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The character of free topological groups I

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Abstract. A systematic analysis is made of the character of the free and free abelian topological groups on uniform spaces and on topological spaces. In the case of the free abelian topological group on a uniform space, expressions are given for the character in terms of simple cardinal invariants of the family of uniformly continuous pseudometrics of the given uniform space and of the uniformity itself. From these results, others follow on the basis of various topological assumptions. Amongst these: (i) if X is a compact Hausdorff space, then the character of the free abelian topological group on X lies between $w(X)$ and $w(X)^{\aleph_0}$, where $w(X)$ denotes the weight of X; (ii) if the Tychonoff space X is not a P-space, then the character of the free abelian topological group is bounded below by the "small cardinal" $\mathfrak{d}$; and (iii) if X is an infinite compact metrizable space, then the character is precisely $\mathfrak{d}$.
In the non-abelian case, we show that the character of the free abelian topological group is always less than or equal to that of the corresponding free topological group, but the inequality is in general strict. It is also shown that the characters of the free abelian and the free topological groups are equal whenever the given uniform space is ω-narrow.
A sequel to this paper analyses more closely the cases of the free and free abelian topological groups on compact Hausdorff spaces and metrizable spaces.

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1. Introduction

In 1948, Graev [5] proved that the free topological group $F(X)$ and the free abelian topological group $A(X)$ on a Tychonoff space X are metrizable only if X is discrete, in which case the groups are themselves discrete. For our present purposes, we may rephrase Graev's result as saying that when X is a non-discrete Tychonoff space, the groups $F(X)$ and $A(X)$ have uncountable character (= minimal size of a base at the identity of the group). It appears that no other estimates of the characters of these groups (except for those valid in the context of general topological groups) have been found to date.

In this paper and its sequel [14], we investigate these characters systematically and in some detail. Most of our results are in fact for free and free abelian topological groups on uniform spaces, since this gives maximum generality and allows the derivation at will of bounds for the characters of free and free abelian groups over topological spaces.

In the abelian case, the free topology has a rather straightforward description in terms of the family of uniformly continuous pseudometrics on the given uniform space, and another in terms of the given uniformity itself, and our initial results express the character of the corresponding group in terms of simple cardinal invariants of the family of pseudometrics (Theorem 2.3) or of the uniformity (Theorem 2.9). An immediate corollary of these results is that if X is a compact Hausdorff space, then the character of $A(X)$ lies between $w(X)$ and $w(X)^{\aleph_0}$, where $w(X)$ denotes the weight of X.

The well known "small cardinal" $\mathfrak{d}$ (see [3] and our discussion below) plays a role in several other results. For example, we show that if the Tychonoff space X is not a P-space, then $\mathfrak{d}$ is a lower bound for the character of $A(X)$ (Corollary 2.16); this applies in particular if X contains an infinite compact subset or a proper dense Lindelöf subspace. Further, if X is an infinite compact metrizable space, then the character of $A(X)$ is precisely $\mathfrak{d}$ (Corollary 2.22).

In the non-abelian case, the situation is intrinsically more complex and description of the free topology in terms of pseudometrics or entourages, for example, is now far from straightforward (see [21] and [16]). Our main results on the characters in the non-abelian case make use of a new description of a neighborhood base at the identity in the free topological group on an arbitrary uniform space (Theorem 3.6). While it is easy to see that the character of the free abelian topological group is always less than or equal to that of the corresponding free topological group (Lemma 3.1), the inequality is in general strict, and the two characters may indeed differ arbitrarily largely (see [14]). Using our new description of the topology of the free topological group, however, we show that the characters of the free abelian and the free (non-abelian) topological groups are equal whenever the underlying uniform space is ω-narrow (Theorem 3.15).

The sequel [14] to this paper analyses more closely the cases of the free and free abelian topological groups on compact Hausdorff spaces and metrizable spaces.

1.1. **Notation and terminology.** We denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers.

All topological spaces are hypothesised implicitly to be completely regular, but are not taken to be Hausdorff (and therefore Tychonoff) without explicit indication. Similarly, our uniform spaces are not taken to be Hausdorff (or separated) unless this is explicitly signalled.

We next establish our conventions for certain cardinal invariants of topological spaces and uniform spaces. In some formulations, such as that of [8], for example, cardinal invariants are taken always to be infinite, that is, to have a minimum value of $\aleph_0$, because this tends to simplify the statements of theorems. For us, however, it is convenient not to follow this convention. Thus, if x is a point in a space X, then $\chi(x, X)$ denotes the minimum cardinal of a local base at x, and then $\chi(X)$, the character of X, is the supremum of the values $\chi(x, X)$ for $x \in X$. More generally, if Y is a subspace of X, then we write $\chi(Y, X)$ for the minimum cardinal of a base at Y in X. We introduce the *ad hoc* notation $\chi_\Delta(X)$ to denote the least cardinal of a basis at the diagonal Δ in $X \times X$; that is, $\chi_\Delta(X) = \chi(\Delta, X \times X)$. For a space X, the weight $w(X)$ of X is defined to be the smallest cardinal of an open base for X, and for a uniform space $(X,\mathcal{U})$, we denote by $w(X,\mathcal{U})$ the least cardinal of a base of $(X,\mathcal{U})$.

A space X is a k_ω*-space* if there exists a sequence of compact subsets $X_n \subseteq X$ for $n \in \mathbb{N}$ such that $X = \bigcup_{n=1}^\infty X_n$ and such that X has the weak topology with respect to the family $\{X_n : n \in \mathbb{N}\}$ (that is, such that $U \subseteq X$ is open in X if and only if $U \cap X_n$ is open in X_n for each $n \in \mathbb{N}$). In this situation, we call the collection $\{X_n : n \in \mathbb{N}\}$, or the corresponding expression $X = \bigcup_{n=1}^\infty X_n$ for X, a k_ω*-decomposition* of X. We may take the sets X_n without loss of generality to be non-decreasing.

If $\mathcal{U}$ is a uniformity on a set X, and if $U \in \mathcal{U}$ and $n \geq 1$, we use nU to denote the n-fold relational composite $U \circ U \circ \cdots \circ U$. If $-U$ denotes the relational inverse of U, then we call U *symmetric* if $U = -U$. For $x \in X$ and $U \in \mathcal{U}$, we denote by $B(x, U)$ the set $\{y \in X : (x, y) \in U\}$.

