Characterizing a class of completable fuzzy metric spaces

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Abstract

In this paper we give a characterization of the class of completable strong (non-Archimedean) fuzzy metric spaces, in the sense of George and Veeramani.

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1. Introduction

The problem of constructing a satisfactory theory of fuzzy metric spaces has been investigated by several authors from different points of view. Here we use the concept of fuzzy metric space that George and Veeramani [1, 3] introduced and studied with the help of continuous t-norms. In [2, 6], it is proved that the class of topological spaces which are fuzzy metrizable agrees with the class of metrizable spaces. This result allows to restate some classical

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theorems on metrics in the realm of fuzzy metric spaces. Nevertheless, the theory of fuzzy metric completion is, in this context, very different from the classical theory of metric completion. Indeed, Gregori and Romaguera proved that there exist fuzzy metric spaces which are not completable [7]. Later, the same authors gave a characterization of those fuzzy metric spaces that are completable, which we reformulate, for our convenience, as follows:

Theorem 1.1. *[8] A fuzzy metric space $(X, M, *)$ is completable if and only if for each pair of Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in X the following three conditions are fulfilled:*

(c1) $\lim_n M(a_n, b_n, s) = 1$ *for some* $s > 0$ *implies* $\lim_n M(a_n, b_n, t) = 1$ *for all* $t > 0$.

(c2) $\lim_n M(a_n, b_n, t) > 0$ *for all* $t > 0$.

(c3) The assignment $t \to \lim_n M(a_n, b_n, t)$ *for each* $t > 0$ *is a continuous function on* $]0, \infty[$*, provided with the usual topology of* $\mathbb{R}$.

There were in the literature examples of non-completable strong fuzzy metrics that do not satisfy $(c1)$ or $(c2)$ [7, 8], and recently [4], the authors have constructed a non-completable fuzzy metric space which does not satisfy $(c3)$.

In this paper we first observe that $(c1) - (c3)$ constitute an independent axiomatic system and then we will proof, after several lemmas, that strong fuzzy metrics satisfy $(c3)$, or in other words (Theorem 4.7): A strong fuzzy metric space $(X, M, *)$ is completable if and only if M satisfies $(c1)$ and $(c2)$. Several corollaries can be obtained from this theorem, for instance a characterization of completable fuzzy ultrametrics (Corollary 4.9) and also we could obtain that metric spaces admit a unique completion, but we do not insist on it because it is well-known from the properties of the standard fuzzy metric. Several examples illustrate our results.

The structure of the paper is as follows. After the preliminaries section, in Section 3 we prove that $(c1) - (c3)$ constitute an independent axiomatic system. In Section 4 we give a characterization for the class of completable strong fuzzy metrics.

2. Preliminaries

Definition 2.1. (George and Veeramani [1].) A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times]0, \infty[$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

(GV1) $M(x, y, t) > 0$

(GV2) $M(x, y, t) = 1$ if and only if $x = y$

(GV3) $M(x, y, t) = M(y, x, t)$

(GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$

(GV5) $M(x, y, \_) :]0, \infty[\rightarrow]0, 1]$ is continuous.

If $(X, M, *)$ is a fuzzy metric space, we will say that $(M, *)$ (or simply M) is a *fuzzy metric* on X.

Remark 2.2. *$M(x, y, \_)$ is non-decreasing for all $x, y \in X$.*

George and Veeramani proved in [1] that every fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x, \epsilon, t) : x \in X, 0 < \epsilon < 1, t > 0\}$, where $B_M(x, \epsilon, t) = \{y \in X : M(x, y, t) > 1 - \epsilon\}$ for all $x \in X$, $\epsilon \in]0, 1[$ and $t > 0$.

Let (X, d) be a metric space and let M_d a fuzzy set on $X \times X \times]0, \infty[$ defined by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

Then $(X, M_d, \cdot)$ is a fuzzy metric space, [1], and M_d is called the *standard fuzzy metric* induced by d. The topology on X deduced from d agrees with τ_{M_d}.

Definition 2.3. (Gregori and Romaguera [8].) A fuzzy metric M on X is said to be *stationary* if M does not depend on t, i.e., if for each $x, y \in X$, the function $M_{x,y}(t) = M(x, y, t)$ is constant. In this case we write $M(x, y)$ instead of $M(x, y, t)$.

