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On separation axioms of uniform bundles and sheaves

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Abstract. In the context of the theory of uniform bundles in the sense of J. Dauns and K. H. Hofmann, the topology of the fiber space of a uniform bundle depends on the assumption of upper semicontinuity of its defining set of pseudometrics when composed with local sections. In this paper we show that the additional hypothesis of lower semicontinuity of these functions secures that the fiber space of the uniform bundle is Hausdorff, regular or completely regular provided that the base space has the corresponding separation axiom. Similar results for the particular important case of sheaves of sets follow suit.

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1. Introduction

In the general theory of uniform bundles laid down by K. H. Hofmann and J. Dauns [1], Theorem I, page 23, the topology of the fiber space E of a uniform bundle (E, p, B), where B is the base space and $p : E \longrightarrow B$ is a surjection, is constructed in terms of the data provided by a family of local selections (functions from a variable open subset S of B, that when composed with p give the identity of S) and a uniformity on E. This construction require some conditions to be carried out successfully, conditions that in the simplest case on bundles of metric spaces (when the uniformity is associated with a metric on E) amount to the requirement that the distance functions $s \longmapsto d(\sigma(s), \tau(s)) : S \longrightarrow \mathbb{R}$ are upper semicontinuous, where σ and τ are local selections.

A basic question that has been pending in this theory is the significance of the additional hypothesis of lower semicontinuity and consequently of the more

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stringent condition of continuity of these distance functions. The aim of this paper is to answer this question in a general manner.

In the section of Preliminaries we establish notation and recall an existence theorem of uniform bundles indispensable in the different examples presented in the paper. In the third section on Separation Axioms appear the main results of the paper: Theorem 3.10 establishes that, under the ground assumption of lower semicontinuity of the distance functions, if the base space B is Hausdorff, then the fiber space E is also Hausdorff, Theorems 3.15 and 3.19 contain similar results in the case of regularity and complete regularity respectively.

The concept of sheaf of sets can be regarded as a particular case of the concept of uniform bundle by the simple expedient of considering each fiber equipped with the discrete metric. This allows us to examine the upper and lower semicontinuity of the distance functions. As in the general case, the first condition determines the topology of E, while the second has to do with the separation axioms of the fiber space.
It is well known that the fiber space of the sheaf of germs of holomorphic functions is a Tychonoff space. As an application of the results presented in this paper we obtain an alternative proof of this property.

2. Preliminaries

Let E and B be topological spaces, $p : E \longrightarrow B$ be a surjective function. For each $t \in B$, the set $E_t = p^{-1}(t) = \{a \in E : p(a) = t\}$ is called the *fiber above* t. Note that E is the disjoint union $\coprod_{t\in B} E_t$ of the family $(E_t)_{t\in B}$.
A *local selection* for p is a function $\sigma : Q \longrightarrow E$ such that $Q \subset B$ is an open set and $p \circ \sigma$ is the identity map Id_Q of Q. A *local section for* p is by definition a continuous local selection.
Let $\Gamma_Q(p) := \{\sigma : Q \longrightarrow E \mid Q \subset B \text{ is open, } \sigma \text{ is continuous and } p\circ\sigma = Id_Q\}$. If $Q = B$, then σ is a *global section* and we write $\Gamma(p)$ instead of $\Gamma_B(p)$. A set Σ of local sections is called *full* if for every $x \in E$ there exists $\sigma \in \Sigma$ such that $\sigma(p(x)) = x$.

Let $E \bigvee E := \{(u, v) \in E \times E : p(u) = p(v)\}$. The function $d : E \bigvee E \longrightarrow \mathbb{R}$ is called a *pseudometric for* p provided that the restriction of d to $E_t \times E_t$ is a pseudometric on E_t, for each $t \in B$. A family $(d_i)_{i\in I}$ of pseudometrics for p is *directed* if for each pair $i_1,\ i_2 \in I$ there exists $i \in I$ such that $d_{i_1}(u, v) \leq d_i(u, v)$ and $d_{i_2}(u, v) \leq d_i(u, v)$, for every $(u, v) \in E \bigvee E$.

Definition 2.1. *Let $(d_i)_{i\in I}$ be a directed family of pseudometrics for p and consider a local selection σ, $i \in I$ and $\epsilon > 0$. The set $\mathcal{T}^i_\epsilon(\sigma) = \{u \in E : d_i(u, \sigma(p(u))) < \epsilon\}$ is called the ϵ-tube around σ with respect to d_i.*

Definition 2.2. *Let E and B be topological spaces, $p : E \longrightarrow B$ a surjective function and $(d_i)_{i\in I}$ a family of pseudometrics for p. The triple (E, p, B) is called a bundle of uniform spaces or, for short, a uniform bundle, provided that:*

1. *For every $u \in E$, every $\epsilon > 0$ and every $i \in I$, there exists a local section σ such that $u \in \mathcal{T}^i_\epsilon(\sigma)$.*

2. *The tubes around all the local sections for p form a base for the topology of E.*

The space B is called *the base space* and the space E is called *the bundle space.*

Note that if (E, p, B) is a uniform bundle, then the function p is continuous and open.

From Definition 2.2 we obtain the upper semicontinuity of the distance functions $s \longmapsto d(\sigma(s), \tau(s)) : Q \longrightarrow \mathbb{R}_+$, Q being an open subset of $Dom\,\sigma \cap Dom\,\tau$ and σ and τ arbitrary local sections for p.

The following theorem on existence of uniform bundles will be used freely when required and without previous notice in the constructions outlined in the examples presented in this work. We sketch its proof since the reference [4], containing it, is not easily accesible.

Theorem 2.3. *Let B be a topological space and $p : E \longrightarrow B$ be a surjective function. Denote by Σ a set of local selections for p and let $(d_i)_{i\in I}$ be a directed family of pseudometrics for p. We make the following assumptions:*

a) *For every $u \in E$, every $i \in I$ and every $\epsilon > 0$, there exists $\alpha \in \Sigma$ such that $u \in \mathcal{T}^i_\epsilon(\alpha)$.*
b) *For every $i \in I$ and every $(\alpha, \beta) \in \Sigma\times\Sigma$ the function $s \longmapsto d_i(\alpha(s), \beta(s)) : Dom\,\alpha \cap Dom\,\beta \longrightarrow \mathbb{R}$ is upper semicontinuous.*

Then E can be equipped with a topology $\mathcal{S}$ such that:

1) *$\mathcal{S}$ has a base consisting of the sets of the form $\mathcal{T}^i_\epsilon(\alpha_Q)$, where $i \in I$, $\epsilon > 0$, $\alpha \in \Sigma$, Q is an open subset of $Dom\,\alpha$ and α_Q denotes the restriction of α to Q.*
2) *Each $\alpha \in \Sigma$ is a section.*
3) *(E, p, B) is a uniform bundle.*