If $(X,\mathcal{U})$ is a uniform space and $\tau \geq \aleph_0$ is a cardinal, then we say that $(X,\mathcal{U})$ is τ*-narrow* if for each $U \in \mathcal{U}$, there is a set $\{x_\alpha : \alpha < \tau\} \subseteq X$ such that $X = \bigcup_{\alpha<\tau} B(x_\alpha, U)$. Similarly, we say that a topological group G is τ*-narrow* if G can be covered by τ many translates of every neighborhood of the identity (the groups with this property was called τ*-bounded* in [6]).

For a non-empty set X, we use $F_a(X)$ and $A_a(X)$ to denote the abstract free group and abstract free abelian group on X. If $n \in \omega$, then $F_n(X)$ and $A_n(X)$ are the subsets of $F_a(X)$ and $A_a(X)$, respectively, which consist of all elements whose length with respect to the basis X does not exceed n.

If X is a space, then $F(X)$ and $A(X)$ stand respectively for the free topological group and free abelian topological group on X. In this case, $F_n(X)$ and $A_n(X)$ refer to the corresponding subspaces of $F(X)$ and $A(X)$, respectively. Given a subset Y of X, $F(Y, X)$ denotes the subgroup of $F(X)$ generated by Y, and $A(Y, X)$ has a similar meaning.

If $(X,\mathcal{U})$ is a uniform space, then $F(X,\mathcal{U})$ and $A(X,\mathcal{U})$ stand for the free topological group and free abelian topological group on $(X,\mathcal{U})$, and other related notations follow those already introduced for the free and free abelian topological groups on a topological space.

If X and Y are sets, we write ${}^X Y$ for the set of functions from X to Y. If α and β are cardinals, we write cardinal exponentiation in the form β^α; thus, $\left|{}^X Y\right| = |Y|^{|X|}$. Also, $\mathfrak{c} = 2^{\aleph_0}$ is the power of the continuum.

In any topological group G, we denote by $\mathcal{N}(e)$ the family of all open neighborhoods of the neutral element e in G.

1.2. **Quasi-ordered sets.** Many of the arguments below make use of the notion of a quasi-ordered set, and related ideas. We establish here the relevant terminology and notation.

We say that a pair $(P,\le)$ is a *quasi-ordered set* if $\le$ is a reflexive transitive relation on the set P. If $(P,\le)$ has the additional property of antisymmetry, then it is a *partially ordered set.* A set $D\subseteq P$ is called *dominating* or *cofinal* in the quasi-ordered set $(P,\le)$ if for every $p\in P$ there exists $q\in D$ such that $p\le q$. Similarly, a subset E of P is said to be *dense* in $(P,\le)$ if for every $p\in P$ there exists $q\in E$ with $q\le p$. The minimal cardinality of a dominating family in $(P,\le)$ is denoted by $D(P,\le)$, while we use $d(P,\le)$ for the minimal cardinality of a dense set in $(P,\le)$. The notions of dominating and dense sets are dual: if a set S is dense in $(P,\le)$, then it is dominating in $(P,\ge)$ and vice versa. Therefore, $d(P,\le) = D(P,\ge)$ and $D(P,\le) = d(P,\ge)$. Note in any topological group G we have $d(\mathcal{N}(e),\subseteq) = \chi(G)$.

If $(P,\le)$ and $(Q,\ll)$ are quasi-ordered sets, then a mapping $f\colon P\to Q$ is called *order-preserving* if $x\le y$ implies $f(x)\ll f(y)$, where $x,y\in P$. Similarly, f is *order-reversing* if $x\le y$ implies $f(x)\gg f(y)$. If the mapping f is a bijection between P and Q and if f and f^{-1} both are order-preserving, then f is called an (*order-*)*isomorphism* of $(P,\le)$ onto $(Q,\ll)$.

Lemma 1.1. *Let $(P,\le)$ and $(Q,\ll)$ be quasi-ordered sets, and let $f\colon P\to Q$ be an order-preserving mapping. If $f(P)$ is a dominating set in Q, then $D(Q)\le D(P)$.*

Proof. Let D be an arbitrary dominating set in P. For each $q\in Q$, there exists $p\in P$ such that $q\le f(p)$. Also, there exists $d\in D$ such that $p\le d$, from which we have $q\le f(p)\le f(d)$. Hence $f(D)$ is a dominating set in Q, and so $D(Q)\le |f(D)|\le |D|$. Finally, taking D to be a dominating set in P of minimal cardinality gives $D(Q)\le D(P)$, as required. □

Observe that because of the duality noted above between dominating sets and dense sets, there are dualised versions of the lemma: a version for dense sets rather than dominating sets, and others for order-reversing mappings rather than order-preserving ones. We will make frequent use of the lemma and its unstated variants, usually without explicit reference.

We will deal later with many different quasi-ordered sets, most, though not all, of which will in fact be partially ordered sets. Generally, we will give each ordering distinctive notation by using an appropriate subscript, since some arguments make use of several quasi-orderings simultaneously. If $\ll$ is an ordering on a set X, then we will usually denote the order defined coordinatewise on a set of functions ${}^Y X$ by attaching an asterisk as a superscript, as in $\ll^*$. An exception is when $X = \omega$, when the ordering defined coordinatewise using $\leq$ on sets such as ${}^\omega\omega$ and ${}^\mathbb{N}\omega$ will again be denoted just by $\leq$.

