)$ is contained in every free ideal of $\mathfrak{C}(X, F)$ or $\mathfrak{C}^*(X, F)$.*

The proof depends on the lemma :

Lemma 3.3. *An ideal I in $\mathfrak{C}(X, F)$ or $\mathfrak{C}^*(X, F)$ is free, if and only if, for any compact subset $A \subseteq X$ there is an $f \in I$ such that $\mathcal{Z}_{X,F}(f) \cap A = \varnothing$.*

Proof.

Proof of the Sufficiency part: Suppose that I is a free ideal of $\mathfrak{A}$, where $\mathfrak{A}$ is any of the rings $\mathfrak{C}(X,F)$ or $\mathfrak{C}^*(X,F)$. Let $A \subseteq X$ be compact.

Since I is free, for every $x \in A$, there exists an $f_x \in I$ with $f_x(x) \neq 0$. Then the set $\mathfrak{H} = \{\mathrm{coz}(f_x) : x \in A\}$ is an open cover of A and therefore from compactness of A, there exists a finite sub-cover $\mathfrak{H}_1 = \{f_{x_1}, f_{x_2}, \ldots, f_{x_n}\}$ of A. Let $f = \sum_{i=1}^{n} f_{x_i}^2$. Then $f \in I$ and $\mathcal{Z}_{X,F}(f) \cap A = \varnothing$.

Proof of the Necessity part: Trivial.

□

We now prove Theorem 3.2 :

Proof. If I is a free ideal of $\mathfrak{A}$, where $\mathfrak{A}$ is any one of $\mathfrak{C}(X,F)$ or $\mathfrak{C}^*(X,F)$, and $f \in \mathfrak{C}_{\mathfrak{k}}(X,F)$, then from Lemma 3.3 it follows that there exists a $g \in I$ such that $\mathcal{Z}_{X,F}(g) \cap \mathrm{supp}_X(f) = \varnothing$. Consequently, $\mathcal{Z}_{X,F}(g) \subseteq X \setminus \mathrm{supp}_X(f) \subseteq X \setminus \mathrm{coz}(f) = \mathcal{Z}_{X,F}(f)$, so that $\mathcal{Z}_{X,F}(f)$ is a zero set neighbourhood of $\mathcal{Z}_{X,F}(g)$, and therefore by Theorem 2.3 it follows that g divides f. Consequently, $f \in I$ as $g \in I$. This proves that $\mathfrak{C}_{\mathfrak{k}}(X,F) \subseteq I$. □

The main objective in this section is to show that in the class of completely F regular topological spaces both locally compact spaces and nowhere locally compact spaces can be characterised in terms of the ring $\mathfrak{C}_{\mathfrak{k}}(X,F)$.

Theorem 3.4. *A non-compact completely F regular topological space X is locally compact, if and only if, $\mathfrak{C}_{\mathfrak{k}}(X,F)$ is a free ideal in both the rings $\mathfrak{C}(X,F)$ and $\mathfrak{C}^*(X,F)$.*

Proof. In view of Theorem 3.1 it is enough to consider any of the rings $\mathfrak{C}(X,F)$ or $\mathfrak{C}^*(X,F)$. The non-compactness of the space X is necessary and sufficient to ensure the inequality $\mathfrak{C}_{\mathfrak{k}}(X,F) \subset \mathfrak{C}^*(X,F)$.

Proof of the Sufficiency part: Let $x \in X$. Since X is locally compact it follows that there exists an open neighbourhood $V \in \mathfrak{N}_x^X$ so that $\mathrm{cl}_X(V)$ is compact. Since X is completely F regular, $\mathfrak{C}^*(X,F)$ separates points and closed subsets of X, so that there exists some $f \in \mathfrak{C}^*(X,F)$ such that $f(x) = 1$ and $X \setminus V \subseteq \mathcal{Z}_{X,F}(f)$. Thus $\mathrm{coz}(f) \subseteq V \Rightarrow \mathrm{supp}_X(f) \subseteq \mathrm{cl}_X(V)$, so that $\mathrm{supp}_X(f)$ is a compact set, entailing thereby that $f \in \mathfrak{C}_{\mathfrak{k}}(X,F)$. Thus there exists some $f \in \mathfrak{C}_{\mathfrak{k}}(X,F)$ so that $f(x) \neq 0$; and as this holds for any $x \in X$, it follows that $\mathfrak{C}_{\mathfrak{k}}(X,F)$ is a free ideal of any of the function rings $\mathfrak{C}(X,F)$ or $\mathfrak{C}^*(X,F)$.

