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Relative Collectionwise Normality

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Abstract. In this paper we study properties of relative collectionwise normality type based on relative properties of normality type introduced by Arhangel'skii and Genedi.
Theorem Suppose Y is strongly regular in the space X. If Y is paracompact in X then Y is collectionwise normal in X.
Example A T_2 space X having a subspace which is $1-$ paracompact in X but not collectionwise normal in X.
Theorem Suppose that Y is s- regular in the space X. If Y is metacompact in X and strongly collectionwise normal in X then Y is paracompact in X.

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1. Introduction

In this paper properties of relative collectionwise normality type based on relative properties of normality type introduced in [2] and [3] are studied. Our study focusses on the following well known theorems and relative properties of paracompactness type introduced in [1] and [4].

Theorem 1.1 (Bing). *Every paracompact space is collectionwise normal.*

Theorem 1.2 (Michael-Nagami). *Every metacompact collectionwise normal space is paracompact.*

A theorem concerning the relative properties of a subspace Y in a space X becomes a theorem about the corresponding global properties of X by letting $Y = X$. It is not surprising when the proof of a result concerning relative properties is a straight forward modification of the usual proof of the corresponding global result. For example we show that if Y is strongly star normal in X then Y is strongly collectionwise normal in X, Theorem 3.7. The proof is the natural relative version of the standard proof that T_2 paracompact spaces are collectionwise normal using the fully normal characterization of paracompactness. However this is not always true. For example there exist a good number of non-equivalent relative properties of paracompactness type, see [1], [2], [6], [7] and [8]. Some of these properties are preserved by closed maps (cp-paracompact in X, [7]) and some are not (paracompact in X from outside, [7]). Some imply that the subspace Y is paracompact (strongly star normal in X, [4]) while others do not (1- paracompact in X, [6]). We give an example of a T_2 space having a subspace which is $1-$ paracompact in X but not collectionwise normal in X, Example 5.4. Thus to obtain an analog of Bing′s Theorem for subspaces Y paracompact in X it is necessary to assume that Y satisfies relative separation properties not implied by the space X being a T_2 space and Y being paracompact in X. If Y is paracompact in X and strongly regular in X then Y is collectionwise normal in X, Theorem 3.3.

We give several relative versions of the Michael-Nagami Theorem. If Y is $s-$ regular in X, metacompact in X and strongly collectionwise normal in X then Y is paracompact in X, Theorem 4.4. If Y is closed, $s-$ regular in X, collectionwise normal in X and metacompact then Y is paracompact in X, Corollary 4.5.

Throughout this paper all spaces are assumed to be Hausdorff. Suppose X is a space and Y a subspace of X. When a set U is said to be open, we mean open with respect to the topology on X even if U happens to be a subset of Y. For a set X, $x \in X$, a subset A of X and a collection $\mathcal{U}$ of subsets of X, $(\mathcal{U})_x = \{U \in \mathcal{U} : x \in U\}$, $(\mathcal{U})_A = \{U \in \mathcal{U} : A \cap U \neq \phi\}$, $st(x,\mathcal{U}) = \cup(\mathcal{U})_x$ and $st(A,\mathcal{U}) = \cup(\mathcal{U})_A$.

2. Definitions and Lemma

Suppose Y is a subset of the space X. The subset Y is 1. *regular in* X, 2. *super regular in* X, 3. *strongly regular in* X, 4. *s- regular in* X, 5. *normal in* X , 6. *s- normal in* X, 7. *strongly normal in* X provided

1. for each $x \in Y$ and every subset F of $X\backslash\{x\}$ closed in X there are disjoint open sets U and V such that $x \in U$ and $F \cap Y \subseteq V$ [3].
2. for each $x \in Y$ and every subset F of $X\backslash\{x\}$ closed in X there are disjoint open sets U and V such that $x \in U$ and $F \subseteq V$ [3].
3. for each $x \in X$ and every subset F of $X\backslash\{x\}$ closed in X there are disjoint open sets U and V such that $x \in U$ and $F \cap Y \subseteq V$ [3].
4. Y is both super regular and strongly regular in X.
5. for each pair E and F of disjoint closed subsets of X there are disjoint open sets U and V such that $E \cap Y \subseteq U$ and $F \cap Y \subseteq V$ [3].
6. for each pair, E and F of disjoint closed subsets of X, there are disjoint open subsets of X, U and V such that $E \subseteq U$ and $F \cap Y \subseteq V$ [10].
7. for each pair E and F of disjoint closed (in Y) subsets of Y there are disjoint open sets U and V such that $E \subseteq U$ and$F \subseteq V$ [2].

Suppose Y is a subset of a space X. If Y is super regular or strongly regular in X ($s-$ normal or strongly normal in X) then Y is regular (normal) in X. However in general there is no implication between these two stronger conditions. Also if Y is normal (s- normal) in X then Y is regular (s- regular) in X. If X is a regular (normal) space then every subspace of X is $s-$ regular (s- normal but not necessarily strongly normal) in X. The subspace Y can be strongly normal in X without being strongly regular in X.

Suppose Y is a subset of a space X. A collection $\mathcal{U}$ is said to be locally finite on Y provided for every $y \in Y$ there is an open V containing y such that $(\mathcal{U})_V$ is finite. A collection $\mathcal{F}$ of closed subsets of X is said to be *weakly closure reserving with respect to* Y provided for all $\mathcal{F}' \subseteq (\mathcal{F})_Y$, $(\cup\mathcal{F}') \cap Y = (\overline{\cup\mathcal{F}'}) \cap Y$, [7]. The following lemmas from [7] are frequently used when working with collections that are locally finite with respect to a subset Y of a space X.

Lemma 2.1. *Suppose $Y \subseteq X$ and $\mathcal{U}$ is a collection of open subsets of the space X locally finite on Y. Then the collection $\{\overline{U} : U \in \mathcal{U}\}$ is weakly closure preserving with respect to Y and locally finite on Y.*

Lemma 2.2. *Suppose that $Y \subseteq X$ and $\mathcal{F}$ is a collection of closed subsets of the space X weakly closure preserving with respect to Y.*
1. If $B \subseteq X$ is closed then $\{F \cap B : F \in \mathcal{F}\}$ is weakly closure preserving with respect to Y.
2. If $A \subseteq Y$ then $A \subseteq X\backslash\overline{\cup(\mathcal{F}\backslash(\mathcal{F})_A)}$. In particular, for all $y \in Y$, $y \notin \overline{\cup\{F \in \mathcal{F} : y \notin F\}}$.