Proof. We first show that the collection of all sets $\mathcal{T}^i_\epsilon(\alpha_Q)$, with the specifications given in conclusion 1), is a base for a topology $\mathcal{S}$ in E.
Given two such tubes $\mathcal{T}^i_\epsilon(\alpha_Q)$ and $\mathcal{T}^j_\delta(\beta_P)$ and $u \in \mathcal{T}^i_\epsilon(\alpha_Q) \cap \mathcal{T}^j_\delta(\beta_P)$ let $\rho = \min\{\frac{1}{4}(\epsilon - d_i(u, \alpha(p(u)))), \frac{1}{4}(\delta - d_j(u, \beta(p(u))))\}$. Let $k \in I$ such that $d_i(u_1, u_2) \le d_k(u_1, u_2)$ and $d_j(u_1, u_2) \le d_k(u_1, u_2)$ for every $(u_1, u_2) \in E \bigvee E$ and let $\xi \in \Sigma$ such that $u \in \mathcal{T}^k_\rho(\xi) = \{v \in E : d_k(v, \xi(p(v))) < \rho\}$, then $p(u) \in \{s \in B : d_i(\xi(s), \alpha(s)) < \epsilon_i\}$, where $\epsilon_i = \frac{1}{2}(d_i(u, \alpha(p(u))) + \epsilon)$, in fact, since $u \in \mathcal{T}^i_\epsilon(\alpha_Q)$ it follows that $d_i(u, \alpha(p(u))) < \epsilon$ and thus $d_i(u, \alpha(p(u))) < \frac{3}{4}d_i(u, \alpha(p(u))) + \frac{1}{4}\epsilon$. On the other hand, the relation $u \in \mathcal{T}^k_\rho(\xi)$ implies $d_i(u, \xi(p(u))) < \frac{1}{4}(\epsilon - d_i(u, \alpha(p(u))))$ and therefore $d_i(\xi(p(u)), \alpha(p(u))) < \epsilon_i$. Similarly, $p(u) \in \{s \in B : d_j(\xi(s), \beta(s)) < \delta_j\}$ where $\delta_j = \frac{1}{2}(d_j(u, \beta(p(u))) + \delta)$. By the semicontinuity hypothesis the sets $\{s \in B : d_i(\xi(s), \alpha(s)) < \epsilon_i\}$ and $\{s \in B : d_j(\xi(s), \beta(s)) < \delta_j\}$ are open, it follows that $S = P \cap Q \cap \{s \in B : d_i(\xi(s), \alpha(s)) < \epsilon_i\} \cap \{s \in B : d_j(\xi(s), \beta(s)) < \delta_j\}$ is a neighborhood of $p(u)$ in the space B and $\mathcal{T}^k_\rho(\xi_S) \subset \mathcal{T}^i_\epsilon(\alpha)$, indeed, the relation

$v \in \mathcal{T}_\rho^k(\xi_S)$ implies $d_i(v, \xi(p(v))) < \rho < \frac{1}{2}(\epsilon - d_i(u, \alpha(p(u))))$, but $p(v) \in S$, then $d_i(\xi(p(v)), \alpha(p(v))) < \frac{1}{2}(d_i(u, \alpha(p(u))) + \epsilon)$, thus $d_i(v, \alpha(p(v))) < \epsilon$ and therefore $v\ in \mathcal{T}_\epsilon^i(\alpha_Q)$. The inclusion $\mathcal{T}_\rho^k(\xi_S) \subset \mathcal{T}_\delta^j(\beta_P)$ is obtained in the same manner.

2) Let $\alpha \in \Sigma$ and $t \in Dom\,\alpha$. A fundamental neighborhood of $\alpha(t)$ in E is of the form $\mathcal{T}_\epsilon^i(\beta_Q)$, where $\beta \in \Sigma$, $Q \subset Dom\,\alpha$ is open in B, $\epsilon > 0$, $i \in I$ and $\alpha(t) \in \mathcal{T}_\epsilon^i(\beta_Q)$. By hypothesis b), the set $\alpha^{-1}(\mathcal{T}_\epsilon^i(\beta_Q)) = \{s \in Q : d_i(\alpha(s), \beta(s)) < \epsilon\}$ is open in B, therefore α is a section.

3) The tubes around arbitrary local sections are open, in fact, let $u \in E$ and let σ be a local section for p (not necessarily in Σ) such that $u \in \mathcal{T}_\epsilon^i(\sigma)$. To prove that (E, p, B) is a uniform bundle, we must exhibit $\eta > 0$ and $\alpha \in \Sigma$ such that $u \in \mathcal{T}_\eta^i(\alpha)$ and $\mathcal{T}_\eta^i(\alpha_P) \subset \mathcal{T}_\epsilon^i(\sigma)$ for some neighborhood P of $p(u)$ in B.

Let $\eta = \frac{1}{4}(\epsilon - d_i(u, \sigma(p(u))))$ and let $\alpha \in \Sigma$ be such that $u \in \mathcal{T}_\eta^i(\alpha)$. Since $u \in \mathcal{T}_\epsilon^i(\sigma)$ we have $d_i(u, \sigma(p(u))) < \epsilon$ and thus $d_i(u, \sigma(p(u))) < \frac{3}{4} d_i(u, \sigma(p(u))) + \frac{1}{4}\epsilon$. On the other hand, the relation $u \in \mathcal{T}_\eta^i(\alpha)$ implies $d_i(u, \alpha(p(u))) < \eta = \frac{1}{4}(\epsilon - d_i(u, \sigma(p(u))))$, therefore $d_i(\sigma(p(u)), \alpha(p(u))) < \frac{1}{2} d_i(u, \sigma(p(u))) + \frac{1}{2}\epsilon$, then $p(u) \in \sigma^{-1}(\mathcal{T}_{\epsilon_i}^i(\alpha))$, where $\epsilon_i = \frac{1}{2}(d_i(u, \sigma(p(u))) + \epsilon)$. Since σ is continuous, $\sigma^{-1}(\mathcal{T}_{\epsilon_i}^i(\alpha))$ is an open neighborhood P of $p(u)$, then $v \in \mathcal{T}_\eta^i(\alpha_P)$ implies $p(v) \in P$ and hence $d_i(\alpha(p(v)), \sigma(p(v))) < \frac{1}{2}(d_i(u, \sigma(p(u))) + \epsilon)$, we also have

$$d_i(v, \alpha(p(v))) < \eta < \frac{1}{2}(\epsilon - d_i(u, \sigma(p(u)))),$$

thus $d_i(v, \sigma(p(v))) < \epsilon$, that is $v \in \mathcal{T}_\epsilon^i(\sigma)$. □

Definition 2.4. *Let E and B be topological spaces and let $p : E \longrightarrow B$ be a surjective function. A triple (E, p, B) is said to be a sheaf of sets provided that p is a local homeomorphism, that is, each point $a \in E$ has an open neighborhood which is mapped homeomorphically by p onto an open subset of B.*

Recall that if (E, p, B) is a sheaf of sets we have:

(1) The ranges of the local sections for p form a base of the topology of E.
(2) If two local sections intersect at a point t, they agree on an open neighborhood of t.
(3) The discrete metric $d : E \bigvee E \longrightarrow \mathbb{R}$ defined by

$$d(m, n) = \begin{cases} 0 & \text{if } m = n \\ 1 & \text{if } m \neq n \end{cases}$$

is in particular a pseudometric for p, and it can be seen that the sheaf (E, p, B), with the family of pseudometrics reduced to the single discrete pseudometric, becomes a uniform bundle, indeed:

i. Consider $m \in E$, $\epsilon > 0$ and an open neighborhood M of m in E such that $p \restriction_M$ is a homeomorphism from M onto an open set of B. It is clear that $(p \restriction_M)^{-1}$ is a local section such that $m \in \mathcal{T}_\epsilon((p \restriction_M)^{-1}) = \{n \in E : d(n, (p \restriction_M)^{-1}(p(n))) < \epsilon\}$ because $m = (p \restriction_M)^{-1}(p(m))$.

ii. If $\sigma : Q \longrightarrow E$ is a local section for p and $\epsilon > 0$, then $\mathcal{T}_\epsilon(\sigma) = \{m \in E : d(m, \sigma(p(m))) < \epsilon\}$ is the range of σ provided that $\epsilon \leq 1$, but $\mathcal{T}_\epsilon(\sigma) = p^{-1}(Q)$ if $\epsilon > 1$. Hence the tubes around all the local sections for p form a base for the topology of E.