Proof of the Necessity part: If $\mathfrak{C}_{\mathfrak{k}}(X,F)$ is a free ideal of $\mathfrak{C}^*(X,F)$ then for any $x \in X$ there exists a $g \in \mathfrak{C}_{\mathfrak{k}}(X,F)$ so that $g(x) \neq 0$. Since $\mathrm{supp}_X(g)$ is a compact neighbourhood of x, it follows that every point of X has a pre-compact neighbourhood. Hence X is locally compact.

□

Theorem 3.5. *A completely F regular topological space X is nowhere locally compact, if and only if,* $\mathfrak{C}_{\mathfrak{k}}(X,F)=\{\mathbf{0}\}$.

Proof.

Proof of the Sufficiency part: If X is nowhere locally compact then for any non-zero $f\in\mathfrak{C}(X,F)$, $\mathrm{coz}(f)$ is a non-empty open set in X with $\mathrm{supp}_X(f)$ non-compact — for otherwise $\mathrm{supp}_X(f)$ will be a compact neighbourhood for every point of $\mathrm{coz}(f)$, contradicting the fact that X is nowhere locally compact. Consequently, $f\notin\mathfrak{C}_{\mathfrak{k}}(X,F)$. This implies that $\mathfrak{C}_{\mathfrak{k}}(X,F)=\{0\}$.

Proof of the Necessity part: It is enough to show that for any $x\in X$ there cannot exist any open $U\in\mathfrak{N}_x^X$ with $\mathrm{cl}_X(U)$ compact.

Choose and fix any $x\in X$ and any open $U\in\mathcal{N}_x^X$. Then since X is completely F regular there exists an $f\in\mathfrak{C}(X,F)$ such that $f(x)=1$ and $X\setminus U\subseteq\mathcal{Z}_{X,F}(f)$. Consequently, $\mathrm{coz}(f)\subseteq U$, and thus if $\mathrm{cl}_X(U)$ is compact then $\mathrm{supp}_X(f)$ will be compact implying thereby that $\mathfrak{C}_{\mathfrak{k}}(X,F)$ contains a non-zero member, namely f, contradicting the hypothesis $\mathfrak{C}_{\mathfrak{k}}(X,F)=\{\mathbf{0}\}$. Hence $\mathrm{cl}_X(U)$ is non-compact.

□

4. Structure Space of $\mathfrak{C}_{\mathfrak{k}}(X,F)$

With any commutative ring with unity one can associate a topological space, called its structure space, [3, Ex. 7A (page 108), Ex. M and Ex. N (page 111)]. The result that in a commutative ring with unity every maximal ideal is prime is a crucial tool in showing the set of all sets of maximal ideals that contain a given point of the ring is a base for the topology of the structure space. Since $\mathfrak{C}_{\mathfrak{k}}(X,F)$ does not have units and a maximal ideal need not be prime, the classical construction fails. We show here that the set of those maximal ideals which are also prime is the right analogue for building the structure space. We will use the term *prime maximal ideal* for a maximal ideal which is prime.

Theorem 4.1. *If X is locally compact then the set* $\mathfrak{B}=\{\mathcal{Z}_{X,F}(f):f\in\mathfrak{C}_{\mathfrak{k}}(X,F)\}$ *is a base for the closed subsets of X.*

Proof. Let A be a closed subset of X with $x\in X\setminus A$. Since X is locally compact there exists an open set $V\in\mathfrak{N}_x^X$ so that $\mathrm{cl}_X(V)$ is compact and that $\mathrm{cl}_X(V)\subseteq X\setminus A$.

Since V is open and $x\notin X\setminus V$, it follows from complete F regularity of X that there exists an $f\in\mathfrak{C}(X,F)$ such that $f(x)=1$ and $X\setminus V\subseteq\mathcal{Z}_{X,F}(f)$. Consequently, $\mathrm{coz}(f)\subseteq V\Rightarrow\mathrm{supp}_X(f)\subseteq\mathrm{cl}_X(V)$, and thus $\mathrm{supp}_X(f)$ is a compact subset of X, i.e., $f\in\mathfrak{C}_{\mathfrak{k}}(X,F)$.