Definition 2.4. (Gregori et al. [5], Istrăţescu [9].) A fuzzy metric space $(X, M, *)$ is said to be *strong* (non-Archimedean) if for all $x, y, z \in X$ and all $t > 0$ satisfies

$$M(x, z, t) \geq M(x, y, t) * M(y, z, t).$$

A strong fuzzy metric for the minimum t-norm is called a *fuzzy ultrametric*.

Proposition 2.5. *(George and Veeramani [1].) Let* $(X, M, *)$ *be a fuzzy metric space. A sequence* $\{x_n\}$ *in* X *converges to* x *if and only if* $\lim_n M(x_n, x, t) = 1$*, for all* $t > 0$*.*

Definition 2.6. (George and Veeramani [1].) A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be *M-Cauchy*, or simply Cauchy, if for each $\epsilon \in]0, 1[$ and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $n, m \geq n_0$ or, equivalently, $\lim_{n,m} M(x_n, x_m, t) = 1$ for all $t > 0$. X is said to be *complete* if every Cauchy sequence in X is convergent with respect to τ_M. In such a case M is also said to be complete.

Definition 2.7. (Gregori and Romaguera [7].) Let $(X, M, *)$ and $(Y, N, \diamond)$ be two fuzzy metric spaces. A mapping f from X to Y is said to be an *isometry* if for each $x, y \in X$ and $t > 0$, $M(x, y, t) = N(f(x), f(y), t)$ and, in this case, if f is a bijection, X and Y are called *isometric*. A *fuzzy metric completion* of (X, M) is a complete fuzzy metric space (X^*, M^*) such that (X, M) is isometric to a dense subspace of X^*. X is said to be *completable* if it admits a fuzzy metric completion.

A t-norm $*$ is called integral (positive) if $x * y > 0$ whenever $x, y \in]0, 1]$.

Theorem 2.8. *(Gregori et al. [5, Theorem 35]) Let* $(X, M, *)$ *be a strong fuzzy metric space and suppose that* $*$ *is integral. If* $\{x_n\}$ *and* $\{y_n\}$ *are Cauchy sequences in* X *and* $t > 0$ *then* $\{M(x_n, y_n, t)\}_n$ *converges in* $]0, 1]$*.*

3. Non-completable fuzzy metric spaces

In this section we will show that the axioms $(c1) - (c3)$ constitute an independent axiomatic system. To that end, we show three examples of non-completable fuzzy metric space, which do not satisfy anyone of these three axioms but they satisfy the other two.

Example 3.1. (Gregori and Romaguera [8, Example 2].) Let $\{x_n\}$ and $\{y_n\}$ be two strictly increasing sequences of positive real numbers, which converge to 1 with respect to the usual topology of $\mathbb{R}$, with $A \cap B = \emptyset$, where $A = \{x_n : n \in \mathbb{N}\}$ and $B = \{y_n : n \in \mathbb{N}\}$. Put $X = A \cup B$ and define a fuzzy set M on $X \times X \times]0, \infty[$ by:

$M(x_n, x_n, t) = M(y_n, y_n, t) = 1$ for all $n \in \mathbb{N}, t > 0$,
$M(x_n, x_m, t) = x_n \wedge x_m$ for all $n, m \in \mathbb{N}$ with $n \neq m, t > 0$,
$M(y_n, y_m, t) = y_n \wedge y_m$ for all $n, m \in \mathbb{N}$ with $n \neq m, t > 0$,
$M(x_n, y_m, t) = M(y_m, x_n, t) = x_n \wedge y_m$ for all $n, m \in \mathbb{N}, t \geq 1$,
$M(x_n, y_m, t) = M(y_m, x_n, t) = x_n \wedge y_m \wedge t$ for all $n, m \in \mathbb{N}, t \in]0, 1[$.

As pointed out in [8], an easy computation shows that $(X, M, *)$ is a fuzzy metric space, where $*$ is the minimum t-norm, and it satisfies conditions $(c2)$ and $(c3)$ of Theorem 1.1. But M does not satisfy condition $(c1)$ of Theorem 1.1. Indeed, in [8] it was observed that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X such that $\lim_n M(x_n, y_n, t) = 1$ for all $t \geq 1$, but $\lim_n M(x_n, y_n, t) = t$ for all $t \in]0, 1[$.