For a space X and $Y \subseteq X$, a collection $\mathcal{A}$ of subsets of the space X is said to be *discrete with respect to* Y provided for all $x \in Y$ there is an open neighborhood U of x that intersects at most one member of $\mathcal{A}$. We say that Y is *collectionwise normal in* a space X provided for every discrete collection $\mathcal{F}$ of closed subsets of X, there is a collection of open subsets of X, $\mathcal{U} = \{U(F) : F \in \mathcal{F}\}$ discrete with respect to Y such that for all $F \in \mathcal{F}$, $F \cap Y \subseteq U(F) \subseteq X\backslash\cup(\mathcal{F}\backslash\{F\})$. Notice that a collection of subsets of a space X which is discrete with respect to a subspace Y of X need not be pairwise disjoint. However in the case of collectionwise normality in X this is not a problem as seen in the following lemma.

Lemma 2.3. *Suppose $Y \subseteq X$ and $\mathcal{U}$ is a collection of open subsets of the space X discrete with respect to Y. For each $U \in \mathcal{U}$ let $V(U) = U\backslash\overline{\cup(\mathcal{U}\backslash\{U\})}$. Then the collection $\{V(U) : U \in \mathcal{U}\}$ is a pairwise disjoint collection of open subsets of X discrete with respect to Y such that for all $U \in \mathcal{U}$, $U \cap Y = V(U) \cap Y$.*

Theorem 2.4. *If Y is collectionwise normal in the space X then Y is normal in X.*

We say that a subspace Y is *strongly collectionwise normal in* the space X provided for every collection $\mathcal{F}$ of closed subsets of X which is discrete with respect to Y there is a collection of open subsets of X, $\mathcal{U} = \{U(F) : F \in \mathcal{F}\}$ discrete with respect to Y such that for all $F \in \mathcal{F}$, $F \cap Y \subseteq U(F) \subseteq$

$X\backslash\overline{\cup(\mathcal{F}\backslash\{F\})}$. By Lemma 2.3 the members of $\mathcal{U}$ can be taken to be pairwise disjoint and discrete with respect to Y if we choose. Notice that if Y is a closed subset of X and $\mathcal{F}$ is a collection of closed subsets of X which is discrete with respect to Y then $\{F \cap Y : F \in \mathcal{F}\}$ is a discrete collection of closed subsets of X.

Theorem 2.5. *If Y is strongly collectionwise normal in the space X then Y is strongly normal in X and a collectionwise normal subspace of X. If Y is a closed subset of X then Y is strongly collectionwise normal in X if and only if Y is collectionwise normal in X.*

A closed collectionwise normal subspace of a space X need not be collectionwise normal in X, Example 5.2.

3. Relative paracompact implies relative collectionwise normality

The following definitions of the most natural properties of relative paracompactness type are from [2]. The subspace Y is said to be $1-$ *paracompact in* X provided every open cover of X has an open refinement locally finite on Y. The subspace Y is *paracompact in* X provided every open cover of X has an open partial refinement covering Y and locally finite on Y. In [6] it is observed that if Y is strongly regular in X and paracompact in X then Y is normal in X. If Y is closed and paracompact in X then Y is normal in X. However a closed subset of a regular space X can be paracompact in X and not $s-$ normal in X, Example 5.3. Although it is readily seen that if Y is $1-$ paracompact in X then Y is super- regular in X it need not be strongly regular in X, Example 5.1. The following Theorem shows that $s-$ normality in X is a relative property of normality type that relates to $1-$ paracompactness in X.

Theorem 3.1. *Suppose Y is strongly regular in the space X. If Y is $1-$ paracompact in X then Y is $s-$normal in X.*

Proof. Suppose E and F are disjoint closed subsets of X. Since Y is strongly regular in X, for every $x \in E$ there are disjoint open sets $W(x)$ and $G(x)$ such that $x \in W(x)$ and $F \cap Y \subseteq G(x)$. Let $\mathcal{W} = \{W(x) : x \in E\} \cup \{X\backslash E\}$ and $\mathcal{V}$ be and open refinement of $\mathcal{W}$ locally finite on Y. For each $V \in (\mathcal{V})_E$ let $x(V) \in E$ such that $V \subseteq W(x(V))$.

Let $U = \cup(\mathcal{V})_E$ and note that since $\mathcal{V}$ is a cover of X, $E \subseteq U$. Let $O = X\backslash\overline{U}$. Suppose $x \in F \cap Y$. Since $\mathcal{V}$ is locally finite on Y, let Q be an open neighborhood of x meeting only finitely many members of $\mathcal{V}$. Let $\mathcal{V}' = \{V \in (\mathcal{V})_E : Q \cap V \neq \phi\}$ and note that $\mathcal{V}'$ is finite. If $\mathcal{V}' = \phi$ then $Q \cap U = \phi$ and so $x \in O$. Suppose $\mathcal{V}' \neq \phi$, say $\mathcal{V}' = \{V_1, V_2, .., V_n\}$. Then $Q \cap G(x(V_1)) \cap ... \cap G(x(V_n))$ is an open neighborhood of x missing U and so again $x \in O$. Therefore $F \cap Y \subseteq O$. $\square$

A space X can have a subspace which is $1-$ paracompact in X but not collectionwise normal in X, Example 5.4.This example is not regular and the subspace Y is not closed.

Theorem 3.2. *Suppose that Y is closed and paracompact in the space X. Then Y is strongly collectionwise normal in X.*

Proof. By Theorem 2.5 we need only show that Y is collectionwise normal in X. Let $\{F_\alpha : \alpha \in \Gamma\}$ be a discrete collection of closed subsets of X such that if $\alpha, \beta \in \Gamma$ with $\alpha \neq \beta$ then $F_\alpha \neq F_\beta$. For each $x \in X$, let U_x be and open neighborhood of x meeting at most one member of $\mathcal{F}$. Let $\mathcal{V}$ be and open partial refinement of $\{U_x : x \in X\}$ covering Y locally finite on Y. For each $\alpha \in \Gamma$ let $V_\alpha = \cup\{V \in \mathcal{V} : Y \cap V \cap F_\alpha \neq \phi\}$. Then $\{V_\alpha : \alpha \in \Gamma\}$ is a collection of open subsets of X locally finite on Y such that for all $\alpha \in \Gamma$, $Y \cap F_\alpha \subseteq V_\alpha \subseteq X \backslash \cup (\mathcal{F} \backslash \{F_\alpha\})$. Since Y is closed and paracompact in X it is normal in X. For all $\alpha \in \Gamma$ let G_α and W_α be disjoint open subsets of X such that $Y \cap F_\alpha \subseteq G_\alpha$ and $Y \cap (\cup(\mathcal{F} \backslash \{F_\alpha\})) \subseteq W_\alpha$.