Consider ${}^\omega\omega$, the collection of all functions from ω to ω. Then following [3], we define two quasi-orders $\leq^*$ and $\leq$ on ${}^\omega\omega$, by specifying that if $f, g \in {}^\omega\omega$, then $f \leq^* g$ if $f(n) \leq g(n)$ for all except finitely many $n \in \omega$, and $f \leq g$ if $f(n) \leq g(n)$ for all $n \in \omega$; the quasi-order $\leq$ is of course in fact a partial order. (The use of the asterisk in $\leq^*$ is inconsistent with the notational convention just established, but we will not in fact use this ordering again after this paragraph.) The least cardinal of a dominating set in the quasi-ordered set $({}^\omega\omega, \leq^*)$ is denoted by $\mathfrak{d}$, and in the partially ordered set $({}^\omega\omega, \leq)$ by $\mathfrak{d}_1$. It is shown in Theorem 3.6 of [3] that $\mathfrak{d} = \mathfrak{d}_1$. It is also known that $\aleph_1 \leq \mathfrak{d} \leq \mathfrak{c}$, but that the value of $\mathfrak{d}$ depends on extra axioms of set theory [3]. Below, we will find it most useful to use the characterization of $\mathfrak{d}$ as the least cardinal of a dominating set with respect to the relation $\leq$: thus, $\mathfrak{d} = D({}^\omega\omega, \leq)$.

2. The character of free abelian topological groups

Graev's proof in [5] of the fact that the free topological group on a Tychonoff space X is Hausdorff proceeds by a construction which extends each continuous pseudometric on X to an invariant pseudometric on the underlying abstract free group $F_a(X)$, and an argument which shows that the group topology induced by all the extensions (referred to as *Graev's topology*) is Hausdorff and weaker than the free topology. It is well known that Graev's topology is only equal to the free topology in somewhat pathological cases [9, 19, 22]. In the abelian case, a parallel construction was also outlined by Graev, and in this case Graev's topology is always the free topology (see [13, 11, 20]; cf. also 4.4 and 4.8 of [10]). (It seems unlikely that this fact was unknown to Graev, but it is not mentioned in his paper.)

In what follows, we deal mostly with free and free abelian topological groups on uniform spaces, for convenience and generality, deducing applications to free and free abelian topological groups on topological spaces when appropriate. We therefore need the uniform space analog of the result just noted: that Graev's pseudometrics generate a neighborhood base at the neutral element of the free abelian topological group on a uniform space.

For a given pseudometric d on a set X, we denote by $\widehat{d}$ and $\widehat{d}_A$ Graev's extension [5] of d to the abstract groups $F_a(X)$ and $A_a(X)$, respectively. Note that algebraically the free uniform groups $F(X,\mathcal{U})$ and $A(X,\mathcal{U})$ are $F_a(X)$ and $A_a(X)$, respectively [12, 15]. If $(X,\mathcal{U})$ is a uniform space and d is a pseudometric on X, then we write

$$V_d = \{g \in A(X,\mathcal{U}) : \widehat{d}_A(g, 0) < 1\},$$

where 0 is the neutral element of $A(X,\mathcal{U})$. The following theorem is the uniform space analog of the result of [13, 20]; the arguments in the topological case apply with minimal adjustment in the uniform space case, and we therefore omit the proof.

Theorem 2.1. *Let $(X,\mathcal{U})$ be a uniform space. Then for every uniformly continuous pseudometric d on $(X,\mathcal{U})$ the set V_d is open in $A(X,\mathcal{U})$, and for every neighborhood U of the neutral element in $A(X,\mathcal{U})$ there exists a uniformly continuous pseudometric d on $(X,\mathcal{U})$ such that $V_d \subseteq U$.*

Instead of using uniformly continuous pseudometrics, one can directly use the entourages $U \in \mathcal{U}$ belonging to a uniform space $(X,\mathcal{U})$ to construct a neighborhood base at the identity of $A(X,\mathcal{U})$. The theorem below gives a simple, explicit description in these terms of the topology of the free abelian topological group on a uniform space, equivalent in a sense to Theorem 2.1. This result has certainly been known since the early 1980s, when the first description of the topology of the (non-abelian) free topological group on a uniform space was published [16]. Though the paper [20] contains a result equivalent to ours stated in the language of pseudometrics, the first occurrence of essentially our result in the literature appears to have been in [26], though it is formulated there in a context less general than ours.

First, we introduce the following notation. If $(X,\mathcal{U})$ is a uniform space and if $\langle U_0, U_1, \ldots\rangle$ is a sequence of elements of $\mathcal{U}$, then we denote by $N(\langle U_0, U_1, \ldots\rangle)$ the set of all elements of $A(X,\mathcal{U})$ of the form

$$x_0 - y_0 + x_1 - y_1 + \cdots + x_n - y_n,$$

where $n \in \omega$ and where $(x_i, y_i) \in U_i$ for $i = 0, 1, \ldots, n$.

Theorem 2.2. *Let $(X,\mathcal{U})$ be a uniform space. Then the collection of all sets of the form $N(\langle U_0, U_1, \ldots\rangle)$, as $\langle U_0, U_1, \ldots\rangle$ runs over all sequences of elements of $\mathcal{U}$, is an open basis at the neutral element in $A(X,\mathcal{U})$.*

Denote by $\mathcal{P}(X,\mathcal{U})$ the family of all uniformly continuous pseudometrics on $(X,\mathcal{U})$ bounded by 1. For $d_1, d_2 \in \mathcal{P}(X,\mathcal{U})$, we write $d_1 \le d_2$ if $d_1(x,y) \le d_2(x,y)$ for all $x, y \in X$. We express our first result on the character of $A(X,\mathcal{U})$) in terms of the partially ordered set $(P(X,\mathcal{U}), \le)$.

Theorem 2.3. *If $(X,\mathcal{U})$ is a uniform space, then $\chi(A(X,\mathcal{U})) = D(\mathcal{P}(X,\mathcal{U}), \le)$.*

Proof. By Theorem 2.1, there exists a natural correspondence between the family $\mathcal{P}(X,\mathcal{U})$ and a base at the neutral element 0 of $A(X,\mathcal{U})$. In fact, the mapping $d \mapsto V_d$ from $(\mathcal{P}(X,\mathcal{U}), \le)$ to the partially ordered set $(\mathcal{N}(0), \supseteq)$ of open neighborhoods of 0 in $A(X,\mathcal{U})$ ordered by reverse inclusion is order-preserving and maps $\mathcal{P}(X,\mathcal{U})$ to a base at 0 in $A(X,\mathcal{U})$, that is, a dominating set in $(\mathcal{N}(0), \supseteq)$. This immediately implies (by Lemma 1.1) that $\chi(A(X,\mathcal{U})) \le D(\mathcal{P}(X,\mathcal{U}), \le)$.