Example 3.2. (Gregori and Romaguera [7, Example 2].) Let $\{x_n\}$ and $\{y_n\}$ be two sequences of distinct points such that $A \cap B = \emptyset$, where $A = \{x_n : n \geq 3\}$ and $B = \{y_n : n \geq 3\}$. Put $X = A \cup B$ and define a fuzzy set M on $X \times X \times]0, \infty[$ by:

$M(x_n, x_m, t) = M(y_n, y_m, t) = 1 - \left[\frac{1}{n \wedge m} - \frac{1}{n \vee m}\right]$,

$M(x_n, y_m, t) = M(y_m, x_n, t) = \frac{1}{n} + \frac{1}{m}$,

for all $n, m \geq 3$. In [7], it was proved that $(X, M, *)$ is a fuzzy metric space, where $*$ is the Luckasievicz t-norm ($a * b = \max\{0, a + b - 1\}$), for which both $\{x_n\}_{n \geq 3}$ and $\{y_n\}_{n \geq 3}$ are Cauchy sequences. Clearly,

$$\lim_n M(x_n, y_n, t) = \lim_n \left(\frac{1}{n} + \frac{1}{n}\right) = 0.$$

Therefore, M does not satisfy condition $(c2)$.

On the other hand, M is a stationary fuzzy metric on X, and so it satisfies conditions $(c1)$ and $(c3)$, since, obviously, this two conditions are satisfied for stationary fuzzy metrics.

Example 3.3. (Gregori et al. [4, Proposition 9].) Let d be the usual metric on $\mathbb{R}$ restricted to $]0, 1]$ and consider the standard fuzzy metric M_d induced by d. Put $X =]0, 1]$ and define a fuzzy set M on $X \times X] \times]0, \infty[$ by

$$M(x, y, t) = \begin{cases} M_d(x, y, t), & 0 < t \leq d(x, y) \\ M_d(x, y, 2t) \cdot \frac{t - d(x,y)}{1 - d(x,y)} + M_d(x, y, t) \cdot \frac{1 - t}{1 - d(x,y)}, & d(x, y) < t \leq 1 \\ M_d(x, y, 2t), & t > 1 \end{cases}$$

In [4] it is proved that $(X, M, *)$ is a fuzzy metric space, where $*$ is the usual product. Also, it is obtained that for the Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in X, given by $a_n = \frac{1}{n}$ and $b_n = 1$ for all $n \in \mathbb{N}$, the assignment

$$\lim_n M(a_n, b_n, t) = \begin{cases} \frac{t}{t+1}, & 0 < t < 1 \\ \\ \frac{2t}{2t+1}, & t \geq 1 \end{cases}$$

is a well-defined function on $]0, \infty[$ which is not continuous at $t = 1$. Therefore, M does not satisfy condition $(c3)$.

Next, we will see that M satisfies conditions $(c1)$ and $(c2)$.

For proving that M satisfies $(c1)$, we suppose that $\{a_n\}$ and be $\{b_n\}$ are two Cauchy sequences in $]0, 1]$ such that $\lim_n M(a_n, b_n, s) = 1$ for some $s > 0$. By Remark 2.2, we can find $t_0 > 1$, with $t_0 > s$, such that $\lim_n M(a_n, b_n, t_0) = 1$. Then,

$$\lim_n M(a_n, b_n, t_0) = \lim_n M_d(a_n, b_n, 2t_0) = \lim_n \frac{2t_0}{2t_0 + |a_n - b_n|} = 1$$

and thus $\lim_n |a_n - b_n| = 0$.

Let $t > 0$. We distinguish two cases:

(1) If $t \in]0, 1]$, then there exists $n_0 \in \mathbb{N}$ such that $|a_n - b_n| < t$ for all $n \geq n_0$, since $\lim_n |a_n - b_n| = 0$. Then

$$\lim_n M(a_n, b_n, t) = \lim_n \left(\frac{2t}{2t + |a_n - b_n|} \cdot \frac{t - |a_n - b_n|}{1 - |a_n - b_n|} + \frac{t}{t + |a_n - b_n|} \cdot \frac{1 - t}{1 - |a_n - b_n|} \right) =$$

$$= t + 1 - t = 1$$

(2) If $t > 1$, then

$$\lim_n M(a_n, b_n, t) = \lim_n \frac{2t}{2t + |a_n - b_n|} = 1$$

Therefore, $\lim_n M(a_n, b_n, t) = 1$ for all $t > 0$, and so M satisfies $(c1)$.