For all $\alpha \in \Gamma$ let $H_\alpha = G_\alpha \cap V_\alpha$ and $U_\alpha = H_\alpha \backslash \overline{\cup\{H_\beta : \beta \in \Gamma \backslash \{\alpha\}\}}$. The collection $\mathcal{U} = \{U_\alpha : \alpha \in \Gamma\}$ is a pairwise disjoint collection of open subsets of X. Since for all $\alpha \in \Gamma$, $U_\alpha \subseteq V_\alpha$ the collection $\mathcal{U}$ is locally finite on Y and $U_\alpha \subseteq X \backslash \cup (\mathcal{F} \backslash \{F_\alpha\})$. Thus we need only show that $F_\alpha \cap Y \subseteq U_\alpha$. Note that for all $\alpha \in \Gamma$, $F_\alpha \cap Y \subseteq H_\alpha$ and since $H_\alpha \subseteq V_\alpha$ the collection $\{H_\alpha : \alpha \in \Gamma\}$ is also locally finite on Y. Thus by Lemma 2.1 the collection $\{\overline{H_\alpha} : \alpha \in \Gamma\}$ is weakly closure preserving with respect to Y and so for all $\alpha \in \Gamma$

$$Y \cap (\overline{\cup\{H_\beta : \beta \in \Gamma \backslash \{\alpha\}\}}) = Y \cap (\cup\{\overline{H_\beta} : \beta \in \Gamma \backslash \{\alpha\}\}).$$

Suppose $\alpha \in \Gamma$, $x \in Y \cap F_\alpha$ and $\lambda \in \Gamma \backslash \{\alpha\}$. Since $\lambda \neq \alpha$ and $x \in Y \cap F_\alpha$, $x \in Y \cap (\cup(\mathcal{F} \backslash \{F_\lambda\})) \subseteq W_\lambda$. Since $H_\lambda \subseteq G_\lambda$ and $G_\lambda \cap W_\lambda = \phi$, $x \notin \overline{H_\lambda}$. Hence $(Y \cap F_\alpha) \cap (\cup\{\overline{H_\beta} : \beta \in \Gamma \backslash \{\alpha\}\}) = \phi$ and so $Y \cap F_\alpha \subseteq U_\alpha$.

We now proceed much as in Theorem 5.1.17 of [5]. Let $F = Y \cap (\cup \mathcal{F})$ and $K = Y \backslash \cup \mathcal{U}$. Since F and K are disjoint subsets of Y closed in X and Y is normal in X there exist disjoint open sets W and W'such that $F \subseteq W$, $K \subseteq W'$.

Clearly for all $\alpha \in \Gamma$, $Y \cap F_\alpha \subseteq W \cap U_\alpha$ and the collection $\{W \cap U_\alpha : \alpha \in \Gamma\}$ is pairwise disjoint. Suppose $y \in Y$. If $\alpha \in \Gamma$ and $y \in U_\alpha$ then U_α is an open neighborhood of y meeting at most one member of $\{W \cap U_\alpha : \alpha \in \Gamma\}$, (that member being $W \cap U_\alpha$) If $y \notin U_\alpha$ for all $\alpha \in \Gamma$ then $y \in K$ and so W' is an open neighborhood of y missing all members of $\{W \cap U_\alpha : \alpha \in \Gamma\}$. Thus the collection $\{W \cap U_\alpha : \alpha \in \Gamma\}$ is a pairwise disjoint collection of open sets discrete on Y such that for all $\alpha \in \Gamma$, $Y \cap F_\alpha \subseteq W \cap U_\alpha \subseteq X \backslash \cup (\mathcal{F} \backslash \{F_\alpha\})$. □

The following is a natural relative version of Bing's Theorem. In light of Example 5.4, we need to assume that the subspace Y is relatively regular in X.

Theorem 3.3. *Suppose Y is strongly regular in the space X. If Y is paracompact in X then Y is collectionwise normal in X.*

Proof. Let $\mathcal{F} = \{F_\alpha : \alpha \in \Gamma\}$ be a discrete collection of closed subsets of X such that if $\alpha, \beta \in \Gamma$ with $\alpha \neq \beta$ then $F_\alpha \neq F_\beta$. Using the fact that Y is strongly regular in X, for each $x \in \cup\{F_\alpha : \alpha \in \Gamma\}$ let U_x be an open neighborhood of x such that $|\{\alpha \in \Gamma : U_x \cap F_\alpha \neq \phi\}| = 1$ and $|\{\alpha \in \Gamma : \overline{U_x} \cap F_\alpha \cap Y \neq \phi\}| = 1$. For each $x \in X \setminus \cup\{F_\alpha : \alpha \in \Gamma\}$, let U_x be an open neighborhood of x such that $\overline{U_x} \cap \cup\{F_\alpha \cap Y : \alpha \in \Gamma\} = \phi$. Let $\mathcal{U} = \{U_x : x \in X\}$. Since Y is paracompact in X, there is an open partial refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V}$ covers Y and $\mathcal{V}$ is locally finite on Y. Note that since $\mathcal{V}$ is a partial refinement of $\mathcal{U}$, $|\{\alpha \in \Lambda : \overline{V} \cap F_\alpha \cap Y \neq \phi\}| \leq 1$ for all $V \in \mathcal{V}$. For each $y \in Y$, let $V_y \in \mathcal{V}$ such that $y \in V_y$. For each $\alpha \in \Gamma$, let $Y_\alpha = Y \cap F_\alpha$. For each $y \in \cup\{Y_\alpha : \alpha \in \Gamma\}$, let W_y be an open neighborhood of y such that $W_y \subseteq V_y$ and $|\{V \in \mathcal{V} : W_y \cap V \neq \phi\}| < \aleph_0$. Also, for each $y \in \cup\{Y_\alpha : \alpha \in \Gamma\}$, let $O_y = W_y \setminus \cup\{\overline{V} : V \in \mathcal{V},\ W_y \cap V \neq \varnothing, \text{ and } y \notin \overline{V}\}$. For each $\alpha \in \Gamma$, let $O_\alpha = \cup\{O_y : y \in Y_\alpha\}$. Clearly, $F_\alpha \cap Y = Y_\alpha \subseteq O_\alpha \subseteq X \setminus \cup (\mathcal{F} \setminus \{F_\alpha\})$ for all $\alpha \in \Gamma$. It remains to show that $\{O_\alpha : \alpha \in \Gamma\}$ is discrete with respect to Y. To see this, let $z \in Y$, and $\beta, \gamma \in \Gamma$ with $\beta \neq \gamma$. It suffices to show that either $V_z \cap O_\beta = \phi$ or $V_z \cap O_\gamma = \phi$. By the choice of V_z, either $\overline{V_z} \cap Y_\beta = \phi$ or $\overline{V_z} \cap Y_\gamma = \phi$. Without loss of generality, suppose that $\overline{V_z} \cap Y_\gamma = \phi$. To see that $V_z \cap O_\gamma = \phi$, let $u \in Y_\gamma$. Either $W_u \cap V_z = \phi$ or $O_u \subseteq W_u \setminus \overline{V_z}$. In either case, $O_u \cap V_z = \phi$. Since u was chosen arbitrarily, $V_z \cap O_y = \phi$ for all $y \in Y_\gamma$. Therefore, $V_z \cap O_\gamma = \phi$, as desired. □