Conversely, if (E, p, B) is a uniform bundle with the directed family of pseudometrics $(d_i)_{i \in I}$ generating the discrete uniform structure on E, that is, if the diagonal $\Delta_E = \{(m, m) : m \in E\}$ belongs to the uniform structure, then there exist $j \in I$ and $\epsilon > 0$ such that Δ_E is precisely the set $U_\epsilon^j = \{(m, n) : d_j(m, n) < \epsilon\}$. It is apparent that for each local section σ for p, we have $\mathcal{T}_\epsilon^j(\sigma) = Ran\,\sigma$. Given $m \in E$, since (E, p, B) is a uniform bundle there exists a local section σ for p such that $m \in \mathcal{T}_\epsilon^j(\sigma)$, then $\sigma(p(m)) = m$, it follows that $\mathcal{T}_\epsilon^j(\sigma)$ is an open neighborhood of m and that the restriction q of p to $\mathcal{T}_\epsilon^j(\sigma)$ is a homeomorphism from $\mathcal{T}_\epsilon^j(\sigma)$ onto the domain of σ whose inverse is σ. Then p is a local homeomorphism and thus (E, p, B) is a sheaf of sets. In particular, if the family generating the uniformity for p reduces to the one pseudometric whose restriction to each fiber is the discrete metric, then (E, p, B) is a sheaf of sets.

The following example shows that the requirement that each fiber E_t of a uniform bundle (E, p, B) has the discrete topology does not guarantee it to be a sheaf of sets.

Example 2.5. Let $B = \mathbb{R}$ with the usual topology, let E be the subset of the euclidean plane defined by $E = \{(x, y) : y = x \text{ or } y = -x\}$ and let $p : E \longrightarrow B$ be the map such that $p(x, y) = x$.

Consider the family of pseudometrics for p reduced to the pseudometric d defined on each fiber by $d((x, y_1), (x, y_2)) = |y_1 - y_2|$. Let $\Sigma = \{\sigma_1, \ \sigma_2\}$ be the full set of global selections for p defined by $\sigma_1(x) = (x, x)$ and $\sigma_2(x) = (x, -x)$. The function $\varphi : B \longrightarrow \mathbb{R}$ defined by $\varphi(x) = d(\sigma_1(x), \sigma_2(x))$ is upper semicontinuous, thus the tubes $\mathcal{T}_\epsilon(\sigma)$, where $\epsilon > 0$ and σ is the restriction of any one of the elements of Σ to an open set of B, form a base for a topology on E that gives to the triple (E, p, B) the structure of a uniform bundle in which σ_1 and σ_2 are sections. Each fiber E_x is discrete, but (E, p, B) is not a sheaf of sets since $\sigma_1(0) = \sigma_2(0)$ but for each open interval J containing 0, $\sigma_1(x) \neq \sigma_2(x)$ if $x \in J$ and $x \neq 0$.

3. Separation axioms

The proofs of the next two elementary lemmas are straighforward and are omitted.

Lemma 3.1. *Let E, B be topological spaces and $p : E \longrightarrow B$ be a continuous map. Assume that B is a T_0 (resp. T_1) space and that for each $t \in B$ the subspace $p^{-1}(t)$ is T_0 (resp. T_1), then E is also a T_0 (resp. T_1) space.*

Lemma 3.2. *Let E, B be topological spaces, $p : E \longrightarrow B$ be a continuous function and $\sigma : B \longrightarrow E$ be a global section for p, then σ is an embedding. If in addition E is supposed to be a T_2 space, then σ is a closed embedding.*

Proposition 3.3. *Let (E, p, B) be a uniform bundle whose uniformity is given by the directed family $(d_i)_{i \in I}$ of pseudometrics. If the space E is T_0, then the space B is also T_0.*

Proof. Let t and t' be two different points of B. Choose a point u on the fiber E_t and $i \in I$. Given $\epsilon > 0$, there exists a local section σ such that $u \in \mathcal{T}_\epsilon^i(\sigma)$. If $t' \notin Dom\,\sigma$, then $Dom\,\sigma$ is a neighborhood of t which does not contain t', but if $t' \in Dom\,\sigma$, then $\sigma(t)$ and $\sigma(t')$ are different points of E and thus there exists an open set $M \subset E$ containing only one of the two points either $\sigma(t)$ or $\sigma(t')$, but not both, then $\sigma^{-1}(M)$ is an open subset of B containing one of the points, either t or t' but not both. It follows that B is a T_0 space. □

The next proposition is a direct consequence of Lemma 3.1.

Proposition 3.4. *Let (E, p, B) be a uniform bundle. If the fiber E_t is a T_0 space for each $t \in B$ and the space B is T_0, then the space E is also T_0.*

Since the fibers in a bundle of metric spaces (that is, a bundle whose uniformity is given by a metric), particularly in a sheaf of sets, are Hausdorff spaces, we have the following corollary.

Corollary 3.5. *Let (E, p, B) be a bundle of metric spaces (resp. a sheaf of sets). The space B is T_0 if and only if E is T_0.*

In the next two propositions we examine how the property of being T_1 is inherited from E to B and vice versa.

Proposition 3.6. *Let (E, p, B) be a uniform bundle whose uniformity is given by the directed family $(d_i)_{i \in I}$ of pseudometrics. If the space E is T_1, then the space B is also T_1.*

Proof. Let t and t' be two different points of B. Choose a point u on the fiber E_t above t and take $i \in I$. For $\epsilon > 0$, let σ be a local section such that $u \in \mathcal{T}_\epsilon^i(\sigma)$. If t and t' belong to $\sigma^{-1}(\mathcal{T}_\epsilon^i(\sigma))$ we have that $\sigma(t)$ and $\sigma(t')$ are two different points of E and thus there exists an open neighborhood V of $\sigma(t)$ with $\sigma(t') \notin V$ and there exists an open neighborhood W of $\sigma(t')$ with $\sigma(t) \notin W$.

Therefore $\sigma^{-1}(V)$ is an open neighborhood of t such that $t' \notin \sigma^{-1}(V)$ and $\sigma^{-1}(W)$ is an open neighborhood of t' such that $t \notin \sigma^{-1}(W)$. If $t' \notin \sigma^{-1}(\mathcal{T}_\epsilon^i(\sigma))$ we choose a point $v \in E_{t'}$ and a local section τ such that $v \in \mathcal{T}_\epsilon^i(\tau)$. If t and t' belong to $\tau^{-1}(\mathcal{T}_\epsilon^i(\tau))$ we are in the previous case, but if t does not belong to $\tau^{-1}(\mathcal{T}_\epsilon^i(\tau))$, then $\sigma^{-1}(\mathcal{T}_\epsilon^i(\sigma))$ is an open neighborhood of t which does not contain t' and $\tau^{-1}(\mathcal{T}_\epsilon^i(\tau))$ is an open neighborhood of t' that does not contain t. Then B is a T_1 space. □

Conversely, the following property follows directly from Lemma 3.1.

Proposition 3.7. *Let (E, p, B) be a uniform bundle. If the fiber E_t is a T_1 space for each $t \in B$ and if the space B is T_1, then the space E is T_1.*

In the context of bundles of metric spaces (resp. sheaves of sets) we have the following result.

Corollary 3.8. *Let (E, p, B) be a bundle of metric spaces (resp. a sheaf of sets). The space B is T_1 if and only if E is T_1.*

In the next two statements it is shown how the property of being a Hausdorff space is transferred from the bundle space to the base space and vice versa.