Now, we will prove that M satisfies $(c2)$. Suppose the contrary, i.e., there exist two Cauchy sequences $\{a_n\}$ and $\{b_n\}$ such that $\lim_n M(a_n, b_n, s) = 0$ for some $s > 0$. First, we claim that M-Cauchy sequences are Cauchy for the usual metric d of $\mathbb{R}$ restricted to $]0, 1]$. Indeed, if $\{a_n\}$ is a Cauchy sequence in

$(X, M, *)$, then $\lim_{n,m} M(a_n, a_m, t) = 1$ for all $t > 0$. In particular, for $t > 1$ we have that $\lim_{n,m} M(a_n, a_m, t) = \lim_{n,m} \frac{2t}{2t+|a_n - a_m|} = 1$, and so $\lim_{n,m} |a_n - a_m| = 0$, i.e., $\{a_n\}$ is Cauchy in $(\mathbb{R}, d)$.

Then, there exist $a, b \in [0, 1]$ such that $\{a_n\}$ and $\{b_n\}$ converge to a and b, respectively, for the usual topology of $\mathbb{R}$ restricted to $[0, 1]$. Therefore, $\lim_n |a_n - b_n| = |a - b|$.

We distinguish two cases:

(1) Suppose that $|a - b| = 0$. Then for $t_0 > 1$ we have that

$$\lim_n M(a_n, b_n, t_0) = \lim_n \frac{2t_0}{2t_0 + |a_n - b_n|} = \frac{2t_0}{2t_0 + |a - b|} = 1.$$

So $M(a_n, b_n, t) = 1$ for all $t > 0$, since M satisfies condition $(c1)$, a contradiction.

(2) Suppose that $|a - b| \in]0, 1]$. Taking into account our assumption and Remark 2.2, we can find $0 < t_0 < |a - b|$, with $t_0 < s$, such that $\lim_n M(a_n, b_n, t_0) = 0$. Then there exists $n_0 \in \mathbb{N}$ such that $|a_n - b_n| > t_0$ for all $n \geq n_0$, and so

$$\lim_n M(a_n, b_n, t_0) = \lim_n \frac{t_0}{t_0 + |a_n - b_n|} = \frac{t_0}{t_0 + |a - b|} > 0,$$

a contradiction.

Therefore, M satisfies $(c2)$.

Consequently, $(c1) - (c3)$ constitute an independent axiomatic system.

4. Completable strong fuzzy metrics

In this section we will show that condition $(c3)$ in Theorem 1.1 can be omitted when $(X, M, *)$ is a strong fuzzy metric space.

We begin this section giving five lemmas.

Lemma 4.1. *Let $(X, M, *)$ be a strong fuzzy metric space and let $\{a_n\}$, $\{b_n\}$ be two Cauchy sequences in X. For each $t > 0$, the sequence $\{M(a_n, b_n, t)\}_n$ converges in $[0, 1]$ with the usual topology of $\mathbb{R}$ restricted to $[0, 1]$.*

Proof.

Fix $t > 0$. Let $\{a_n\}$ and $\{b_n\}$ be two Cauchy sequences in X. Since $[0,1]$ is compact the sequence $M(a_n, b_n, t) \in [0,1]$ has a subsequence $\{M(a_{n_k}, b_{n_k}, t)\}_k$ that converges to some $c \in [0,1]$. We will see that $\{M(a_n, b_n, t)\}_n$ converges to c.

Contrary, suppose that $\{M(a_n, b_n, t)\}_n$ does not converge to c. Then, we can find a subsequence $\{M(a_{m_i}, b_{m_i}, t)\}_i$ of $\{M(a_n, b_n, t)\}_n$ converging to $a \in [0,1]$, with $a \neq c$.

Now, since M is strong, for each $i, k \in \mathbb{N}$ we have that

$$M(a_{n_k}, b_{n_k}, t) \geq M(a_{n_k}, a_{m_i}, t) * M(a_{m_i}, b_{m_i}, t) * M(b_{m_i}, b_{n_k}, t)$$

and taking limit as $i, k \to \infty$, we have that

$$\lim_k M(a_{n_k}, b_{n_k}, t) \geq \lim_i M(a_{m_i}, b_{m_i}, t).$$

With a similar argument, we can also obtain

$$\lim_i M(a_{m_i}, b_{m_i}, t) \geq \lim_k M(a_{n_k}, b_{n_k}, t).$$

So, $c = \lim_k M(a_{n_k}, b_{n_k}, t) = \lim_i M(a_{m_i}, b_{m_i}, t) = a$, a contradiction.

Therefore, $\lim_n M(a_n, b_n, t) = c$.