It is not clear as to how one might modify the definition of collectionwise normality in a space X to obtain a stronger version that would be implied by being $1-$ paracompact in X but not by being paracompact in X. A space X is said to be *discretely expandable* if every discrete collection of subsets of X is expandable to a locally finite open collection, [9]. A normal space is collectionwise normal if and only if it is discretely expandable, [9]. For a space X and $Y \subseteq X$,we say that Y is $(1-)$ *discretely expandable in* X provided every discrete collection of closed subsets of X, $\mathcal{F}$ there is a collection of open subsets of X, $\{U(F) : F \in \mathcal{F}\}$ locally finite on Y such that for all $F \in \mathcal{F}$, $Y \cap F \subseteq U(F) \subseteq X \setminus \cup (\mathcal{F} \setminus \{F\})$, $(F \subseteq U(F) \subseteq X \setminus \cup (\mathcal{F} \setminus \{F\}))$. Clearly, if Y is $(1-)$ paracompact in X then Y is $(1-)$ discretely expandable in X.

Theorem 3.4. *Suppose Y is $s-$ normal in the space X. If Y is $1-$ discretely expandable in X then Y is collectionwise normal in X.*

Proof. Let $\mathcal{F} = \{F_\alpha : \alpha \in \Gamma\}$ be a discrete collection of closed subsets of X such that if $\alpha, \beta \in \Gamma$ with $\alpha \neq \beta$ then $F_\alpha \neq F_\beta$. For each $x \in X$ let $U(x)$ be an open neighborhood of x meeting at most one member of $\mathcal{F}$. Let $\mathcal{V} = \{V_\alpha : \alpha \in \Gamma\}$ be a collection of open subsets of X locally finite on Y such that for all $\alpha \in \Gamma$, $F_\alpha \subseteq V_\alpha \subseteq X \setminus \cup (\mathcal{F} \setminus \{F_\alpha\})$.

Since Y is $s-$ normal in X, for all $\alpha \in \Gamma$ there exist open sets W_α and M_α such that $Y \cap F_\alpha \subseteq W_\alpha \subseteq \overline{W_\alpha} \subseteq V_\alpha$ and $Y \cap \overline{W_\alpha} \subseteq M_\alpha \subseteq \overline{M_\alpha} \subseteq V_\alpha$. For all $\alpha \in \Gamma$ let $G_\alpha = W_\alpha \setminus \overline{\cup\{M_\beta \cup W_\beta : \beta \in \Gamma \setminus \{\alpha\}\}}$. Note that for all $\alpha, \beta \in \Gamma$, if $\alpha \neq \beta$ then $G_\alpha \cap G_\beta = \phi$.

Suppose $\alpha \in \Gamma$ and $x \in F_\alpha \cap Y$. Since the collection $\{M_\gamma \cup W\gamma : \gamma \in \Gamma\}$ is locally finite on Y, if $x \in \overline{\cup\{M_\beta \cup W_\beta : \beta \in \Gamma \setminus \{\alpha\}\}}$ then $x \in \overline{M_\beta \cup W_\beta} \cap Y = (\overline{M_\beta} \cup \overline{W_\beta}) \cap Y = \overline{M_\beta} \cap Y$ for some $\beta \in \Gamma \setminus \{\alpha\}$. However $\overline{M_\beta} \subseteq V_\beta$ and $V_\beta \cap F_\alpha = \phi$ for all $\beta \in \Gamma \setminus \{\alpha\}$ a contradiction. Hence $x \notin \overline{\cup\{M_\beta \cup W_\beta : \beta \in \Gamma \setminus \{\alpha\}\}}$ and so $F_\alpha \cap Y \subseteq G_\alpha \subseteq X \setminus \cup (\mathcal{F} \setminus \{F_\alpha\})$ for all $\alpha \in \Gamma$.

Suppose that $x \in Y$. Since the collection $\{W_\alpha : \alpha \in \Gamma\}$ is locally finite on Y, if $x \in \overline{\cup\{W_\alpha : \alpha \in \Gamma\}}$ then there is an $\alpha^* \in \Gamma$ such that $x \in \overline{W_{\alpha^*}}$. Thus M_{α^*} is an open neighborhood of x meeting at most one member of $\{G_\alpha : \alpha \in \Gamma\}$, i.e. G_{α^*}. Hence the collection $\{G_\alpha : \alpha \in \Gamma\}$ is discrete with respect to Y. □

Theorem 3.5. *Suppose Y is closed and $s-$ normal in the space X. If Y is discretely expandable in X then Y is strongly collectionwise normal in X.*

Proof. Proceed as in Theorem 3.4 replacing the closed discrete collection $\{F_\alpha : \alpha \in \Gamma\}$ with the closed discrete collection $\{Y \cap F_\alpha : \alpha \in \Gamma\}$. □

Theorem 3.6. *Suppose Y is strongly normal in the space X. If Y is discretely expandable in X then Y is collectionwise normal in X.*

Proof. Let $\mathcal{F} =\{F_\alpha : \alpha \in \Gamma\}$ be a discrete collection of closed subsets of X such that if $\alpha, \beta \in \Gamma$ with $\alpha \neq \beta$ then $F_\alpha \neq F_\beta$. Let $\mathcal{V} =\{V_\alpha : \alpha \in \Gamma\}$ be a collection of open subsets of X locally finite on Y such that for all $\alpha \in \Gamma$, $F_\alpha \cap Y \subseteq V_\alpha \subseteq X \setminus \cup(\mathcal{F}\setminus\{F_\alpha\})$.