Proposition 3.9. *Let (E, p, B) be a uniform bundle. If E is a Hausdorff space and if there exists a global section for p, then B is also a Hausdorff space.*

Proof. It follows from Lemma 3.2. □

Theorem 3.10. *Let (E, p, B) be a uniform bundle, $(d_i)_{i\in I}$ be a directed family of pseudometrics inducing the uniformity for p and suppose that the fiber E_t is Hausdorff for every $t \in B$. If B is a Hausdorff space and if the functions $t \longmapsto d_i(\sigma(t), \tau(t)) : Q \longrightarrow \mathbb{R}$ are continuous for each open set $Q \subset B$, each $i \in I$ and each pair $\sigma,\ \tau \in \Gamma_Q(p)$, then E is also a Hausdorff space.*

Proof. Let $u,\ v \in E$ such that $u \neq v$. If $p(u) \neq p(v)$, then there exist disjoint open neighborhoods Q and R of $p(u)$ and $p(v)$ respectively. Since p is a continuous function, $p^{-1}(Q)$ is an open neighborhood of u, $p^{-1}(R)$ is an open neighborhood of v and $p^{-1}(Q) \cap p^{-1}(R) = \varnothing$, but if $p(u) = p(v) = t$ one can find $i \in I$ such that $d_i(u, v) > 0$. For $\epsilon,\ \delta > 0$ such that $\epsilon + \delta < \dfrac{d_i(u, v)}{2}$, there exist two sections σ and τ with domain P, where P is an open neighborhood of $p(u)$, such that $u \in \mathcal{T}^i_\epsilon(\sigma)$ and $v \in \mathcal{T}^i_\delta(\tau)$. It then follows that

$$\begin{aligned} 2(\epsilon + \delta) < d_i(u, v) &\leq d_i(u, \sigma(t)) + d_i(\sigma(t), v) \\ &\leq d_i(u, \sigma(t)) + d_i(\sigma(t), \tau(t)) + d_i(\tau(t), v) \\ &< \epsilon + \delta + d_i(\sigma(t), \tau(t)). \end{aligned}$$

Therefore $d_i(\sigma(t), \tau(t)) > \epsilon + \delta$, and since the function $\varphi : Q \longrightarrow \mathbb{R}$, $s \longmapsto d_i(\sigma(s), \tau(s))$ is continuous, there exists an open neighborhood S of t such that $\varphi(s) > \epsilon + \delta$, for each $s \in S$. Let $\sigma' := \sigma \restriction_S$ and $\tau' := \tau \restriction_S$, one has that $u \in \mathcal{T}^i_\epsilon(\sigma')$, $v \in \mathcal{T}^i_\delta(\tau')$ and these tubes are disjoint because if $z \in \mathcal{T}^i_\epsilon(\sigma') \cap \mathcal{T}^i_\delta(\tau')$, then $\epsilon + \delta < d_i(\sigma'(p(z), \tau'(p(z)))) \leq d_i(\sigma'(p(z)), z) + d_i(z, \tau'(p(z))) < \epsilon + \delta$, which is a contradiction. □

In the special case of sheaves of sets we have:

Corollary 3.11. *Let (E, p, B) a sheaf of sets and assume that B is a Hausdorff space. The space E is Hausdorff if and only if the functions $t \longmapsto d(\sigma(t), \tau(t)) : Q \longrightarrow \mathbb{R}$ are continuous for each open subset Q of B and each $\sigma,\ \tau \in \Gamma_Q(p)$, d being the discrete metric on each fiber.*

Proof. It remains to prove that if E is a Hausdorff space, then the functions $t \longmapsto d(\sigma(t), \tau(t)) : Q \longrightarrow \mathbb{R}$ are lower semicontinuous. To this end, consider $a > 0$ such that $d(\sigma(t), \tau(t)) > a$. It follows that $a < 1$ and $\sigma(t) \neq \tau(t)$. Since (E, p, B) is a sheaf of sets and E is Hausdorff there exist an open neighborhood R of t and local sections σ_1 and τ_1 such that $\sigma(r) = \sigma_1(r)$ and $\tau(r) = \tau_1(r)$ for each $r \in R$ and $Ran\, \sigma_1 \cap Ran\, \tau_1 = \varnothing$. Then $d(\sigma(r), \tau(r)) = 1 > a$ for each $r \in R$. □

The following example shows that, in the general case of uniform bundles, the continuity hypothesis made on the functions $t \longmapsto d(\sigma(t), \tau(t))$, where σ and τ are local sections defined in an open set $Q \subset B$, is a sufficient but not a necessary condition for E to be a Hausdorff space.

Example 3.12. Let $B = \mathbb{R}$ with the usual topology, let E be the subset of the euclidean plane defined by

$$E = \{(x, y) : x \neq 0 \ \text{ and } \ (y = \frac{1}{2} \ \text{ or } \ y = 1)\} \cup \{(0, \frac{1}{4}), (0, \frac{5}{4})\}$$

and let $p : E \longrightarrow B$ be the map such that $p(x, y) = x$.

Consider the family of pseudometrics for p reduced to the pseudometric d defined on each fiber by $d((x, y_1), (x, y_2)) = |y_1 - y_2|$. Let $\Sigma = \{\sigma_1,\ \sigma_2\}$ be the full set of global selections for p defined by

$$\sigma_1(x) = \begin{cases} (x, 1) & \text{if } x \neq 0 \\ (0, \frac{5}{4}) & \text{if } x = 0 \end{cases}$$

$$\sigma_2(x) = \begin{cases} (x, \frac{1}{2}) & \text{if } x \neq 0 \\ (0, \frac{1}{4}) & \text{if } x = 0. \end{cases}$$

The function $\varphi : B \longrightarrow \mathbb{R}$ defined by $\varphi(t) = d(\sigma_1(t), \sigma_2(t))$ is upper semicontinuous. In fact

$$\varphi(t) = \begin{cases} \frac{1}{2} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$$

thus the tubes $\mathcal{T}_\epsilon(\sigma)$, where $\epsilon > 0$ and σ is the restriction of any one of the elements of Σ to an open set of B, form a base for a topology on E that gives to the triple (E, p, B) the structure of a uniform bundle, actually of a sheaf of sets, in which σ_1 and σ_2 are sections. Although the space E is Hausdorff φ fails to be continuous.

It is interesting to remark that the chosen metric is not the discrete metric despite the topology of the fibers being the discrete one. If we had constructed the bundle by means of the discrete metric on the fibers, we had obtained that the function φ would be constant (equal to 1) and consequently continuous.

In the following example the space B is Hausdorff while the space E is not.

Example 3.13. Let $B = \mathbb{R}$ with the usual topology, let E be the subset of the euclidean plane $E = \{(x, 0) : x < 0\} \cup \{(x, y) : x \geq 0 \ \text{ and } \ y = \pm 1\}$ and let $p : E \longrightarrow B$ be the function defined by $p((x, y)) = x$.

Consider the family of pseudometrics for p with the only pseudometric d that is the discrete metric on each fiber. Let $\Sigma = \{\sigma_1,\ \sigma_2\}$ be the set of global selections for p given by

$$\sigma_1(x) = \begin{cases} (x, 0) & \text{if } x < 0 \\ (x, 1) & \text{if } x \geq 0 \end{cases}$$

$$\sigma_2(x) = \begin{cases} (x, 0) & \text{if } x < 0 \\ (x, -1) & \text{if } x \geq 0. \end{cases}$$

So we have that if $(x, y) \in E$, then either $(x, y) = \sigma_1(x)$ or $(x, y) = \sigma_2(x)$. The function $\varphi : B \longrightarrow \mathbb{R}$, $t \longmapsto d(\sigma_1(t), \sigma_2(t))$ is upper semicontinuous, in fact,

$$\varphi(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0. \end{cases}$$

Therefore the tubes $\mathcal{T}_\epsilon(\sigma)$, where $\epsilon > 0$ and σ is a restriction of an element of Σ to an open set of B, form a base for a topology on E such that (E, p, B) is a uniform bundle, actually a sheaf of sets, and Σ is a set of sections for p.
The points $(0, 1)$ and $(0, -1)$ of the space E can not be separated by disjoint open sets because every open set in E containing one of these points contains the set $\{(x, 0) : \xi < x < 0\}$ for some $\xi < 0$; therefore E is not a Hausdorff space.