□

Lemma 4.2. *Let $(X, M, *)$ be a fuzzy metric space, let $\{a_n\}$ be a Cauchy sequence in X and let $\{t_n\}$ be a strictly increasing (decreasing) sequence of positive real numbers converging to $t_0 > 0$ (for the usual topology of $\mathbb{R}$). Then $\lim_{n,m} M(a_n, a_m, t_n) = 1$.*

Proof. It is immediate.

□

Lemma 4.3. *Let $(X, M, *)$ be a strong fuzzy metric space. Let $\{a_n\}$, $\{b_n\}$ be two Cauchy sequences in X and let $\{t_n\}$ be a strictly increasing (decreasing) sequence of positive real numbers converging to $t_0 > 0$ (for the usual topology of $\mathbb{R}$). Then, the sequence $\{M(a_n, b_n, t_n)\}_n$ converges in $[0,1]$, with the usual topology of $\mathbb{R}$ restricted to $[0,1]$.*

Proof. Let $\{a_n\}$, $\{b_n\}$ be two Cauchy sequences in X and let $\{t_n\}$ be a strictly increasing sequence of positive real numbers converging to $t_0 > 0$. Consider the sequence $\{M(a_n, b_n, t_n)\}_n \subset [0,1]$. Since $[0,1]$ is compact then, there exists a subsequence $\{M(a_{n_k}, b_{n_k}, t_{n_k})\}_k$ of $\{M(a_n, b_n, t_n)\}_n$ converging to $c \in [0,1]$.

Suppose that $\{M(a_n, b_n, t_n)\}_n$ does not converge to c. Then, we can find a subsequence $\{M(a_{m_i}, b_{m_i}, t_{m_i})\}_i$ of $\{M(a_n, b_n, t_n)\}_n$ converging to $a \in [0,1]$, with $a \neq c$.

Suppose, without loss of generality, that $a > c$. We will construct, by induction, two subsequences $\{M(a_{n_{k_l}}, b_{n_{k_l}}, t_{n_{k_l}})\}_l$ and $\{M(a_{m_{i_j}}, b_{m_{i_j}}, t_{m_{i_j}})\}_j$ of $\{M(a_{n_k}, b_{n_k}, t_{n_k})\}_k$ and $\{M(a_{m_i}, b_{m_i}, t_{m_i})\}_i$, respectively, as follows.

Take $m_{i_1} = m_1 \in \mathbb{N}$. We can choose $n_{k_1} \in \mathbb{N}$ such that $n_{k_1} > m_{i_1}$ and $t_{n_{k_1}} > t_{m_{i_1}}$ (since $\{t_{n_k}\}$ is strictly increasing). By Remark 2.2 and using that M is strong, we have that

$$M(a_{n_{k_1}}, b_{n_{k_1}}, t_{n_{k_1}}) \geq M(a_{n_{k_1}}, b_{n_{k_1}}, t_{m_{i_1}}) \geq$$

$$M(a_{n_{k_1}}, a_{m_{i_1}}, t_{m_{i_1}}) * M(a_{m_{i_1}}, b_{m_{i_1}}, t_{m_{i_1}}) * M(b_{m_{i_1}}, b_{n_{k_1}}, t_{m_{i_1}}).$$

Now, we choose $m_{i_2} \in \mathbb{N}$ such that $m_{i_2} > n_{k_1}$. Given m_{i_2}, we can choose $n_{k_2} \in \mathbb{N}$ such that $n_{k_2} > m_{i_2}$ and $t_{n_{k_2}} > t_{m_{i_2}}$. By Remark 2.2 and using that M is strong, we have that

$$M(a_{n_{k_2}}, b_{n_{k_2}}, t_{n_{k_2}}) \geq M(a_{n_{k_2}}, b_{n_{k_2}}, t_{m_{i_2}}) \geq$$

$$M(a_{n_{k_2}}, a_{m_{i_2}}, t_{m_{i_2}}) * M(a_{m_{i_2}}, b_{m_{i_2}}, t_{m_{i_2}}) * M(b_{m_{i_2}}, b_{n_{k_2}}, t_{m_{i_2}}).$$

Therefore, by induction on j we have that

$$M(a_{n_{k_j}}, b_{n_{k_j}}, t_{n_{k_j}}) \geq$$

$$M(a_{n_{k_j}}, a_{m_{i_j}}, t_{m_{i_j}}) * M(a_{m_{i_j}}, b_{m_{i_j}}, t_{m_{i_j}}) * M(b_{m_{i_j}}, b_{n_{k_j}}, t_{m_{i_j}}).$$

Taking limit as $j \to \infty$, by Lemma 4.2 we have that.

$$c = \lim_j M(a_{n_{k_j}}, b_{n_{k_j}}, t_{n_{k_j}}) \geq \lim_j M(a_{m_{i_j}}, b_{m_{i_j}}, t_{m_{i_j}}) = a,$$

a contradiction.