Thus we have $f\in\mathfrak{C}_{\mathfrak{k}}(X,F)$ such that $x\notin\mathcal{Z}_{X,F}(f)\supseteq A$. Hence the assertion is proved. □

Theorem 4.2. $M_x^{\mathfrak{k}}=\{f\in\mathfrak{C}_{\mathfrak{k}}(X,F):f(x)=0\}$, $x\in X$, *are precisely the fixed maximal ideals of* $\mathfrak{C}_{\mathfrak{k}}(X,F)$.

Furthermore, if $x, y \in X$, $x \neq y$ then $M_x^{\mathfrak{k}} \neq M_y^{\mathfrak{k}}$.

Proof. It is clear from Theorem 3.4 that for each $x \in X$, $M_x^{\mathfrak{k}}$ is a proper ideal of $\mathfrak{C}_{\mathfrak{k}}(X, F)$ and it remains to show that these are maximal.

Choose and fix one $x \in X$ and let $g \in \mathfrak{C}_{\mathfrak{k}}(X, F) \setminus M_x^{\mathfrak{k}}$. It suffices to show that the ideal $(M_x^{\mathfrak{k}}, g)$ generated by $M_x^{\mathfrak{k}}$ and g in $\mathfrak{C}_{\mathfrak{k}}(X, F)$ is the whole of $\mathfrak{C}_{\mathfrak{k}}(X, F)$.

For this purpose let $h \in \mathfrak{C}_{\mathfrak{k}}(X, F) \setminus M_x^{\mathfrak{k}}$. From Theorem 4.1 it follows that there exists an $s \in \mathfrak{C}_{\mathfrak{k}}(X, F)$ such that $s(x) = 1$. Consequently, if $f = s\frac{\mathbf{h(x)}}{\mathbf{g(x)}} \in \mathfrak{C}_{\mathfrak{k}}(X, F)$, then $f(x) = \frac{h(x)}{g(x)}$. Hence, $h - gf \in \mathfrak{C}_{\mathfrak{k}}(X, F)$ with $(h - gf)(x) = 0$, so that $h - gf \in M_x^{\mathfrak{k}}$, implying thereby $h \in (M_x^{\mathfrak{k}}, g)$. Hence, $M_x^{\mathfrak{k}}$ is a maximal ideal of $\mathfrak{C}_{\mathfrak{k}}(X, F)$.

If M is a fixed maximal ideal of $\mathfrak{C}_{\mathfrak{k}}(X, F)$ then there exists an $x \in X$ such that $M \subseteq M_x^{\mathfrak{k}}$, and then the maximality of M ensures $M = M_x^{\mathfrak{k}}$.

The last part regarding the one-to-oneness of the map $x \mapsto M_x^{\mathfrak{k}}$ follows from Theorem 4.1. □

Theorem 4.3. *Every proper prime ideal of $\mathfrak{C}_{\mathfrak{k}}(X, F)$ is also a proper prime ideal of $\mathfrak{C}(X, F)$.*

Proof. It is enough to show that if P is a proper prime ideal in $\mathfrak{C}_{\mathfrak{k}}(X, F)$ then it is a proper ideal of $\mathfrak{C}(X, F)$. Therefore, we show : if $h \in P$ and $g \in \mathfrak{C}(X, F) \setminus \mathfrak{C}_{\mathfrak{k}}(X, F)$, then $gh \in P$.

As $\mathfrak{C}_{\mathfrak{k}}(X, F)$ is an ideal of $\mathfrak{C}(X, F)$ it follows that $gh, g^2h \in \mathfrak{C}_{\mathfrak{k}}(X, F)$ which implies that $(g^2h)h = g^2h^2 \in P$. As P is prime in $\mathfrak{C}_{\mathfrak{k}}(X, F)$, it follows that $gh \in P$. □

Theorem 4.4. *There does not exist any proper free prime ideal in $\mathfrak{C}_{\mathfrak{k}}(X, F)$.*

Proof. If possible, let P be a proper free prime ideal in $\mathfrak{C}_{\mathfrak{k}}(X, F)$. Then by Theorem 4.3 it follows that P is a prime ideal of $\mathfrak{C}(X, F)$; as P is a free ideal of $\mathfrak{C}(X, F)$ it follows from Theorem 3.2 that $\mathfrak{C}_{\mathfrak{k}}(X, F) \subseteq P$ — contradicting the fact that P is a proper ideal of $\mathfrak{C}_{\mathfrak{k}}(X, F)$. This proves the proposition. □

Theorem 4.5. *The entire family of prime maximal ideals in the ring $\mathfrak{C}_{\mathfrak{k}}(X, F)$ is $\{M_x^{\mathfrak{k}} : x \in X\}$. (We state again for emphasis : a maximal ideal in a commutative ring without identity need not be a prime).*