Now we recall that a topological space X is said to be *regular* if for every closed subset K of X and every $x \in X \setminus K$, there are open subsets V and W of X such that $K \subset V$, $x \in W$ and $V \cap W = \varnothing$. Equivalently, X is regular if and only if for every open subset A of X and every $x \in A$ there is an open neighborhood V of x such that $\overline{V} \subset A$.

The following proposition is a direct consequence of Lemma 3.2.

Proposition 3.14. *Let (E, p, B) be a uniform bundle. If E is a regular space and if there exists a global section for p, then B is also a regular space.*

The converse of the above proposition is not so trivial and gives rise to following result.

Theorem 3.15. *Let (E, p, B) be a uniform bundle and let $(d_i)_{i \in I}$ be a directed family of pseudometrics inducing the uniformity for p. If B is a regular space and the functions $t \longmapsto d_i(\sigma(t), \tau(t)) : Q \longrightarrow \mathbb{R}$ are continuous for each open set $Q \subset B$, each $i \in I$ and each pair $\sigma,\ \tau \in \Gamma_Q(p)$, then E is also a regular space.*

Proof. Let M be an open subset of E and $u \in M$. There exist a local section σ for p, $i \in I$ and $\epsilon > 0$ such that $u \in \mathcal{T}^i_\epsilon(\sigma)$ and $\mathcal{T}^i_\epsilon(\sigma) \subset M$. Let $R := Dom\, \sigma$. Since R is an open subset of B containing $p(u)$ and B is by hypothesis a regular space, there exists an open neighborhood $S \subset B$ of $p(u)$ such that

$$\overline{S} \subset R. \tag{1}$$

Let $\delta \in \mathbb{R}$ be such that $\delta > 0$ and $d_i(u, \sigma(p(u))) < \delta < \epsilon$. If $\sigma' := \sigma \restriction_S$, then $\mathcal{T}_\delta^i(\sigma')$ is an open neighborhood of u. To show that $\overline{\mathcal{T}_\delta{}^i(\sigma')} \subset \mathcal{T}_\epsilon^i(\sigma)$ we give the following indirect argument: suppose that $z \notin \mathcal{T}_\epsilon^i(\sigma)$, then either $d_i(z, \sigma(p(z))) \geq \epsilon$ or $p(z) \notin R$. If $d_i(z, \sigma(p(z))) \geq \epsilon$ we choose $\xi > 0$ with $\xi \leq \dfrac{d_i(z, \sigma(p(z))) - \delta}{2}$ and a local section τ such that $z \in \mathcal{T}_\xi^i(\tau)$. Then

$$\begin{aligned} d_i(z, \sigma(p(z))) &\leq d_i(z, \tau(p(z))) + d_i(\tau(p(z)), \sigma(p(z))) \\ &< \xi + d_i(\tau(p(z)), \sigma(p(z))). \end{aligned}$$

Therefore

$$\begin{aligned} d_i(\tau(p(z)), \sigma(p(z))) &> d_i(z, \sigma(p(z))) - \xi \\ &\geq d_i(z, \sigma(p(z))) - \frac{d_i(z, \sigma(p(z))) - \delta}{2} \\ &= \frac{d_i(z, \sigma(p(z))) + \delta}{2} \\ &= \frac{d_i(z, \sigma(p(z))) - \delta}{2} + \delta \\ &\geq \delta + \xi. \end{aligned}$$

On the other hand, by assumption the function

$$t \longmapsto d_i(\sigma(t), \tau(t)) : Dom\,\sigma \,\cap\, Dom\,\tau \longrightarrow \mathbb{R}$$

is continuous, thus there exists an open neighborhood P of $p(z)$ such that if $s \in P$, then

$$d_i(\tau(s), \sigma(s)) > \delta + \xi. \tag{2}$$

Let $\tau' = \tau \restriction_P$. The tubes $\mathcal{T}_\xi^i(\tau')$ and $\mathcal{T}_\delta^i(\sigma)$ are disjoint since $y \in \mathcal{T}_\xi^i(\tau') \cap \mathcal{T}_\delta^i(\sigma)$ implies $p(y) \in Dom\,\tau' = P$ and

$$\begin{aligned} d_i(\tau(p(y)), \sigma(p(y))) &= d_i(\tau'(p(y)), \sigma(p(y))) \\ &\leq d_i(\tau'(p(y)), y) + d_i(y, \sigma(p(y))) \\ &< \xi + \delta \end{aligned}$$

that contradicts (2). Taking into account that $\mathcal{T}_\xi^i(\tau')$ is a neighborhood of z, it follows that $z \notin \overline{\mathcal{T}_\delta^i(\sigma')}$. If $p(z) \notin R$, then, from (1), $p(z) \notin \overline{S}$. Using again the regularity of B, we find two disjoint open sets Q_1 and Q_2 in B such that $p(z) \in Q_1$ and $\overline{S} \subset Q_2$. Let ρ be a local section with $Dom\,\rho \subset Q_1$ such that $z \in \mathcal{T}_\delta^i(\rho)$. Since $Dom\,\rho \,\cap\, Dom\,\sigma' = \varnothing$, then $\mathcal{T}_\delta^i(\rho) \cap \mathcal{T}_\delta^i(\sigma') = \varnothing$ and consequently $z \notin \overline{\mathcal{T}_\delta^i(\sigma')}$. This proves the theorem. □

If there are two sections σ and τ in $\Gamma_Q(p)$ and $i \in I$ such that $\varphi_{\sigma\tau} : Q \longrightarrow \mathbb{R}$, $t \longmapsto d_i(\sigma(t), \tau(t))$ is not continuous, then the space E could fail to be regular even if B is regular. In Example 3.13 we have that $B = \mathbb{R}$ is a regular space, $K = \{(x, 1) : x \geq 0\}$ being the complement of $\mathcal{T}_{\frac{1}{2}}(\sigma_2)$ is a closed subset of E, $(0, -1) \notin K$ and if V and W are open subsets of E such that $K \subset V$ and $(0, -1) \in W$, then there exist $\epsilon,\ \xi,\ \delta > 0$ and sections $\sigma,\ \tau$ which are defined in the interval $(-\delta, \delta)$ such that $\mathcal{T}_\epsilon(\sigma) \subset V$ and $\mathcal{T}_\xi(\tau) \subset W$.

Then the points $(x,0)$ with $-\delta < x < 0$ belong to $V \cap W$ and therefore V and W are not disjoints.

In the particular case of sheaves of sets we have the following result.

Corollary 3.16. *Let (E,p,B) be a sheaf of sets and suppose that B is a regular space. The space E is a regular space if and only if the functions $t \longmapsto d(\sigma(t),\tau(t)) : Q \longrightarrow \mathbb{R}$ are continuous, for each open set $Q \subset B$ and each pair $\sigma,\ \tau \in \Gamma_Q(p)$, d being the discrete metric on each fiber.*

Proof. It remains to prove that if E is a regular space the functions $t \longmapsto d(\sigma(t),\tau(t)) : Q \longrightarrow \mathbb{R}$ are lower semicontinuous, for each open set $Q \subset B$ and each pair $\sigma,\ \tau \in \Gamma_Q(p)$. To this end, let $t \in B$, $a > 0$ and suppose that $d(\sigma(t),\tau(t)) > a$, then $a < 1$ and $\sigma(t) \neq \tau(t)$. Since $Ran\,\sigma$ is an open neighborhood of $\sigma(t)$ there exist an open neighborhood R of t and local sections σ_1 and τ_1 such that $\sigma(r) = \sigma_1(r)$ for each $r \in R$, $\tau(r) = \tau_1(r)$ for each $r \in R$, $\overline{Ran\,\sigma_1} \subset Ran\,\sigma$ and $Ran\,\tau_1 \cap \overline{Ran\,\sigma_1} = \varnothing$, the last identity holds since $\overline{Ran\,\sigma_1}$ is a closed set and $\tau(t) \notin Ran\,\sigma$ implies $\tau(t) \notin \overline{Ran\,\sigma_1}$. Then $d(\sigma(r),\tau(r)) = 1 > a$, for each $r \in R$. □

Recall that a topological space X is *completely regular* if for every closed subset K of X and every $x_0 \in X \setminus K$ there is a continuous function $f : X \longrightarrow \mathbb{R}$ such that $f(K) = 0$ and $f(x_0) = 1$.