Suppose that a subset $\mathcal{Q} \subseteq \mathcal{P}(X,\mathcal{U})$ is such that $\{V_d : d \in \mathcal{Q}\}$ is a base at the neutral element in $A(X,\mathcal{U})$. We claim that for every $\varrho \in \mathcal{P}(X,\mathcal{U})$ there exists

$d \in Q$ such that $\varrho \leq 2d$. Indeed, given $\varrho \in \mathcal{P}(X,\mathcal{U})$, we can find $d \in \mathcal{Q}$ such that $V_d \subseteq V_\varrho$. Let $x, y \in X$. If $d(x,y) < 1$, then $x-y \in V_d \subseteq V_\varrho$, whence $\varrho(x,y) < 1$. Similarly, if $n \in \mathbb{N}$ and $d(x,y) < 2^{-n}$, then $\widehat{d}_A(0, 2^n(x-y)) = 2^n d(x,y) < 1$ by the linearity of the pseudometric $\widehat{d}_A$ (see [23]). Therefore, $2^n(x-y) \in V_d \subseteq V_\varrho$, whence $\varrho(x,y) < 2^{-n}$. We have thus proved that $d(x,y) < 2^{-n}$ implies $\varrho(x,y) < 2^{-n}$, for $n = 0, 1, \ldots$. In particular, $d(x,y) = 0$ implies $\varrho(x,y) = 0$. Therefore, the inequality $\varrho(x,y) \leq 2d(x,y)$ holds if $d(x,y) = 0$. It is also obvious that the same inequality holds if $d(x,y) = 1$. If $0 < d(x,y) < 1$, choose $n \in \omega$ such that $2^{-n-1} \leq d(x,y) < 2^{-n}$. Then $\varrho(x,y) < 2^{-n}$, so that $\varrho(x,y) \leq 2d(x,y)$, and the inequality again holds. Thus, $\varrho \leq 2d$, proving our claim.

For $d \in \mathcal{Q}$, define $d^* \in \mathcal{P}(X,\mathcal{U})$ by setting $d^* = \min\{2d, 1\}$, and consider the family $\mathcal{Q}^* = \{d^* : d \in \mathcal{Q}\} \subseteq \mathcal{P}(X,\mathcal{U})$. By the claim just verified, for all $\varrho \in \mathcal{P}(X,\mathcal{U})$ there exists $d^* \in \mathcal{Q}^*$ such that $\varrho \leq d^*$, and so $\mathcal{Q}^*$ is a dominating family in $\mathcal{P}(X,\mathcal{U})$. Therefore, $D(\mathcal{P}(X,\mathcal{U}), \leq) \leq |\mathcal{Q}^*| \leq |\mathcal{Q}|$.

Let $\mathcal{B}$ be any base at 0 in $A(X,\mathcal{U})$. Then for every $N \in \mathcal{B}$, Theorem 2.1 shows that there exists $d_N \in \mathcal{P}(X,\mathcal{U})$ such that $V_{d_N} \subseteq N$. Set $\mathcal{Q} = \{d_N : N \in \mathcal{B}\}$. Then $\{V_d : d \in \mathcal{Q}\}$ is a base at 0 and satisfies $|\mathcal{Q}| \leq |\mathcal{B}|$, so that $D(\mathcal{P}(X,\mathcal{U}), \leq) \leq |\mathcal{Q}| \leq |\mathcal{B}|$. Hence $D(\mathcal{P}(X,\mathcal{U}), \leq) \leq \chi(A(X,\mathcal{U}))$. Combining this with the reverse inequality obtained earlier, we conclude that $\chi(A(X,\mathcal{U})) = D(\mathcal{P}(X,\mathcal{U}), \leq)$. □

For a topological space X, denote by $\mathcal{P}_X$ the family of all continuous pseudometrics on X bounded by 1. It is clear that $\mathcal{P}_X = \mathcal{P}(X,\mathcal{U}_X)$, where $\mathcal{U}_X$ is the fine uniformity of X. Since $A(X) = A(X,\mathcal{U}_X)$, from Theorem 2.3 we have:

Corollary 2.4. *The equality $\chi(A(X)) = D(\mathcal{P}_X, \leq)$ holds for all Tychonoff spaces X.*

We record in passing a couple of consequences of these last results, not otherwise directly related to our main concerns in this paper.

First, let $(X,\mathcal{U})$ be an arbitrary uniform space, and let $F_G(X,\mathcal{U})$ denote the abstract group $F_a(X)$ equipped with the Graev topology, the topology generated by Graev's extensions of all the uniformly continuous pseudometrics on $(X,\mathcal{U})$. Then we have:

Theorem 2.5. *If $(X,\mathcal{U})$ is a uniform space, then $\chi(F_G(X,\mathcal{U})) = \chi(A(X,\mathcal{U}))$.*

Proof. By Theorem 2.3, $\chi(A(X,\mathcal{U})) = D(\mathcal{P}(X,\mathcal{U}), \leq)$. Now if $\{d_\alpha : \alpha \in A\}$ is a dominating family of pseudometrics in $(\mathcal{P}(X,\mathcal{U}), \leq)$, then it is clear that the corresponding family of Graev extensions $\{\widehat{d_\alpha} : \alpha \in A\}$ on the group $F_a(X)$ induces an open base at the identity, giving $\chi(F_G(X,\mathcal{U})) \leq \chi(A(X,\mathcal{U}))$. Conversely, the natural continuous homomorphism from $F_G(X,\mathcal{U}))$ onto $A(X,\mathcal{U})$ is a quotient, from which the inequality $\chi(A(X,\mathcal{U})) \leq \chi(F_G(X,\mathcal{U}))$ follows, giving the result. □

In particular, if X is an arbitrary topological space and $F_G(X)$ denotes the group $F_a(X)$ equipped with the Graev topology, then we have:

Corollary 2.6. $\chi(F_G(X)) = \chi(A(X))$.