For all $\alpha \in \Gamma$, since Y is strongly normal in X and $F_\alpha \cap Y \subseteq V_\alpha$, there exist open sets W_α and M_α such that $F_\alpha \cap Y \subseteq W_\alpha \subseteq V_\alpha$, $\overline{W_\alpha} \cap Y \subseteq M_\alpha \subseteq V_\alpha$ and $\overline{M_\alpha} \cap Y \subseteq V_\alpha$. For all $\alpha \in \Gamma$ let $G_\alpha = W_\alpha \setminus \overline{\cup\{M_\beta \cup W_\beta : \beta \in \Gamma\setminus\{\alpha\}\}}$. Then as in Theorem 3.4 for all $\alpha \in \Gamma$, $F_\alpha \cap Y \subseteq G_\alpha \subseteq X \setminus \cup(\mathcal{F}\setminus\{F_\alpha\})$ and the collection $\{G_\alpha : \alpha \in \Gamma\}$ is discrete with respect to Y. □

Question 1 *Suppose Y is s- normal in the space X and discretely expandable in X. Is Y collectionwise normal in X?*

For a normal space X, a subspace Z can be collectionwise normal in X without being $1-$ discretely expandable in X, Example 5.2. A subspace Y of a normal space X can be $1-$ paracompact in X but not strongly collectionwise normal in X. In fact a subspace of a compact space X need not be strongly collectionwise normal in X, Example 5.5. In [4] a relative property of paracompactness type which does imply strongly collectionwise normality in X is introduced. Suppose X is a set, $\mathcal{U}$, $\mathcal{V}$ collections of subsets of X and $y \in X$. The collection $\mathcal{V}$ is said to star refine $\mathcal{U}$ at y provided there is a $U \in \mathcal{U}$ such that $st(y, \mathcal{V}) \subseteq U$. For a space X, a subspace Y is said to be *strongly star - normal in* X provided for every collection $\mathcal{U}$ of open subsets of X covering Y there is a collection $\mathcal{V}$ of open subsets of X covering Y which star refines $\mathcal{U}$ at every point of $\cup\mathcal{V}$.

Theorem 3.7. *If Y is strongly star normal in the space X then Y is strongly collectionwise normal in X.*

Proof. (Proceed as in Theorem 5.1.18 of [5]) Let $\mathcal{F} =\{F_\alpha : \alpha \in \Gamma\}$ be a collection of closed subsets of X which is discrete with respect to Y such that if $\alpha, \beta \in \Gamma$ with $\alpha \neq \beta$ then $F_\alpha \neq F_\beta$. For each $y \in Y$ let U_y be and open neighborhood of y meeting at most one member of $\mathcal{F}$. Let $\mathcal{W}$ be a collection of open subsets of X covering Y which star refines $\mathcal{U} = \{U_x : x \in Y\}$ at every point of $\cup\mathcal{W}$ and $\mathcal{V}$ be a collection of open subsets of X which covers

Y and star refines $\mathcal{W}$ at every point of $\cup\mathcal{V}$. Then using the same argument as in Lemma 5.1.15 of [5], we see that $\mathcal{V}$ is a collection of open subsets of X covering Y such that for every $V \in \mathcal{V}$ there is a $U \in \mathcal{U}$ with $st(V, \mathcal{V}) \subseteq U$. For each $\alpha \in \Gamma$ let $V_\alpha = \cup\{V \in \mathcal{V} : V \cap F_\alpha \neq \phi\}$ and note that for all $\alpha \in \Gamma$ $F_\alpha \cap Y \subseteq V_\alpha \subseteq X \setminus \cup(\mathcal{F} \setminus \{F_\alpha\})$ and the collection $\{V_\alpha : \alpha \in \Gamma\}$ is discrete with respect to Y. □

4. Relative versions of the Michael-Nagami Theorem

By replacing "locally finite" with "point finite" in the definitions of $(1-)$ paracompactness we obtain relative metacompact analogs [7]. The subspace Y of X is *strongly metacompact in* X provided every open cover of X has an open refinement point finite on Y. The subspace Y of a space X is *metacompact in* X provided every open cover of X has an open partial refinement point finite on Y. Clearly for a space X strongly metacompactness in X is a natural relatively metacompact analog of $1-$ paracompactness in X and metacompactness in X is the corresponding relative metacompact analog of paracompactness in X.

Before presenting several relative versions of the Michael - Nagami Theorem here are several examples clarifying the limitations of what we can expect. A closed discrete subspace of a normal space X is always strongly metacompact in X and collectionwise normal but need not be paracompact in X, Example 5.2. In Example 5.6 we give a regular space X having an open subspace Y which is strongly collectionwise normal in X and strongly metacompact in X but not $1-$ paracompact in X. In Example 5.7 we give a non regular space X having a closed subspace Y which is super regular in X, strongly metacompact in X and $1-$ discretely expandable in X but not $1-$ paracompact in X.

Question 2 *Suppose Y is strongly metacompact in X and $1-$ discretely expandable in the space X. Is Y paracompact in X?*

The proof of Theorem 5.3.3 (Michael-Nagami Theorem) of [5] can be readily modified to prove the following relative version.

Theorem 4.1. *Suppose that X is a regular space and $Y \subseteq X$. If Y is strongly metacompact in X and $1-$ discretely expandable in X then every open cover of X has an open partial refinement covering Y which is the countable union of collections locally finite on Y.*

Question 3 *Suppose that X is a regular space, $Y \subseteq X$ and every open cover of X has an open partial refinement covering Y which is the countable union of collections locally finite on Y. Is Y paracompact in X?*

For a closed subspace Y of a space X, Y is paracompact in X if and only if every open cover of X has an open partial refinement covering Y which is the countable union of collections locally finite on Y, [8]. Also for a closed subset Y, Y is strongly metacompact in X if and only if Y is a metacompact subspace of X, [7].

Corollary 4.2. *Suppose Y is closed in the regular space X. If Y is 1- discretely expandable and metacompact then Y is paracompact in X. (Is Y 1- paracompact in X?)*

Question 4 *Suppose Y is (strongly) metacompact in X and collectionwise normal in X. Is Y paracompact in X?*

In Question 3 if locally finite on Y is replaced with discrete with respect to Y the answer is yes.