From Lemma 3.2 it follows:

Proposition 3.17. *Let (E,p,B) be a uniform bundle. If E is completely regular and there exists a global section for p, then B is also completely regular.*

The following result plays a crucial role in establishing the upcoming Theorem 3.19 on complete regularity.

Lemma 3.18. *Let (E,p,B) be a uniform bundle and $(d_i)_{i\in I}$ a family of pseudometrics that induces the uniformity of E. Let $i \in I$, Q be an open subset of B and $\sigma \in \Gamma_Q(p)$ be a fixed local section for p. The function $\varphi : Q \longrightarrow \mathbb{R}$ given by $\varphi(t) = d_i(\sigma(t),\tau(t))$ is continuous for each $\tau \in \Gamma_Q(p)$ if and only if the function $\psi : p^{-1}(Q) \longrightarrow \mathbb{R}$, defined by $\psi(x) = d_i(x,\sigma(p(x)))$ is continuous.*

Proof. Let $i \in I$, Q be an open set of B, $\sigma \in \Gamma_Q(p)$ be a fixed local section for p and suppose that for each $\tau \in \Gamma_Q(p)$ the function φ is continuous. The function ψ is upper semicontinuous, indeed, if $a > 0$, then $\{x \in p^{-1}(Q) : d_i(x,\sigma(p(x)) < a\} = \mathcal{T}^i_a(\sigma)$ is an open set. To see that ψ is lower semicontinuous, let $a \in \mathbb{R}$ with $a > 0$ and let $u \in p^{-1}(Q)$ such that $d_i(u,\sigma(p(u)) > a$. Choose $b \in \mathbb{R}$ such that $d_i(u,\sigma(p(u))) > b > a$ and let $\delta = \dfrac{b-a}{2}$. There exists an open neighborhood P of $p(u)$ in B and a local section τ with domain P such that $d_i(u,\tau(p(u))) < \delta$. Therefore

$$\begin{aligned} d_i(u,\sigma(p(u))) &\le d_i(u,\tau(p(u))) + d_i(\sigma(p(u)),\tau(p(u))) \\ &< \delta + d_i(\sigma(p(u)),\tau(p(u))), \end{aligned}$$

thus $d_i(\sigma(p(u)),\tau(p(u))) > b-\delta = \dfrac{b+a}{2}$. Let S be an open neighborhood of $p(u)$ contained in $Q\cap P$ such that if $t\in S$, then $d_i(\sigma(t),\tau(t)) > \dfrac{b+a}{2}$. The existence of such an S is secured by the lower semicontinuity of φ. For $y\in \mathcal{T}^i_\delta(\tau\restriction_S)$ one has

$$\begin{aligned} d_i(y,\sigma(p(y))) &\geq d_i(\sigma(p(y)),\tau(p(y))) - d_i(y,\tau(p(y))) \\ &> \frac{b+a}{2} - \frac{b-a}{2} \\ &= a. \end{aligned}$$

Then ψ is lower semicontinuous.

Conversely, suppose now that the function ψ is continuous and let τ be a section in $\Gamma_Q(p)$. Since (E,p,B) is a uniform bundle, φ is upper semicontinuous. It remains to show that φ is lower semicontinuous. Let $a>0$ and $t_0\in Q$ such that $d_i(\sigma(t_0),\tau(t_0)) > a$. Since the function $u\longmapsto d_i(u,\sigma(p(u))) : p^{-1}(Q)\longrightarrow \mathbb{R}$ is continuous at $\tau(t_0)$ there exists an open neighborhood M of $\tau(t_0)$ in $p^{-1}(Q)$ such that $d_i(u,\sigma(p(u))) > a$ for each $u\in M$. Since τ is continuous there exists an open neighborhood R of t_0 such that $\tau(t)\in M$ if $t\in R$. Therefore $d_i(\tau(t),\sigma(t)) > a$ for each $t\in R$.

This establishes the lower semicontinuity of φ and proves the lemma. □

Theorem 3.19. *Let (E,p,B) be a uniform bundle whose uniformity is given by the directed family $(d_i)_{i\in I}$ of pseudometrics. If B is a completely regular space and if the functions $\varphi : Q\longrightarrow\mathbb{R}$ defined by $\varphi(t) = d_i(\sigma(t),\tau(t))$ are continuous, for each open set $Q\subset B$, each pair σ, $\tau\in\Gamma_Q(p)$ of local sections and each $i\in I$, then E is also a completely regular space.*

Proof. Let K be a closed subset of E and let z_0 be a point of E such that $z_0\notin K$. There exist $\epsilon>0$, $i\in I$ and a local section σ such that $z_0\in\mathcal{T}^i_\epsilon(\sigma)$ and $\mathcal{T}^i_\epsilon(\sigma)\subset E\smallsetminus K$. Let $Q := Dom\,\sigma$ and take $a = \dfrac{\epsilon}{\epsilon - d_i(z_0,\sigma(p(z_0)))}$. Since $B\smallsetminus Q$ is a closed set, $p(z_0)\notin B\smallsetminus Q$ and B is completely regular, there exists a continuous function $f : B\longrightarrow [0,a]$ such that $f(p(z_0)) = a$ and $f(B\smallsetminus Q) = 0$. Consider the function $g : [0,+\infty[\longrightarrow [0,1]$ defined by

$$g(t) = \begin{cases} \dfrac{\epsilon - t}{\epsilon} & \text{if } t<\epsilon \\ 0 & \text{if } t\geq\epsilon \end{cases}$$

and the function $h : p^{-1}(Q)\longrightarrow\mathbb{R}$ given by $h(u) = d_i(u,\sigma(p(u)))$, whose continuity was established in Lemma 3.18. Define $\zeta : E\longrightarrow [0,1]$ by

$$\zeta(u) = \begin{cases} g(h(u))f(p(u)) & \text{when } u\in p^{-1}(Q) \\ 0 & \text{otherwise.} \end{cases}$$

The function ζ is continuous in $p^{-1}(Q)\cup(E\smallsetminus p^{-1}(Q))^\circ$. It remains to show the continuity of ζ at the points of the boundary of $p^{-1}(Q)$.

Let $y\in\overline{p^{-1}(Q)}\smallsetminus p^{-1}(Q)$. Taking into account that $f(p(y)) = 0$, since f is continuous, for a given $\delta > 0$ there exists an open neighborhood P of $p(y)$ such that $|f(t)| < \delta$ for each $t \in P$. Moreover, from the continuity of p we have that $p^{-1}(P)$ is an open neighborhood of y. It remains to see that $|\zeta(w)| < \delta$ for every $w \in p^{-1}(P)$. For such a w consider two cases: $w \notin p^{-1}(Q)$ and $w \in p^{-1}(Q)$. If $w \notin p^{-1}(Q)$, then $\zeta(w) = 0$ and if $w \in p^{-1}(Q)$ and $h(w) \geq \epsilon$, then $g(h(w)) = 0$ and $\zeta(w) = 0$, but if $h(w) < \epsilon$, then $g(h(w)) = \dfrac{\epsilon - h(w)}{\epsilon} \leq 1$ and thus $|\zeta(w)| = |g(h(w))f(p(w))| \leq |f(p(w))| < \delta$. It follows that $\zeta : E \longrightarrow [0,1]$ is continuous, $\zeta(z_0) = 1$ and $\zeta(K) = 0$ since $\zeta(E \setminus \mathcal{T}^i_\epsilon(\sigma)) = 0$. Hence E is also a completely regular space. □

In the case of sheaves of sets we have the following corollary.