Proof. The assertion follows from Theorem 4.2 and Theorem 4.4. □

We set $\mathfrak{M}_{X,F}^{\mathfrak{k}}$ to be the set of all prime maximal ideals of the ring $\mathfrak{C}_{\mathfrak{k}}(X, F)$. For $f \in \mathfrak{C}_{\mathfrak{k}}(X, F)$, let $\mathfrak{M}_{X,F}^{\mathfrak{k}}(f) = \{M \in \mathfrak{M}_{X,F}^{\mathfrak{k}} : f \in M\}$. Then :

Theorem 4.6. *$\{\mathfrak{M}_{X,F}^{\mathfrak{k}}(f) : f \in \mathfrak{C}_{\mathfrak{k}}(X, F)\}$ is a base for the closed subsets of some topology on $\mathfrak{M}_{X,F}^{\mathfrak{k}}$.*

Proof. Since for $f, g \in \mathfrak{C}_{\mathfrak{k}}(X, F)$ and $M \in \mathfrak{M}^{\mathfrak{k}}_{X,F}$:

$$\begin{aligned} M \in \mathfrak{M}^{\mathfrak{k}}_{X,F}(fg) &\Leftrightarrow fg \in M \\ &\Leftrightarrow f \in M \text{ or } g \in M \end{aligned}$$

(since M is prime)

$$\Leftrightarrow M \in \mathfrak{M}^{\mathfrak{k}}_{X,F}(f) \cup \mathfrak{M}^{\mathfrak{k}}_{X,F}(g),$$

so that $\mathfrak{M}^{\mathfrak{k}}_{X,F}(fg) = \mathfrak{M}^{\mathfrak{k}}_{X,F}(f) \cup \mathfrak{M}^{\mathfrak{k}}_{X,F}(g)$. The remaining statements follow exactly as in the case of a commutative ring with unity. □

Definition 4.7. *We shall call the set* $\mathfrak{M}^{\mathfrak{k}}_{X,F}$ *equipped with the topology described in Theorem 4.6 the* structure space *of the ring* $\mathfrak{C}_{\mathfrak{k}}(X, F)$.

Theorem 4.8. *If* X *is a locally compact non-compact topological space then* $\mathfrak{p}^{\mathfrak{k}}_{X,F} : X \xrightarrow{\simeq} \mathfrak{M}^{\mathfrak{k}}_{X,F}$, *where* $\mathfrak{p}^{\mathfrak{k}}_{X,F} : x \mapsto M^{\mathfrak{k}}_x$.

Proof. It follows from Theorem 4.2 and Theorem 4.5 that $X \xrightarrow{\mathfrak{p}^{\mathfrak{k}}_{X,F}} \mathfrak{M}^{\mathfrak{k}}_{X,F}$ is a bijection from X onto $\mathfrak{M}^{\mathfrak{k}}_{X,F}$. Furthermore, for $f \in \mathfrak{C}_{\mathfrak{k}}(X, F)$ and $x \in X$, $x \in \mathcal{Z}_{X,F}(f) \Leftrightarrow f(x) = 0 \Leftrightarrow f \in M^{\mathfrak{k}}_x \Leftrightarrow M^{\mathfrak{k}}_x \in \mathfrak{M}^{\mathfrak{k}}_{X,F}(f)$, so that using Theorem 4.1 and Theorem 4.6 the map $\mathfrak{p}^{\mathfrak{k}}_{X,F}$ establishes a one-to-one map between the basic closed sets in the two spaces concerned and therefore $\mathfrak{p}^{\mathfrak{k}}_{X,F} : X \xrightarrow{\simeq} \mathfrak{M}^{\mathfrak{k}}_{X,F}$. □

It is well known that the structure space of a commutative ring with identity is a compact topological space. The proof for this assertion heavily depends on the existence of the identity element in the ring. Theorem 4.8 sounds something contrary in this regard — the structure space of a commutative ring without identity may fail to be compact — indeed for a locally compact non-compact topological space X the structure space $\mathfrak{M}^{\mathfrak{k}}_{X,F}$ of $\mathfrak{C}_{\mathfrak{k}}(X, F)$ is locally compact without being compact.

Nevertheless we are now in a position to formulate the principal theorem of this paper.