Second, as we have noted, it is well known [13, 20] that Graev's topology is the free topology on a free abelian topological group $A(X)$, and Theorem 2.1 extended this to the case of a free abelian topological group $A(X,\mathcal{U})$. On the other hand, it has been noted more than once [13, 17] that the fact that a certain family of (uniformly) continuous pseudometrics are sufficient together to define the topology (or uniformity) of X does not imply in general that the corresponding family of Graev extensions defines the free topology. This is clear, for example, from the observation of Graev [5] noted at the start of the first section: since the free abelian topological group $A(X)$ on a Tychonoff space X is metrizable only when X is discrete, the metrizability of X implies the metrizability of $A(X)$ only when X is discrete. We can adapt the proof of Theorem 2.3 to obtain a necessary and sufficient condition on a family of uniformly continuous pseudometrics under which their Graev extensions do indeed define the free topology. The proof is essentially a recapitulation of the second paragraph of the proof of Theorem 2.3, and we omit the details.

Theorem 2.7. *Let $(X,\mathcal{U})$ be a uniform space and let $\mathcal{Q} \subseteq \mathcal{P}(X,\mathcal{U})$. Then the collection of open sets*

$$\{g \in A(X,\mathcal{U}) : \widehat{d}_A(g,0) < \epsilon\}, \quad \textit{for } d \in \mathcal{Q} \textit{ and } \epsilon > 0,$$

is a base at 0 *for the topology of the free abelian topological group $A(X,\mathcal{U})$ if and only if for every $\varrho \in \mathcal{P}(X,\mathcal{U})$ there exists $d \in Q$ such that $\varrho \le 2d$.*

Let $(X,\mathcal{U})$ be a uniform space. Given two sequences $s = \langle U_n : n \in \omega\rangle$ and $t = \langle V_n : n \in \omega\rangle$ in ${}^\omega\mathcal{U}$, we write $s \le t$ provided that $U_n \subseteq V_n$ for each $n \in \omega$. It is easy to see that the correspondence $s \mapsto N(s)$, where $N(s)$ is as defined immediately before Theorem 2.2, is an order-reversing mapping of $({}^\omega\mathcal{U}, \le)$ to $(\mathcal{N}(0), \supseteq)$. Since by Theorem 2.2 the family $\{N(s) : s \in {}^\omega\mathcal{U}\}$ is a base at 0 in $A(X,\mathcal{U})$, we conclude that $\chi(A(X,\mathcal{U})) \le d({}^\omega\mathcal{U}, \le)$. In fact, we will show shortly that the latter inequality is equality.

Lemma 2.8. $d({}^\omega\mathcal{U}, \le) \le D(\mathcal{P}(X,\mathcal{U}), \le)$ *for every uniform space $(X,\mathcal{U})$.*

Proof. For $d \in \mathcal{P}(X,\mathcal{U})$ and $n \in \omega$, put

$$O_n(d) = \{(x,y) \in X \times X : d(x,y) \le 2^{-n}\}.$$

It is clear that the correspondence $d \mapsto \langle O_n(d) : n \in \omega\rangle$ is an order-reversing mapping of $(\mathcal{P}(X,\mathcal{U}), \le)$ to $({}^\omega\mathcal{U}, \le)$.

Consider an arbitrary sequence $\langle U_n : n \in \omega\rangle \in {}^\omega\mathcal{U}$. Clearly, there exists a sequence $\langle V_n : n \in \omega\rangle \in {}^\omega\mathcal{U}$ such that V_n is symmetric and $3V_{n+1} \subseteq V_n \subseteq U_n$ for each $n \in \omega$. By Theorem 8.1.10 of [4], we can find $d \in \mathcal{P}(X,\mathcal{U})$ such that $O_n(d) \subseteq V_n$ for each $n \in \omega$, so that the function $d \mapsto \langle O_n(d) : n \in \omega\rangle$ maps $\mathcal{P}(X,\mathcal{U})$ to a dense set in ${}^\omega\mathcal{U}$. It follows, as required, that $d({}^\omega\mathcal{U}, \le) \le D(\mathcal{P}(X,\mathcal{U}), \le)$. (We note that [4] uses the standing assumption that all uniform spaces are Hausdorff (or separated), but it is well known and easy to see that this assumption is unnecessary for the validity of Theorem 8.1.10.) □

Using the observation made before the lemma together with the lemma itself, we have

$$\chi(A(X,\mathcal{U})) \leq d({}^{\omega}\mathcal{U}, \leq) \leq D(\mathcal{P}(X,\mathcal{U}), \leq).$$

But by Theorem 2.3, we also have $\chi(A(X,\mathcal{U})) = D(\mathcal{P}(X,\mathcal{U}), \leq)$, so we have proved both of the following results.

Theorem 2.9. *If $(X,\mathcal{U})$ is a uniform space, then $\chi(A(X,\mathcal{U})) = d({}^{\omega}\mathcal{U}, \leq)$.*

Theorem 2.10. *If $(X,\mathcal{U})$ is a uniform space, then $d({}^{\omega}\mathcal{U}, \leq) = D(\mathcal{P}(X,\mathcal{U}), \leq)$.*

It is clear that every uniform space $(X,\mathcal{U})$ satisfies

$$w(X,\mathcal{U}) = d(\mathcal{U}, \subseteq) \leq d({}^{\omega}\mathcal{U}, \leq) \leq d(\mathcal{U}, \subseteq)^{\aleph_0} = w(X,\mathcal{U})^{\aleph_0}.$$

Hence Theorem 2.9 implies the following bounds for the character of the free abelian topological group on $(X,\mathcal{U})$:

Corollary 2.11. *Let $(X,\mathcal{U})$ be a uniform space. Then*

$$w(X,\mathcal{U}) \leq \chi(A(X,\mathcal{U})) \leq w(X,\mathcal{U})^{\aleph_0}.$$

In the case of a compact Hausdorff space X, we have $w(X,\mathcal{U}) = w(X)$, so the bounds can be simplified as follows.

Corollary 2.12. *If X is a compact Hausdorff space, then*

$$w(X) \leq \chi(A(X)) \leq w(X)^{\aleph_0}.$$

In Theorem 2.15 and Corollary 2.16 below we present a different lower bound for the character of most free abelian topological groups. We shall see later that this bound, the cardinal $\mathfrak{d}$, is the exact value of the character of $A(X)$ (and indeed of $F(X)$) when X is an infinite compact metrizable space.