Corollary 3.20. *Let (E, p, B) be a sheaf of sets and suppose that B is a completely regular space. The space E is a completely regular space if and only if the functions $t \longmapsto d(\sigma(t), \tau(t)) : Q \longrightarrow \mathbb{R}$ are continuous for each open set $Q \subset B$ and each pair $\sigma,\ \tau \in \Gamma_Q(p)$, d being the discrete metric on each fiber.*

Proof. Since every completely regular space is a regular space, the corollary follows from Theorem 3.19 and Corollary 3.16. □

Remark 3.21. From Proposition 3.6 and Proposition 3.17 follows that if E is a Tychonoff space, that is, completely regular and T_1, and if there exists a global section, then B is also a Tychonoff space. From Proposition 3.7 and Theorem 3.19 follows that if E_t is a T_1 space for each $t \in B$, if the functions $t \longmapsto d_i(\sigma(t), \tau(t)) : Q \longrightarrow \mathbb{R}$ are continuous for each $i \in I$ and each pair $\sigma,\ \tau \in \Gamma_Q(p)$ of local sections and if B is a Tychonoff space, then E is also a Tychonoff space.

Recall that a topological space X is *normal* if for each pair $K,\ L$ of closed subsets in X there are disjoint open subsets V and W of X such that $K \subset V$ and $L \subset W$.

Proposition 3.22. *Let (E, p, B) be a uniform bundle whose uniformity is given by the directed family $(d_i)_{i \in I}$ of pseudometrics. If E is a normal space and if there exists a global section for p, then B is a normal space.*

Proof. Let $\sigma : B \longrightarrow E$ be a global section and let K and L be two disjoint closed subsets of B, then $p^{-1}(K)$ and $p^{-1}(L)$ are closed subsets of E without common points, thus there exist open and disjoint subsets V and W of E such that $p^{-1}(K) \subset V$ and $p^{-1}(L) \subset W$. It follows that the sets $\sigma^{-1}(V)$ and $\sigma^{-1}(W)$ are open and disjoint subsets of B, $K \subset \sigma^{-1}(V)$ and $L \subset \sigma^{-1}(W)$. □

Remark 3.23. In the absence of a global section in the uniform bundle (E, p, B), as assumed in Proposition 3.9, one may suppose that given two distinct points of the base space, there exists a local section whose domain contains them, the conclusion, that the base space satisfies the Hausdorff axiom, still holds.

In Propositions 3.14 and 3.17 such hypothesis can be replaced by the assumption that given a point and a closed subset of B that does not contain the point, there exists a local section whose domain contains both the point and the closed subset, one can still conclude the regularity and completely regularity respectively. The arguments rest on the fact that a local section is a homeomorphism from its domain onto its range, as stated in Lemma 3.2, and that the subspaces of a topological space inherit the properties of being Hausdorff, regular or completely regular. Regarding Proposition 3.22, the authors do not know if a similar or somewhat weaker assumption could replace the condition of existence of a global section and still secures the normality of the base space.

The normality of B by itself does not guarantee the normality of E. In Example 3.13 the space $B = \mathbb{R}$ is a normal space, $K = \{(x, 1) : x \geq 0\}$ and $L = \{(x, -1) : x \geq 0\}$ are closed subsets of E without common points, and if V and W are open subsets of E such that $K \subset V$ and $L \subset W$, then there exist points $(x, 0)$ with $x < 0$ belonging to $V \cap W$.
The next example exibits a uniform bundle (E, p, B) whose uniformity for p is given by the directed family $(d_i)_{i \in I}$ of pseudometrics, such that B is a normal space, the functions $\varphi : Q \longrightarrow \mathbb{R}$ given by $\varphi(t) = d_i(\sigma(t), \tau(t))$ are continuous for each $i \in I$, each open set $Q \subset B$ and each pair $\sigma,\ \tau \in \Gamma_Q(p)$ of local section, but E fails to be a normal space. We cite first the following result of F. B. Jones [2] required in the example.

Lemma 3.24. *If X contains a dense set D and a closed, relatively discrete subspace S such that $|S| \geq 2^{|D|}$, then X is not normal.*

Example 3.25. Let $E = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$ be the Moore plane. Here the basic neighborhoods of each point $(x, y) \in E$ with $y > 0$, are the intersections of E with the open disks in $\mathbb{R}^2$ that have the center at (x, y) and if $(x, y) \in E$ and $y = 0$, its basic neighborhoods are the sets $\{(x, y)\} \cup A$, where A is an open disk in the upper half plane, tangent to the x-axis at (x, y). The space E is completely regular (hence uniformizable [3], Corollary 17, page 188, [5], Theorem 38.2, page 256 and T_1 [5], Examples 14.5, page 93.)

Let $(d_i)_{i \in I}$ be the caliber of E, that is, the collection of all finite uniformly continuous pseudometrics of E. Consider the topological space $B = \{t\}$ and the map $p : E \longrightarrow B$ defined by $p(x, y) = t$ for each $(x, y) \in E$. Every local section σ for p can be identified with the point $\sigma(t)$ in E and the triple (E, p, B) is a uniform bundle. For each $i \in I$ and each pair $\sigma,\ \tau \in \Gamma_Q(p)$, the map $\varphi : Q \longrightarrow \mathbb{R}$ defined by $\varphi(t) = d_i(\sigma(t), \tau(t))$ is continuous. On the other hand, from Lemma 3.24, the space E is not normal, because if $S = \{(x, 0) : x \in \mathbb{R}\}$ and $D = \{(x, y) \in E : x,\ y \in \mathbb{Q}\}$, then S turns out to be a closed, relatively discrete subspace of E, D is dense in E and $|S| \geq 2^{|D|}$ on account of D being contable and $|S| = \mathfrak{c}$.

Remark 3.26. Let (E, p, B) be a sheaf of sets and σ and τ be local sections for p defined in an open set Q of B. The upper semicontinuity of the function $\varphi : Q \longrightarrow \mathbb{R}$ given by $\varphi(t) = d(\sigma(t), \tau(t))$, d being the discrete metric on each fiber, amounts to the assertion that the set $\{t \in Q : \sigma(t) = \tau(t)\}$ is open and the additional hypothesis that φ is lower semicontinuous amounts to the assertion that $\{t \in Q : \sigma(t) \neq \tau(t)\}$ is also open. Under the assumption that φ is continuous, if σ and τ agree at a point t they agree on the whole connected component of t in Q.

Example 3.27. (The sheaf of germs of holomorphic functions.) Let $\mathbb{C}$ be the field of complex numbers endowed with the usual topology, let $Q \subset \mathbb{C}$ be an open set, $f : Q \longrightarrow \mathbb{C}$ and $z \in Q$. The function f is holomorphic (or regular) at the point z provided that f is complex differenciable in an open disk $D(z, \epsilon) \subset Q$, with center z and radius $\epsilon > 0$.