Lemma 4.3. *Suppose that Y is strongly regular and strongly collectionwise normal in the space X. If every open cover of X has an open partial refinement covering Y which is the countable union of collections discrete with respect to Y then Y is paracompact in X.*

Proof. Let $\mathcal{U}$ be an open cover of X. For all $x \in X$ let W_x be an open neighborhood of x such that $W_x \subseteq U$ and $Y \cap \overline{W_x} \subseteq U$ for some $U \in \mathcal{U}$. Let $\mathcal{W} = \{W_x : x \in X\}$ and $\mathcal{V} = \cup\{\mathcal{V}_n : n < \omega\}$ be an open partial refinement of $\mathcal{W}$ covering Y such that for all $n < \omega$, the collection $\mathcal{V}_n$ is discrete with respect to Y. For all $n < \omega$, since $\mathcal{V}_n$ is discrete with respect to Y, the collection $\{\overline{V} : V \in \mathcal{V}_n\}$ is discrete with respect to Y. For each $n < \omega$ let $\mathcal{G}_n = \{G(V,n) : V \in \mathcal{V}_n\}$ be a collection of open subsets of X discrete with respect to Y such that for all $V \in \mathcal{V}_n$, $\overline{V} \cap Y \subseteq G(V,n)$ and $G(V,n) \subseteq U$ for some $U \in \mathcal{U}$. For each $n < \omega$ let $F_n = \overline{\cup \mathcal{V}_n}$. For each $V \in \mathcal{V}_o$ let $H(V,0) = G(V,0)$. For each $0 < n < \omega$ and $V \in \mathcal{V}_n$ let $H(V,n) = G(V,n) \setminus \cup \{F_k : k < n\}$. For each $n < \omega$ let $\mathcal{H}_n = \{H(V,n) : V \in \mathcal{V}_n\}$ and let $\mathcal{H} = \cup\{\mathcal{H}_n : n < \omega\}$. We now show that $\mathcal{H}$ covers Y and is locally finite with respect to Y.

Let $y \in Y$. Let $n = \min\{k < \omega : y \in F_k\}$. Since $y \in Y \cap F_n$ and $\mathcal{V}_n$ is discrete with respect to Y, there is a $V \in \mathcal{V}_n$ with $y \in \overline{V} \cap Y \subseteq G(V,n)$ and so $y \in H(V,n)$. Let $m = \min\{k < \omega : y \in \cup \mathcal{V}_k\}$ and $V' \in \mathcal{V}_m$ such that $y \in V'$. Since $V' \subseteq F_m$, V' is an open neighborhood of y missing all members of $\mathcal{H}_k$ for all $m < k < \omega$. For all $k \leq m$, since the collection $\mathcal{G}_k$ and hence $\mathcal{H}_k$ is discrete with respect to Y, let O_k be an open neighborhood of y meeting at most one member of $\mathcal{H}_k$. Then $V' \cap O_o \cap ... \cap O_m$ is an open neighborhood of y meeting only finitely many members of $\mathcal{H}$. □

Again the proof of Theorem 5.3.3 (Michael-Nagami Theorem) of [5] can be readily modified to prove the following relative version. We include a proof here to demonstrate the modifications needed for this theorem and in the proof of Theorem 4.1.

Theorem 4.4. *Suppose that Y is strongly regular in the space X. If Y is metacompact in X and strongly collectionwise normal in X then Y is paracompact in X.*

Proof. Let $\mathcal{O}$ be an open cover of X and let $\mathcal{U} = \{U_\alpha : \alpha \in \Gamma\}$ be an open partial refinement of $\mathcal{O}$ covering Y point finite on Y such that if $\alpha, \beta \in \Gamma$ and $\alpha \neq \beta$ then $U_\alpha \neq U_\beta$. Let $\mathcal{V}_o = \{\phi\}$. Suppose $k < \omega$ and for all $i \leq k$ the collection $\mathcal{V}_i$ has been defined and $W_i = \bigcup \mathcal{V}_i$ such that

1. $\mathcal{V}_i$ is an open partial refinement of $\mathcal{U}$ discrete with respect to Y
2. if $x \in Y$ such that $|\{\alpha \in \Gamma : x \in U_\alpha\}| \leq i$ then $x \in \cup\{W_j : j = 0..i\}$.

Let $\mathcal{T}_{k+1} = \{T \subseteq \Gamma : |T| = k+1\}$ and for all $T \in \mathcal{T}_{k+1}$ let
$F_T = (X \setminus \cup\{W_j : j = 0..i\}) \cap (X \setminus \cup\{U_\alpha : \alpha \in \Gamma \setminus T\})$.

Suppose $T \in \mathcal{T}_{k+1}$. If $x \in Y \cap F_T$ then $\{\alpha \in \Gamma : x \in U_\alpha\} \subseteq T$ and $x \notin \cup\{W_j : j = 0..k\}$. Hence $\{\alpha \in \Gamma : x \in U_\alpha\} = T$ and so $Y \cap F_T \subseteq \cap\{U_\alpha : \alpha \in T\}$. Suppose that $x \in Y$. If $|\{\alpha \in \Gamma : x \in U_\alpha\}| \leq k$ then $\cup\{W_j : j = 0..k\}$ is an open neighborhood of x missing all members of $\{F_T : T \in \mathcal{T}_{k+1}\}$. Suppose $|\{\alpha \in \Gamma : x \in U_\alpha\}| \geq k+2$. Let $\alpha_1, \alpha_2, ..., \alpha_{k+2}$ be distinct members of $\{\alpha \in \Gamma : x \in U_\alpha\}$. Then $\cap\{U_{\alpha_i} : i = 1..k+2\}$ is an open neighborhood of x meeting no member of $\{F_T : T \in \mathcal{T}_{k+1}\}$. Suppose $|\{\alpha \in \Gamma : x \in U_\alpha\}| = k+1$. Let $T' = \{\alpha \in \Gamma : x \in U_\alpha\}$ and note $\cap\{U_\alpha : \alpha \in T'\}$ is a neighborhood of x meeting exactly one member of $\{F_T : T \in \mathcal{T}_{k+1}\}$. Hence we see that $\{F_T : T \in \mathcal{T}_{k+1}\}$ is a collection of closed subsets of X which is discrete with respect to Y.