Therefore, $\lim_n M(a_n, b_n, t_n) = c$.

If $\{t_n\}$ is strictly decreasing, it is proved in a similar way.

□

Lemma 4.4. *Let $(X, M, *)$ be a strong fuzzy metric space. Let $\{a_n\}$, $\{b_n\}$ be two Cauchy sequences in X and let $\{t_n\}$, $\{s_n\}$ be two strictly increasing (decreasing) sequences of positive real numbers converging to $t_0 > 0$ (for the usual topology of $\mathbb{R}$). Then, $\lim_n M(a_n, b_n, t_n) = \lim_n M(a_n, b_n, s_n)$.*

Proof. Let $\{a_n\}$, $\{b_n\}$ be two Cauchy sequences in X and let $\{t_n\}$, $\{s_n\}$ be two strictly increasing sequence of positive real numbers converging to $t_0 > 0$. By Lemma 4.3, there exist $a, c \in [0, 1]$ such that $\lim_n M(a_n, b_n, t_n) = a$ and $\lim_n M(a_n, b_n, s_n) = c$. Contrary, suppose that $\lim_n M(a_n, b_n, t_n) \neq \lim_n M(a_n, b_n, s_n)$. Suppose, without loss of generality, that $a < c$.

In a similar way that in the proof of the above lemma, we will construct two subsequences $\{M(a_{n_k}, b_{n_k}, t_{n_k})\}_k$ and $\{M(a_{m_i}, b_{m_i}, s_{m_i})\}_i$ of $\{M(a_n, b_n, t_n)\}_n$ and $\{M(a_n, b_n, s_n)\}_n$, respectively, such that $t_{n_k} > s_{m_k}$ for all $k \in \mathbb{N}$ and we have that

$$M(a_{n_k}, b_{n_k}, t_{n_k}) \geq$$
$$M(a_{n_k}, a_{m_k}, s_{m_k}) * M(a_{m_k}, b_{m_k}, s_{m_k}) * M(b_{m_k}, b_{n_k}, s_{m_k})$$

for each $k \in \mathbb{N}$.

Taking limit as $k \to \infty$, by Lemma 4.2 we have that

$$a = \lim_k M(a_{n_k}, b_{n_k}, t_{n_k}) \geq \lim_k M(a_{m_k}, b_{m_k}, s_{m_k}) = c,$$

a contradiction.

Therefore, $\lim_n M(a_n, b_n, t_n) = \lim_n M(a_n, b_n, s_n)$.

The case in which $\{t_n\}$ and $\{s_n\}$ are strictly decreasing is proved in a similar way.

□

Lemma 4.5. *Let $(X, M, *)$ be a strong fuzzy metric space. Let $\{a_n\}$, $\{b_n\}$ be two Cauchy sequences in X and let $\{t_n\}$ be a strictly increasing (decreasing) sequence of positive real numbers converging to $t_0 > 0$ (for the usual topology of $\mathbb{R}$). Then, $\lim_n M(a_n, b_n, t_n) = \lim_n M(a_n, b_n, t_0)$.*

Proof. Let $\{a_n\}$, $\{b_n\}$ be two Cauchy sequences in X and let $\{t_n\}$ be a strictly increasing sequence of positive real numbers converging to $t_0 > 0$.

By Lemma 4.3, there exists $a \in [0, 1]$ such that $\lim_n M(a_n, b_n, t_n) = a$ and by Lemma 4.1, there exists $c \in [0, 1]$ such that $\lim_n M(a_n, b_n, t_0) = c$. Note that, by Remark 2.2, since $\{t_n\}$ is strictly increasing converging to t_0, we have that for each $n \in \mathbb{N}$ we have that $M(a_n, b_n, t_n) \leq M(a_n, b_n, t_0)$ and so $a \leq c$.