For every complex number z denote by A_z the set of all holomorphic functions at z. In A_z define the equivalence relation R_z by $f\ R_z\ g$ if and only if f and g coincide in an open disk with center z. The class $[f]_z$ of f module R_z is called the germ of the holomorphic function f at z. The set of germs of holomorphic functions at z, that is, the quotient set $E_z = A_z/R_z$ is identified to the set of all sequences $(a_n)_{n\in\mathbb{N}}$ of complex numbers such that $\limsup |a_n|^{\frac{1}{n}} < \infty$, indeed, every $[f]_z \in E_z$ determines the sequence $\left(\dfrac{f^{(n)}(z)}{n!}\right)_n$ with that property, and conversely every such a sequence determines the class module R_z of the holomorphic function defined by the power series $\sum_{n=0}^{\infty} a_n(w-z)^n$ in the open disk with center z and radius $\dfrac{1}{\limsup |a_n|^{\frac{1}{n}}}$.

Let $\widehat{E} = \coprod_{z\in\mathbb{C}} E_z$ be the disjoint union of the family $\{E_z : z \in \mathbb{C}\}$ and let $\widehat{p} : \widehat{E} \longrightarrow \mathbb{C}$ be the function defined by $\widehat{p}(z, [f]_z) = z$ and consider each fiber E_z equipped with the discrete metric.

For every holomorphic function f in an open set Q define $\widehat{f} : Q \longrightarrow \widehat{E}$ by $\widehat{f}(z) = (z, [f]_z)$.

The function $\widehat{f}$ is a local selection for $\widehat{p}$ and if $\widehat{\Sigma} = \{\widehat{f} : f$ is holomorphic in some open set of $\mathbb{C}\}$ and d denotes the pseudometric whose restriction to each fiber is the discrete metric, then, by Theorem 2.3, the triple $(\widehat{E}, \widehat{p}, \mathbb{C})$ is a sheaf of sets and every element of $\widehat{\Sigma}$ is a local section for $\widehat{p}$. Actually, the set of all local sections of this sheaf coincides with the set $\widehat{\Sigma}$, in fact, let $\sigma : Q \longrightarrow \widehat{E}$ be a local section for $\widehat{p}$ and for each $z \in Q$ let f_z be a holomorphic function at z such that $[f_z]_z = \sigma(z)$. Consider the map $f : Q \longrightarrow \mathbb{C}$ defined by $f(z) = f_z(z)$. If $z \in Q$ and g is a holomorphic function at z, the relation $f_z\ R_z\ g$ implies that f_z and g coincide in an open disk with center z, in particular $f_z(z) = g(z)$, hence f is a well defined function. Since f_z is holomorphic at z, there exist $0 < \epsilon < 1$ and a power series $\sum_{n=0}^{\infty} a_n(w-z)^n$ convergent in the disk $D(z, \epsilon)$ such that $f_z(w) = \sum_{n=0}^{\infty} a_n(w-z)^n$ for every $w \in D(z, \epsilon)$. Since σ is a continuous function, there exists $0 < \delta < \epsilon$ such that if $w \in D(z, \delta)$, then $\sigma(w) = [f_w]_w \in \mathcal{T}_\epsilon(\widehat{f_z})$, that is, $d([f_w]_w, [f_z]_w) < \epsilon < 1$, therefore $d([f_w]_w, [f_z]_w) = 0$, thus $[f_w]_w = [f_z]_w$ and $f(w) = f_w(w) = f_z(w) = \sum_{n=0}^{\infty} a_n(w-z)^n$. It follows that f is holomorphic at z. From this argument it also follows that f and f_z coincide in an open disk with center z, then $[f]_z = [f_z]_z$ and therefore $\widehat{f} = \sigma$. Suppose that $\widehat{f},\ \widehat{g} \in \widehat{\Sigma}$, then the map $\varphi : Dom\,\widehat{f} \cap Dom\,\widehat{g} \longrightarrow \mathbb{R}$, $z \longmapsto d([f]_z, [g]_z)$ is continuous.

To obtain the lower semicontinuity of this map observe that $[f]_z \neq [g]_z$ implies that there is an n such that $f^{(n)}(z) \neq g^{(n)}(z)$, then there is an open disk S with center z such that $f^{(n)}(w) \neq g^{(n)}(w)$ and therefore $[f]_w \neq [g]_w$ for each $w \in S$. We conclude that $z \longmapsto d([f]_z, [g]_z)$ is also lower semicontinuous.

Corollaries 3.11, 3.16 and 3.20 guarantee that the space $\widehat{E}$ is Hausdorff, regular and completely regular and consequently a Tychonoff space as it is well known in the literature.

The following example shows a sheaf of sets where local sections σ, τ can be found, such that the function $t \longmapsto d(\sigma(t), \tau(t))$ fails to be continuous.

Example 3.28. For every complex number z denote by C_z the set of all continuous complex valued functions defined in some open set of the complex plane containing z. In C_z define the equivalence relation R_z by $f \; R_z \; g$ if and only if f and g coincide in an open disk with center z and denote by $[f]_z$ the class of f module R_z. Let $E_z = \{[f]_z : f \text{ is continuous at } z\}$, $\widehat{E} = \coprod_{z \in \mathbb{C}} E_z$ and $\widehat{p} : \widehat{E} \longrightarrow \mathbb{C}$ the function defined by $\widehat{p}(z, [f]_z) = z$. Each fiber $\widehat{E}_z$ is considered to be endowed with the discrete metric d. For every continuous function f in the open set Q define $\widehat{f} : Q \longrightarrow \widehat{E}$ by $\widehat{f}(z) = (z, [f]_z)$. The function $\widehat{f}$ is a local selection for $\widehat{p}$ and if

$$\widehat{\Sigma} = \{\widehat{f} : f \text{ is continuous in some open set of } \mathbb{C}\},$$

then the triple $(\widehat{E}, \widehat{p}, \mathbb{C})$ is a uniform bundle, even more, it is a sheaf of sets in which every $\widehat{f} \in \widehat{\Sigma}$ is a local section. We claim that each local section of this sheaf belongs to $\widehat{\Sigma}$, to this effect, let $\sigma : Q \longrightarrow \widehat{E}$ be a local section for $\widehat{p}$ and for each $z \in Q$, let f_z be a continuous function defined in an open set containing z such that $\sigma(z) = [f_z]_z$. It is apparent that the map $f : Q \longrightarrow \mathbb{C}$ defined by $f(z) = f_z(z)$ is a well defined function and once the continuity of f has been established, the relation $\sigma = \widehat{f}$ and the claim will follow. Consider $z \in Q$ and $\epsilon > 0$. Taking into account that f_z is continuous at z, that σ and $\widehat{f_z}$ are local sections and that $\sigma(z) = \widehat{f_z}(z)$, there exists $\delta > 0$ such that $f_z(w) \in D(f_z(z), \epsilon)$ and $\sigma(w) = \widehat{f_z}(w)$ for every $w \in D(z, \delta)$. Then $[f_w]_w = [f_z]_w$ for each $w \in D(z, \delta)$, in particular $f_w(w) = f_z(w)$ for each $w \in D(z, \delta)$, thus $f(w) = f_w(w) \in D(f_z(z), \epsilon)$ for each $w \in D(z, \delta)$ and therefore f is continuous. Consider the continuous functions f, $g : \mathbb{C} \longrightarrow \mathbb{C}$ defined by

$$f(z) = \begin{cases} z & \text{if } |z| \leq 1 \\ \frac{z}{|z|} & \text{if } |z| > 1 \end{cases}$$

and $g(z) = z$. If $|z| < 1$, then $[f]_z = [g]_z$ and if $|z| \geq 1$, then $[f]_z \neq [g]_z$. Thus

$$d([f]_z, [g]_z) = \begin{cases} 0 & \text{if } |z| < 1 \\ 1 & \text{if } |z| \geq 1 \end{cases}$$

and the function $z \longmapsto d([f]_z, [g]_z) : \mathbb{C} \longrightarrow \mathbb{R}$ is not continuous.

Corollaries 3.11, 3.16 and 3.20 back up the assertion that the space $\widehat{E}$ is neither

Hausdorff nor regular nor completely regular and then that it is not a Tychonoff space either.

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