Let $\{G_T : T \in \mathcal{T}_{k+1}\}$ be a collection of open subsets of X discrete with respect to Y such that for all $T \in \mathcal{T}_{k+1}$, $Y \cap F_T \subseteq G_T \subseteq X \setminus (\cup\{F_{T'} : T' \in \mathcal{T}_{k+1} \setminus \{T\}\})$. Also assume that for all $T \in \mathcal{T}_{k+1}$, if $Y \cap F_T = \phi$ then $G_T = \phi$. For all $T \in \mathcal{T}_{k+1}$, let $V_T = G_T \cap (\cap\{U_\alpha : \alpha \in T\})$ and note that $Y \cap F_T \subseteq V_T$. Let $\mathcal{V}_{k+1} = \{V_T : T \in \mathcal{T}_{k+1}\}$ and $W_{k+1} = \bigcup \mathcal{V}_{k+1}$. Then $\mathcal{V}_{k+1}$ is an open partial refinement of $\mathcal{U}$ discrete with respect to Y. Suppose that $x \in Y$ such that $|\{\alpha \in \Gamma : x \in U_\alpha\}| \leq k+1$. Then there is a $T \in \mathcal{T}_{k+1}$ such that $x \in X \setminus \cup\{U_\alpha : \alpha \in \Gamma \setminus T\}$. Thus

$$x \in X \setminus \cup\{U_\alpha : \alpha \in \Gamma \setminus T\} = ((X \setminus \bigcup_{i=0}^{k} W_i) \cup (\bigcup_{i=0}^{k} W_i)) \cap (X \setminus \cup\{U_\alpha : \alpha \in \Gamma \setminus T\})$$
$$= [(X \setminus \bigcup_{i=0}^{k} W_i) \cap (X \setminus \cup\{U_\alpha : \alpha \in \Gamma \setminus T\})] \cup [(\bigcup_{i=0}^{k} W_i) \cap (X \setminus \cup\{U_\alpha : \alpha \in \Gamma \setminus T\})]$$
$$\subseteq F_T \cup \bigcup_{i=0}^{k} W_i.$$

Hence for all $x \in Y$ such that $|\{\alpha \in \Gamma : x \in U_\alpha\}| \leq k+1$, $x \in \bigcup_{i=0}^{k+1} W_i$. Thus, since $\mathcal{U}$ is point finite on Y, $\mathcal{V} = \bigcup_{n<\omega} \mathcal{V}_n$ is an open partial refinement of $\mathcal{U}$ covering Y such that for all $n < \omega$ the collection $\mathcal{V}_n$ is discrete with respect to Y. By Lemma 4.3, Y is paracompact in X. □

Corollary 4.5. *Suppose Y is closed and $s-$ regular in the space X. Then Y is paracompact in X if and only if Y is collectionwise normal in X and metacompact.*

5. Examples

Example 5.1. A T_2 space X having a subspace Y which is $1-$ paracompact in X but not strongly regular in X.

Let $X = \omega \cup (\omega \times \omega) \cup \{*\}$. Define a topology on X as follows:

1. points of $\omega \times \omega$ are isolated,
2. for each $n < \omega$, $\{\{n\} \cup (\{n\} \times (k, \omega)) : k < \omega\}$ is a local base at n,
3. the collection $\{\{*\} \cup ((k, \omega) \times \omega) : k < \omega\}$ is a local base at $*$.

Then X is T_2 and the subspace $Y = \omega$ is $1-$ paracompact in X but the closed set Y cannot be separated from the point $*$ by open subsets of X. Thus Y is not strongly$-$ regular in X.

Example 5.2. Bing's Example G.

Let X be Bing's Example G, Y the nonisolated points of X and Z the isolated points of X. The subset Y is a closed discrete subspace of X and therefore is strongly metacompact in X and collectionwise normal. However Y is not collectionwise normal in X. The subspace Z is an open discrete subspace of X and therefore strongly collectionwise normal in X and paracompact in X but not $1-$ discretely expandable in X.

Example 5.3. A regular space X having an open normal subspace which is collectionwise normal in X but which is not $1-$ discretely expandable in X and a closed subspace Z which is paracompact in X but not $s-$ normal in X.

The space X is a standard modification of the Tychonoff plank. Let $X = [0, \omega_1] \times [0, \omega] \backslash \{(\omega_1, \omega)\}$. Define a topology on X as follows:
1. Points of $\omega_1 \times \omega$ are isolated.
2. For all $n < \omega$ let $\{B(\alpha, n) : \alpha < \omega_1\}$ be a neighborhood base for the point (ω_1, n) where $B(\alpha, n) = (\alpha, \omega_1] \times \{n\}$ for all $\alpha < \omega_1$.
3. For all $\alpha < \omega_1$ let $\{G(\alpha, n) : n < \omega\}$ be a neighborhood base for the point (α, ω) where $G(\alpha, n) = \{\omega_1\} \times (n, \omega]$ for all $n < \omega$.
Clearly X is a regular space.

Let $Y = X \backslash (\{\omega_1\} \times \omega)$. Since Y is an open normal subspace of X it is strongly normal in X. The closed sets $\omega_1 \times \{\omega\}$ and $\{\omega_1\} \times \omega$ cannot be separated by open subsets of X. Thus not only is X not normal but Y is not $s-$ normal in X.

The subset Y is collectionwise normal in X since it is an open subset of X and the direct sum of compact subspaces ($Y = \oplus\{\{\alpha\} \times [0, \omega] : \alpha < \omega_1\}$). However Y is not $1-$ discretely expandable in X. To see this let $C = \{\omega_1\} \times \omega$ and $\mathcal{F} = \{\{r\} : r \in C\}$ and note that $\mathcal{F}$ is a discrete collection of closed subsets of X. Suppose that for all $r \in C$, $U(r)$ is an open neighborhood of r. For all $n < \omega$ let $\beta_n < \omega_1$ such that $B(\beta_n, n) \subseteq U(\omega_1, n)$. Let $\beta^* = \sup\{\beta_n : n < \omega\}$ and note that $\beta^* < \omega_1$. Choose $\beta^* < \gamma < \omega_1$ and let $k < \omega$. Then $(\gamma, m) \in G(\gamma, k) \cap B(\beta_m, m) \subseteq G(\gamma, k) \cap U(\omega_1, m)$ for all $k < m < \omega$. Hence every neighborhood of the point (γ, ω) meets infinitely many members of $\{U(r) : r \in C\}$. Thus the collection $\{U(r) : r \in C\}$ is not locally finite on Y.

Let $Z = \{\omega_1\} \times \omega$. The closed discrete subspace Z is easily seen to be paracompact in X but like Y it is not $s-$ normal in X.

Example 5.4. A T_2 Lindelöf space X having a subspace which is $1-$ paracompact in X but not collectionwise normal in X.