Since $\lim_n M(a_n, b_n, t_0) = c$, for each $\epsilon \in]0,1[$, with $\epsilon < c$, we can find $n_\epsilon \in \mathbb{N}$ such that $M(a_{n_\epsilon}, b_{n_\epsilon}, t_0) \in]c - \epsilon/2, c + \epsilon/2[$. By axiom $(GV5)$ we can find $\delta_{n_\epsilon} > 0$ such that $M(a_{n_\epsilon}, b_{n_\epsilon}, t) \in]c - \epsilon, c + \epsilon[$ for each $t \in]t_0 - \delta_{n_\epsilon}, t_0[$.

Suppose that $c > a$. Taking into account the last paragraph, we will construct a sequence $\{M(a_{n_k}, b_{n_k}, s_k)\}_k$, where $\{a_{n_k}\}$ and $\{b_{n_k}\}$ are subsequences of $\{a_n\}$ and $\{b_n\}$, respectively, converging to c, as follows.

Let $i_1 \in \mathbb{N}$, with $\frac{1}{i_1} < \min\{c, t_0\}$, then there exist $n_1 \in \mathbb{N}$ and $s_1 \in]t_0 - \frac{1}{i_1}, t_0[$ such that $M(a_{n_1}, b_{n_1}, s_1) > c - \frac{1}{i_1}$. Choose $i_2 \in \mathbb{N}$, with $\frac{1}{i_2} < t_0 - s_1$, then we can find $n_2 \in \mathbb{N}$, with $n_2 > n_1$ and $s_2 \in]t_0 - \frac{1}{i_2}, t_0[$, such that $M(a_{n_2}, b_{n_2}, s_2) > c - \frac{1}{i_2}$. Thus, in this way by induction on k, we construct the sequence $\{M(a_{n_k}, b_{n_k}, s_k)\}_k$, which obviously satisfies $\lim_k M(a_{n_k}, b_{n_k}, s_k) = c$. On the other hand, $\{s_k\}$ is a strictly increasing sequence of positive real numbers converging to t_0. Therefore, by Lemma 4.4 $\lim_k M(a_{n_k}, b_{n_k}, r_k) = c$ for each strictly increasing sequence $\{r_k\}$ of positive real numbers converging to t_0. In particular, if we consider the subsequence $\{t_{n_k}\}$ of $\{t_n\}$, then $\lim_k M(a_{n_k}, b_{n_k}, t_{n_k}) = c$, a contradiction, since $\lim_n M(a_n, b_n, t_n) = a < c$.

Therefore, $\lim_n M(a_n, b_n, t_n) = c$.

The case of $\{t_n\}$ strictly decreasing is proved in a similar way.

□

Theorem 4.6. *Let $(X, M, *)$ be a strong fuzzy metric space, and let $\{a_n\}$, $\{b_n\}$ be two Cauchy sequences in X. Then the assignment*

$$t \to \lim_n M(a_n, b_n, t), \text{ for each } t > 0$$

is a continuous function on $]0, \infty[$ provided with the usual topology of $\mathbb{R}$.

Proof. Let $\{a_n\}$ and $\{b_n\}$ be two Cauchy sequences in X. By Lemma 4.1, the assignment $t \to \lim_n M(a_n, b_n, t)$ for each $t > 0$, is a well-defined function on $]0, \infty[$ to $[0, 1]$.

Next, we will see that this function is continuous. First we see that for each $t > 0$ the mentioned function is left-continuous.

Fix $t_0 > 0$. By Lemma 4.1, we have that there exists $c \in [0, 1]$ such that $\lim_n M(a_n, b_n, t_0) = c$. We distinguish two cases:

(1) Suppose that $c = 0$. By Remark 2.2 and Lemma 4.1 we have that $\lim_n M(a_n, b_n, s) = 0$ for all $s \in]0, t_0[$. So, the function $t \to \lim_n M(a_n, b_n, t)$ is left-continuous at t_0.

(2) Suppose that $c \in]0,1]$. Contrary, suppose the function $t \to \lim_n M(a_n, b_n, t)$ is not left-continuous at t_0.

Then, there exists $\epsilon_0 \in]0,1[$ such that for each $\delta \in]0, t_0[$ we can find $t_\delta \in]t_0 - \delta, t_0[$ such that $b_\delta = \lim_n M(a_n, b_n, t_\delta) \notin]c - \epsilon_0, c + \epsilon_0[$. Note that, by Remark 2.2, $b_\delta \leq c$ and so $b_\delta < c - \epsilon_0$.