First, we recall two useful notions. If every G_δ-set in X is open, then X is said to be a *P-space.* Similarly, given a uniform space $(X,\mathcal{U})$, we say that $(X,\mathcal{U})$ is a *uniform P-space* if the intersection of countably many elements of $\mathcal{U}$ is again an element of $\mathcal{U}$. Note that if $(X,\mathcal{U})$ is a uniform P-space, then the underlying topological space X is a P-space.

In some of the arguments which follow, it is natural to consider separately the cases when $(X,\mathcal{U})$ is a uniform P-space and when $(X,\mathcal{U})$ is not a uniform P-space. In fact, the character of $A(X,\mathcal{U})$ in the "exceptional" case of a uniform P-space $(X,\mathcal{U})$ can be dealt with simply and conclusively in the following form.

Theorem 2.13. *If $(X,\mathcal{U})$ is a uniform P-space, then $\chi(A(X,\mathcal{U})) = w(X,\mathcal{U})$.*

Proof. For any uniform space $(X,\mathcal{U})$, the mapping $U \mapsto \langle U, U, \ldots \rangle$ from $(\mathcal{U}, \subseteq)$ to $({}^{\omega}\mathcal{U}, \leq)$ is an order-preserving embedding, and if $(X,\mathcal{U})$ is also a uniform P-space, then $\mathcal{U}$ is mapped to a dense set in ${}^{\omega}\mathcal{U}$, since for an arbitrary $\langle U_0, U_1, \ldots \rangle \in {}^{\omega}\mathcal{U}$ we have $\langle U, U, \ldots \rangle \leq \langle U_0, U_1, \ldots \rangle$, where $U = \bigcap_{n\in\omega} U_n \in \mathcal{U}$. It follows that $d(\mathcal{U}, \subseteq) = d({}^{\omega}\mathcal{U}, \leq)$, and then the conclusion that $\chi(A(X,\mathcal{U})) = w(X,\mathcal{U})$ follows from Theorem 2.9. □

We now turn to the "usual" case when $(X,\mathcal{U})$ is not a uniform P-space.

Lemma 2.14. *If* $(X,\mathcal{U})$ *is not a uniform P-space, then* $\mathfrak{d} \le D(\mathcal{P}(X,\mathcal{U}), \le)$.

Proof. Fix a strictly decreasing sequence $\langle U_n : n \in \omega\rangle \in {}^{\omega}\mathcal{U}$ such that $U_0 = X \times X$ and $\bigcap_{n\in\omega} U_n$ is not in $\mathcal{U}$. For any $s = \langle V_n : n \in \omega\rangle \in {}^{\omega}\mathcal{U}$, we define a function $f_s \in {}^{\omega}\omega$ by

$$f_s(n) = \max\{k \in \omega : V_n \subseteq U_k\},$$

for $n \in \omega$. Since $\bigcap_{n\in\omega} U_n \notin \mathcal{U}$, we have $f_s(n) < \infty$ for each $n \in \omega$, so our definition of f_s is valid. Moreover, it is easy to see that if $\langle V_n : n \in \omega\rangle = s \le t = \langle W_n : n \in \omega\rangle$ then $f_s \ge f_t$, so that the mapping $s \mapsto f_s$ from $({}^{\omega}\mathcal{U}, \le)$ to $({}^{\omega}\omega, \le)$ is order-reversing.

We claim that the set $\{f_s : s \in {}^{\omega}\mathcal{U}\}$ is dominating in $({}^{\omega}\omega, \le)$. Indeed, for any $f \in {}^{\omega}\omega$, let $\widehat{f}$ be a strictly increasing function in ${}^{\omega}\omega$ such that $f \le \widehat{f}$. Now set $s = \langle U_{\widehat{f}(n)} : n \in \omega\rangle \in {}^{\omega}\mathcal{U}$, and note that s is a strictly decreasing sequence of sets. Then

$$f_s(n) = \max\{k \in \omega : U_{\widehat{f}(n)} \subseteq U_k\} = \widehat{f}(n)$$

for all $n \in \omega$, so that $f_s = \widehat{f}$. Therefore, $f \le f_s$, proving our claim. This immediately implies that $\mathfrak{d} = D({}^{\omega}\omega, \le) \le d({}^{\omega}\mathcal{U}, \le) = D(\mathcal{P}(X,\mathcal{U}), \le)$, by Theorem 2.10, as required. □

Theorem 2.3 and Lemma 2.14 imply the following lower bound, complementing Theorem 2.13, for the character of $A(X,\mathcal{U})$.

Theorem 2.15. *If* $(X,\mathcal{U})$ *is not a uniform P-space, then* $\mathfrak{d} \le \chi(A(X,\mathcal{U}))$.

Theorem 2.15 implies several important corollaries.

Corollary 2.16. *If a Tychonoff space X is not a P-space, then* $\mathfrak{d} \le \chi(A(X))$.

Corollary 2.17. *Let* $(X,\mathcal{U})$ *be a Hausdorff uniform space which contains an infinite precompact subset. Then* $\mathfrak{d} \le \chi(A(X,\mathcal{U}))$.

Proof. Suppose that P is an infinite precompact subset of $(X,\mathcal{U})$. Let $(Y,\mathcal{V})$ be the completion of the space $(X,\mathcal{U})$ and let K be the closure of P in Y (we identify X with the corresponding dense subspace of Y). Then K is an infinite compact subset of Y. The group $A(X,\mathcal{U})$ is topologically isomorphic to a dense subgroup of $A(Y,\mathcal{V})$ by Nummela's theorem [15], so that $\chi(A(X,\mathcal{U})) = \chi(A(Y,\mathcal{V}))$. Since the infinite compact set K cannot be a P-space, $(Y,\mathcal{V})$ fails to be a uniform P-space. Therefore, Theorem 2.15 implies that $\mathfrak{d} \le \chi(A(Y,\mathcal{V})) = \chi(A(X,\mathcal{U}))$. □

Corollary 2.18. *If a Tychonoff space X contains an infinite compact set* (*in particular, a non-trivial convergent sequence*), *then* $\mathfrak{d} \le \chi(A(X))$.

Corollary 2.19. *If a Tychonoff space X contains a proper dense Lindelöf subspace, then* $\mathfrak{d} \le \chi(A(X))$.