Theorem 4.9. *Suppose that* X *and* Y *are two locally compact non-compact Hausdorff topological spaces. Then :*

(1) $X \xrightarrow{\simeq} Y$ *as topological spaces, if and only if,* $\mathfrak{C}_{\mathfrak{k}}(X, \mathbb{R}) \xrightarrow{\simeq} \mathfrak{C}_{\mathfrak{k}}(Y, \mathbb{R})$, *as lattice ordered commutative rings.*

(2) *If further,* X *and* Y *are zero dimensional and* F, G *be two ordered fields then* $X \xrightarrow{\simeq} Y$, *as topological spaces, if* $\mathfrak{C}_{\mathfrak{k}}(X, F) \xrightarrow{\simeq} \mathfrak{C}_{\mathfrak{k}}(Y, F)$, *as lattice ordered commutative rings.*

Proof.

(1) If $X \xrightarrow{\simeq} Y$ then it is trivial that $\mathfrak{C}_{\mathfrak{k}}(X,\mathbb{R}) \xrightarrow{\simeq} \mathfrak{C}_{\mathfrak{k}}(Y,\mathbb{R})$. Conversely, $\mathfrak{C}_{\mathfrak{k}}(X,\mathbb{R}) \xrightarrow{\simeq} \mathfrak{C}_{\mathfrak{k}}(Y,\mathbb{R})$ implies that $\mathfrak{M}^{\mathfrak{k}}_{X,\mathbb{R}} \xrightarrow{\simeq} \mathfrak{M}^{\mathfrak{k}}_{Y,\mathbb{R}}$, since the structure spaces of two isomorphic rings are homeomorphic, so that from Theorem 4.8 it follows that $X \xrightarrow{\simeq} Y$.

(2) By Theorem 2.6(3), for $F \neq \mathbb{R}$ the completely F regular topological spaces are precisely the zero dimensional topological spaces; also any zero dimensional Hausdorff topological space is completely regular, i.e., completely $\mathbb{R}$ regular in our terminology.

Thus, $\mathfrak{C}_{\mathfrak{k}}(X,F) \xrightarrow{\simeq} \mathfrak{C}_{\mathfrak{k}}(Y,G)$ implies that $\mathfrak{M}^{\mathfrak{k}}_{X,F} \xrightarrow{\simeq} \mathfrak{M}^{\mathfrak{k}}_{Y,G}$, since the structure spaces of two isomorphic rings are homeomorphic, so that from Theorem 4.8 it follows that $X \xrightarrow{\simeq} Y$.

□

Acknowledgements. The authors are thankful to Professor Hans Peter Künzi, Department of Mathematics and Applied Mathematics, University of Cape Town, for his valuable suggestions regarding the paper. The last author is thankful to the support extended by Dept. of Math and Appl. Math., University of Cape Twon, during his tenure as a Post Doctoral Student there, and further, acknowledges his teacher Professor V. Saroja, without whose constant support the present work would not have been possible.

References

[1] W. Wieslaw, *Topological Fields*, Marcell Dekker (1978).
[2] S. Mrowka and R. Engelking, *On E-compact spaces*, Bull. Acad. Polon. sci. Ser. sci. Math. Astronom. Phys. **6** (1958), 429–435.
[3] L. Gillman and M. Jerison, Rings of Continuous Functions, van Nostrand Reinhold Company, edited by M. H. Stone, L. Nirenberg and S. S. Chern (1960).
[4] S. K. Acharyya, K. C. Chattopadhyaya and P. P. Ghosh, *Constructing Banaschewski Compactification Without Dedekind Completeness Axiom*, to appear in International Journal for Mathematics and Mathematical Sciences.
[5] S. K. Acharyya, K. C. Chattopadhyaya and P. P. Ghosh, *The rings $C_k(X)$ and $C_\infty(X)$, some remarks*, Kyungpook Journal of Mathematics, **43** (2003), 363 - 369.

Received February 2003
Accepted May 2003

S. K. Acharyya (Department of Pure Mathematics, University of Calcutta, 35, Ballygaunge Circular Road, Calcutta 700019, West Bengal, India.)

K. C. Chattopadhyaya (Department of Mathematics, University of Burdwan, Burdwan, West Bengal, India)

Partha Pratim Ghosh (vsatsxc@cal.vsnl.net.in, pghosh@maths.uct.ac.za) Department of Mathematics, St. Xavier's College 30, Park Street, Calcutta 700016, India.

Current Address (Department of Mathematics and Applied Mathematics, University of Cape Town, Rondebosch 7701, Cape Town, South Africa)