Let Y and Z be disjoint subsets of $\mathbb{R}\backslash\mathbb{Q}$ such that for every nonempty open subset U of $\mathbb{R}$ $|U \cap Y| = \omega_1 = |U \cap Z|$. Well order $\mathbb{Q}$, Y, and Z, say

$\mathbb{Q} = \{q_n : n < \omega\}$, $Y = \{y_\alpha : \alpha < \omega_1\}$ and $Z = \{z_\alpha : \alpha < \omega_1\}$.

For any set $A \subseteq \mathbb{R}$ let ${}_qA = \{n < \omega : q_n \in A\}$, ${}_yA = \{\alpha < \omega_1 : y_\alpha \in A\}$ and ${}_zA = \{\alpha < \omega_1 : z_\alpha \in A\}$. Let $X = (\mathbb{R} \times \{0,1\}) \cup (Y \cup Z \cup \mathbb{Q}) \cup (\omega_1 \times \omega \times \{0,1\})$ and define a topology on X as follows:

1. All points of $\omega_1 \times \omega \times \{0,1\}$ are isolated.
2. For all $\alpha < \omega_1$ a basic open neighborhood of y_α $[z_\alpha]$ is of the form $\{y_\alpha\} \cup (\{\alpha\} \times_q U \times \{0\})$ $[\{z_\alpha\} \cup (\{\alpha\} \times_q U \times \{1\})]$ where U is an open neighborhood of y_α $[z_\alpha]$ in $\mathbb{R}$.
3. For all $n < \omega$ a basic open neighborhood of q_n is of the form $\{q_n\} \cup ((\alpha, \omega_1) \times \{n\} \times \{0,1\})$ where $\alpha < \omega_1$.
4. For all $x \in \mathbb{R}$ a basic open neighborhood of $(x,0)$ $[(x,1)]$ is of the form $([x,a) \times \{0\}) \cup ((x,a) \cap (Y \cup \mathbb{Q})) \cup ({}_y(x,a) \times_q (x,a) \times \{0\})$ $\cup((\alpha,\omega_1) \times_q (x,a) \times \{0,1\})$ where $a \in \mathbb{R}$, $x < a$ and $\alpha < \omega_1$.

$$\left[\begin{array}{l} ((b,x] \times \{1\}) \cup ((b,x) \cap (Z \cup \mathbb{Q})) \cup ({}_z(b,x) \times_q (b,x) \times \{1\}) \\ \cup((\beta,\omega_1) \times_q (b,x) \times \{0,1\}) \text{ where } b \in \mathbb{R} \text{ , } b < x \text{ and } \beta < \omega_1. \end{array}\right]$$

The space X is T_2 Lindelöf but not regular. The subspace $Y \cup Z \cup \mathbb{Q}$ is 1− paracompact in X but not collectionwise normal in X.

To see that $Y \cup Z \cup \mathbb{Q}$ is not collectionwise normal in X let $F = (\mathbb{R} \times \{0\}) \cup Y$ and $K = (\mathbb{R} \times \{1\}) \cup Z$. Note F and K are disjoint closed subsets of X. Suppose that U and V are disjoint open subsets of X such that

$F \cap (Y \cup Z \cup \mathbb{Q}) = Y \subseteq U$ and $K \cap (Y \cup Z \cup \mathbb{Q}) = Z \subseteq V$.

Then $\overline{U} \cap \overline{V} \cap Q \neq \phi$.

Example 5.5. A compact space X having a subspace Y which is not strongly collectionwise normal in X.

Let $X = (\omega_1 + 1) \times (\omega + 1)$ with the product topology and $Y = X \backslash \{(\omega_1, \omega)\}$ (Tychonoff plank). Then since X is compact Y (and every other subspace of X) is 1− paracompact in X. The collection of closed subsets of X

$\mathcal{F} = \{(\omega_1 + 1) \times \{\omega\}\} \cup \{\{(\omega, n)\} : n < \omega\}$

is discrete with respect to Y. Using the same argument that the Tychonoff plank is not normal using the closed (in Y) sets $\omega_1 \times \{\omega\}$ and $\{\omega_1\} \times \omega$, one can use $\mathcal{F}$ to show that Y is not strongly collectionwise normal in X.

Example 5.6. A regular space having a subspace which is strongly metacompact in X and strongly collectionwise normal in X but not 1− discretely expandable in X.

Let $X = \mathbb{R} \times \mathbb{R}$, $Y = \mathbb{R} \times \{0\}$ and $Z = X \backslash Y$. Points of Z have their usual open neighborhoods. For each $x \in \mathbb{R}$ a basic neighborhood of $(x,0)$ will be of the form $\{x\} \times (-\epsilon, \epsilon)$ where $\epsilon > 0$. Clearly X is regular and Z is strongly metacompact in X and strongly star normal in X. However the points of the closed discrete subset Y cannot be separated by open subsets of X which are discrete with respect to Z.

Example 5.7. A nonregular space having a subspace which is super regular in X, strongly metacompact in X and $1-$ discretely expandable in X but not $1-$ paracompact in X.

Let $A = \omega_1$ with the order topology. Let $B = [0,1]$ with points of $(0,1]$ having usual open neighborhoods in $[0,1]$ with the order topology and open neighborhoods of 0 are of the form $U \backslash \{\frac{1}{n} : n = 1, 2, ...\}$ where U is a usual open neighborhood of 0 in $[0,1]$ with the order topology. Note that B is T_2 but not regular. Also $\{\frac{1}{n} : n = 1, 2, ...\}$ is closed, $1-$ discretely expandable in B and super regular in B.

The construction of the space X is based on examples in [2] and [7]. Let $X = A \times B$ with the topology defined as follows:

1. for $a \in A$ and $y \in (0,1]$ basic open neighborhoods are of the form $\{a\} \times V$ where V is an open neighborhood of y in B,
2. for $a \in A$ basic open neighborhoods of $(a, 0)$ are of the form $\cup\{\{x\} \times V_x : x \in U\}$ where U is an open neighborhood of a in A and for all $x \in U$, V_x is an open neighborhood of 0 in B.

Let $Y = A \times \{\frac{1}{n} : n = 1, 2, ...\}$ and note that Y is a closed discrete subset of X and therefore strongly metacompact in X. Also note that Y is super regular in X but not strongly regular in X. It is not difficult to show that Y is $1-$ discretely expandable in X. To see that Y ia not $1-$ paracompact in X, let $\mathcal{U} = \{[0, \alpha] \times B : \alpha < \omega_1\}$. Using the Pressing Down Lemma it is easily seen that $\mathcal{U}$ does not have an open refinement that is locally finite on Y.

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