Proof. Suppose that Y is a proper dense Lindelöf subspace of X. Then for every point $x \in X \setminus Y$, there exists a G_δ-set P_x in X containing x such that $P_x \cap Y = \varnothing$. If X were a P-space, the complement $X \setminus Y$ would be open in X, thus contradicting the assumption that Y is dense in X. Hence the conclusion follows from Corollary 2.16. □

One cannot omit the word "proper" in Corollary 2.19, since the character of the free abelian topological group over the one-point Lindelöfication of a discrete space of cardinality $\aleph_1$ is exactly equal to $\aleph_1$ (this follows from [7, Lemma 2.9]).

Lemma 2.20. *Let $(X,\mathcal{U})$ be a uniform space. If $d(\mathcal{U},\subseteq)=\aleph_0$, then $d({}^\omega\mathcal{U},\leq)=\mathfrak{d}$.*

Proof. Choose a dense set $\{U_0,U_1,\ldots\}$ in $(\mathcal{U},\subseteq)$ such that U_n strictly contains U_{n+1} for all $n\in\omega$. Then it is easy to see that the mapping $(n_0,n_1,\ldots)\mapsto(U_{n_0},U_{n_1},\ldots)$ from $({}^\omega\omega,\leq)$ to $({}^\omega\mathcal{U},\leq)$ is order-reversing and maps ${}^\omega\omega$ onto a dense subset of ${}^\omega\mathcal{U}$, which gives $d({}^\omega\mathcal{U},\leq)\leq D({}^\omega\omega,\leq)=\mathfrak{d}$.

Conversely, if we map $(V_0,V_1,\ldots)\in{}^\omega\mathcal{U}$ to the sequence $(n_0,n_1,\ldots)\in{}^\omega\omega$ defined by setting $n_k=\min\{m\in\omega : U_m\subseteq V_k\}$ for all $k\in\omega$, then the mapping is easily seen to be order-reversing and to map ${}^\omega\mathcal{U}$ onto a dominating subset of ${}^\omega\omega$, giving $\mathfrak{d}=D({}^\omega\omega,\leq)\leq d({}^\omega\mathcal{U},\leq)$, and the result. □

The following result is immediate from Lemma 2.20 and Theorem 2.9.

Theorem 2.21. *Let $(X,\mathcal{U})$ be a uniform space satisfying $w(X,\mathcal{U})=\aleph_0$. Then $\chi(A(X,\mathcal{U}))=\mathfrak{d}$.*

Since a uniform space is pseudometrizable if and only if it has a countable base, we have in particular:

Corollary 2.22. *If X is an infinite compact metrizable space, then $\chi(A(X))=\mathfrak{d}$.*

Theorem 2.23. *Let X be a Tychonoff space satisfying $\chi_\Delta(X)\leq\aleph_0$. Then either X and $A(X)$ are discrete or $\chi(A(X))=\mathfrak{d}$.*

Proof. By Theorem 14 in [18], from $\chi_\Delta(X)\leq\aleph_0$ it follows that the set X' of all non-isolated points in X is compact and $\chi(X',X)\leq\aleph_0$. If $X'=\varnothing$, then both X and $A(X)$ are discrete. Suppose, therefore, that $X'\neq\varnothing$. Clearly, X admits a perfect mapping onto a space Y with a single non-isolated point (map X' to a point). Therefore, both Y and X are paracompact, so that every neighborhood of the diagonal in X^2 belongs to the fine uniformity $\mathcal{U}$ of X. In particular, $w(X,\mathcal{U})=\chi_\Delta(X)\leq\aleph_0$. The result now follows from Theorem 2.21. □

3. The character of free topological groups

The next lemma establishes a simple relation between the characters of the groups $A(X,\mathcal{U})$ and $F(X,\mathcal{U})$.

Lemma 3.1. *If $(X,\mathcal{U})$ is a uniform space, then $\chi(A(X,\mathcal{U}))\leq\chi(F(X,\mathcal{U}))$.*

Proof. Since $A(X,\mathcal{U})$ is a quotient group of $F(X,\mathcal{U})$ and continuous open homomorphisms do not raise the character, we have $\chi(A(X,\mathcal{U})) \le \chi(F(X,\mathcal{U}))$. □

Similarly, of course, for a topological space X, we have $\chi(A(X)) \le \chi(F(X))$. Thus, each of the lower bounds we have derived above for $\chi(A(X,\mathcal{U}))$ or $\chi(A(X))$ yields automatically a corresponding lower bound for $\chi(F(X,\mathcal{U}))$ or $\chi(F(X))$, respectively. Corollary 2.19, for example, gives us the bound $\mathfrak{d} \le \chi(A(X)) \le \chi(F(X))$ for a space X containing a proper dense Lindelöf subspace.

In fact, however, the conclusion of Corollary 2.19 can be strengthened to $\mathfrak{d} \le \chi(A(X)) = \chi(F(X))$ (see Corollary 3.18 below), but this is not at all straightforward. Our aim now is to establish the equality $\chi(A(X)) = \chi(F(X))$ for the wide class of $\aleph_0$-*narrow* spaces, which are also known as *pseudo-ω_1-compact* spaces. By definition, a space X is pseudo-ω_1-compact if every discrete family of open sets in X is countable. Our choice of the new name for this class of spaces is motivated (apart from the aesthetic reason) by the fact that X is pseudo-ω_1-compact if and only if the uniform space $(X,\mathcal{U}_X)$ is $\aleph_0$-narrow, where $\mathcal{U}_X$ is the fine uniformity of X (see [24, Assertion 1.2]).

Our arguments require some unpleasant work describing a neighborhood base in a free topological group. Since the cases of the free topological group on a topological space and a uniform space are very similar, we prefer to present the description in the most general form, for free topological groups on uniform spaces.

First we recall some notions related to trees. A partially ordered set $(P,\le)$ is a *tree* if the set $P_x = \{y \in P : y < x\}$ is well ordered by $<$ for each $x \in P$ (where we write $y < x$ if and only if $y \le x$ and $y \ne x$). The *height* of an element $x \in P$, denoted by $h(x)$, is the order type of the set $(P_x,<)$. For an ordinal α, we call the set $P(\alpha) = \{x \in P : h(x) = \alpha\}$ the *αth level* of $(P,\le)$. Finally, the *height* of $(P,\le)$ is defined to be the smallest ordinal α such that $P(\alpha) = \varnothing$.