On the other hand, given $t_\delta \in]t_0 - \delta, t_0[$, since $\lim_n M(a_n, b_n, t_\delta) = b_\delta < c - \epsilon_0$, for $\epsilon_0/2$ we can find $n(\delta) \in \mathbb{N}$ such that $M(a_n, b_n, t_\delta) \in]b_\delta - \epsilon_0/2, b_\delta + \epsilon_0/2[$ for each $n \geq n(\delta)$. Therefore, $M(a_n, b_n, t_\delta) < c - \epsilon_0/2$ for each $n \geq n(\delta)$.

Now, we will construct a sequence $\{M(a_{n_k}, b_{n_k}, t_k)\}_k$, where $\{a_{n_k}\}$ and $\{b_{n_k}\}$ are subsequences of $\{a_n\}$ and $\{b_n\}$, respectively, as follows.

Consider $i_1 \in \mathbb{N}$, with $\frac{1}{i_1} < t_0$. We can find $t_1 \in]t_0 - \frac{1}{i_1}, t_0[$ such that $\lim_n M(a_n, b_n, t_1) < c - \epsilon_0$. Then, we can find $n(i_1) \in \mathbb{N}$ such that $M(a_n, b_n, t_1) < c - \epsilon_0/2$ for each $n \geq n(i_1)$. We choose $n_1 = n(i_1)$.

Consider now, $i_2 \in \mathbb{N}$, with $\frac{1}{i_2} \in]t_1, t_0[$. We can find $t_2 \in]t_0 - \frac{1}{i_2}, t_0[$ such that $\lim_n M(a_n, b_n, t_2) < c - \epsilon_0$. Then, we can find $n(i_2) \in \mathbb{N}$ such that $M(a_n, b_n, t_2) < c - \epsilon_0/2$ for each $n \geq n(i_2)$. We choose $n_2 \geq n(i_2)$, with $n_2 > n_1$.

So, by induction on k we construct the sequence $\{M(a_{n_k}, b_{n_k}, t_k)\}_k$, where $\{a_{n_k}\}$ and $\{b_{n_k}\}$ are subsequences of $\{a_n\}$ and $\{b_n\}$, respectively, such that $M(a_{n_k}, b_{n_k}, t_k) < c - \epsilon_0/2$ for each $k \in \mathbb{N}$. Also, $\{t_k\}$ is a strictly increasing sequence of positive real numbers converging to t_0. Therefore, by Lemma 4.5, we have that $\lim_k M(a_{n_k}, b_{n_k}, t_k) = \lim_k M(a_{n_k}, b_{n_k}, t_0) = \lim_n M(a_n, b_n, t_0) = c$, a contradiction.

So, the above assignment is a left-continuous function at t_0.

In a similar way it is proved that $t \to \lim_n M(a_n, b_n, t)$ is right-continuous at t_0 using a strictly decreasing sequence $\{t_n\}$ converging to t_0 and thus it is continuous at t_0.

Hence, the assignment $t \to \lim_n M(a_n, b_n, t)$ is a continuous function on $]0, \infty[$.

□

Theorem 4.7. *A strong fuzzy metric space $(X, M, *)$ is completable if and only if for each pair of Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in X the following conditions are fulfilled:*

(c1) $\lim_n M(a_n, b_n, s) = 1$ *for some* $s > 0$ *implies* $\lim_n M(a_n, b_n, t) = 1$ *for all* $t > 0$.

(c2) $\lim_n M(a_n, b_n, t) > 0$ *for all* $t > 0$.

Proof. The proof is immediate using Theorem 4.6 and Theorem 1.1. □

By Theorem 2.8 and the fact that the minimum t-norm is integral, the following corollaries are immediate.

Corollary 4.8. *Let* $(X, M, *)$ *be a strong fuzzy metric space and suppose that* $*$ *is integral. Then* $(X, M, *)$ *is completable if and only if for each pair of Cauchy sequences* $\{a_n\}$ *and* $\{b_n\}$ *in* X *the condition* $(c1)$ *is satisfied.*

Corollary 4.9. *Let* $(X, M, *)$ *be a fuzzy ultrametric space. Then* $(X, M, *)$ *is completable if and only if for each pair of Cauchy sequences* $\{a_n\}$ *and* $\{b_n\}$ *in* X *the condition* $(c1)$ *is satisfied.*

Remark 4.10. *We cannot remove the condition that* $*$ *is integral in Corollary* **??** *as shows Example 3.2. In addition, the fuzzy metric of Example 3.1 is a non-completable fuzzy ultrametric which does not satisfy* $(c1)$.

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