In what follows we shall work only with trees of height ω. Clearly, if the height of P is ω, then every $x \in P$ has only finitely many predecessors with respect to $\le$.

Definition 3.2. *Let $(X,\mathcal{U})$ be a uniform space. We say that a tree $(P,\le)$ of height ω is $\mathcal{U}$-*covering *if it satisfies the following conditions*:

- (i) *each element $x \in P$ has the form $x = (U_0,\ldots,U_n)$, where $U_0,\ldots,U_n$ are non-empty open sets in X and $n \in \omega$;*
- (ii) *if $x = (U_0,\ldots,U_n)$ and $y = (V_0,\ldots,V_m)$ are in P, then $x \le y$ if and only if $n \le m$ and $U_i = V_i$ for each $i = 0,\ldots,n$;*
- (iii) *the family $\gamma_P = \{V : (V) \in P(0)\}$ is a $\mathcal{U}$-uniform cover of X;*
- (iv) *if $x = (U_0,\ldots,U_n) \in P$, then the family $\gamma_P(x) = \{V : (U_0,\ldots,U_n,V) \in P\}$ is a $\mathcal{U}$-uniform cover of X.*

Denote by $T(X,\mathcal{U})$ the family of all $\mathcal{U}$-covering trees. For $(P,\le)\in T(X,\mathcal{U})$, we define a subset W_P of $F_a(X)$ as follows:

$$W_P = \bigcup_{\substack{(U_0,\dots,U_n)\in P,\\ \varepsilon_0,\dots,\varepsilon_n=\pm 1,\\ n\in\omega}} U_0^{-\varepsilon_0}\cdots U_n^{-\varepsilon_n}U_n^{\varepsilon_n}\cdots U_0^{\varepsilon_0} \tag{3.1}$$

The next lemma is the first step towards the promised description of a neighborhood base at the identity of free topological groups.

Lemma 3.3. *For every neighborhood O of the identity e in $F(X,\mathcal{U})$, there exists $(P,\le)\in T(X,\mathcal{U})$ such that $W_P\subseteq O$. In addition, if the space $(X,\mathcal{U})$ is τ-narrow for some $\tau\ge\aleph_0$, then one can choose $(P,\le)$ satisfying $|P|\le\tau$.*

Proof. Denote by $\mathcal{O}(e)$ the family of all open symmetric neighborhoods of e in the group $F(X,\mathcal{U})$. For a given $O\in\mathcal{O}(e)$, choose $W\in\mathcal{O}(e)$ such that $W^3\subseteq O$. Suppose that the uniform space $(X,\mathcal{U})$ is τ-narrow for some $\tau\ge\aleph_0$. Then the group $F(X,\mathcal{U})$ is τ-narrow by [6] or [2, Lemma 3.2]. For every $x\in X$, put $U_x = x\cdot W\cap X\cap W\cdot x$. Since the two-sided uniformity of $F(X,\mathcal{U})$ induces on X its original uniformity $\mathcal{U}$ [15], we can find a set $Y\subseteq X$ with $|Y|\le\tau$ such that $X=\bigcup_{x\in Y}U_x$. Put $P(0)=\{(U_x): x\in Y\}$. Then $|P(0)|\le\tau$ and $\gamma_P=\{U_x : x\in Y\}$ is a $\mathcal{U}$-uniform cover of X. This defines the initial level of the required tree P.

Let us describe the second step of the construction. For every $x\in Y$, choose $V_x, W_x\in\mathcal{O}(e)$ such that $x^{-\varepsilon}\cdot V_x\cdot x^{\varepsilon}\subseteq W$ for $\varepsilon=\pm1$ and $W_x^3\subseteq V_x$, and put $U_{x,y}=y\cdot W_x\cap X\cap W_x\cdot y$ for each $y\in X$. Since the space $(X,\mathcal{U})$ is τ-narrow, we can choose, given any $x\in Y$, a set $Y(x)\subseteq X$ with $|Y(x)|\le\tau$ satisfying $X=\bigcup_{y\in Y(x)}U_{x,y}$. Put $\gamma_P(x)=\{U_{x,y}: y\in Y(x)\}$. Then $\gamma_P(x)$ is a $\mathcal{U}$-uniform cover of X for each $x\in Y$, and we define the level $P(1)$ of the tree P by

$$P(1)=\{(U_x,U_{x,y}) : x\in Y,\ y\in Y(x)\}.$$

Clearly, $|P(1)|\le\tau$.

At the third step, for each $x\in Y$ and for each $y\in Y(x)$, choose $V_{x,y}, W_{x,y}\in\mathcal{O}(e)$ such that $y^{-\varepsilon}\cdot V_{x,y}\cdot y^{\varepsilon}\subseteq W_x$ for $\varepsilon=\pm1$ and $W_{x,y}^3\subseteq V_{x,y}$. For $z\in X$, put $U_{x,y,z}=z\cdot W_{x,y}\cap X\cap W_{x,y}\cdot z$ and choose a subset $Y(x,y)$ of X such that $|Y(x,y)|\le\tau$ and $X=\bigcup_{z\in Y(x,y)}U_{x,y,z}$. Put $\gamma_P(x,y)=\{U_{x,y,z}: z\in Y(x,y)\}$. Then $\gamma_P(x,y)$ is a $\mathcal{U}$-uniform cover of X for each $x\in Y$ and each $y\in Y(x)$, and we define the level $P(2)$ of the tree P by

$$P(2)=\{(U_x,U_{x,y},U_{x,y,z}) : x\in Y,\ y\in Y(x),\ z\in Y(x,y)\}.$$

Clearly, $|P(2)|\le|P(1)|\cdot\tau\le\tau$. Continuing this process, we finally obtain the set $P=\bigcup_{n\in\omega}P(n)$, partially ordered according to (ii) of Definition 3.2, and such that $|P|\le\tau$. One easily verifies that $(P,\le)$ satisfies (i)–(iv), so that $(P,\le)\in T(X,\mathcal{U})$.

It remains to show that $W_P\subseteq O$. Let $(U_0,U_1,\dots,U_n)$ be an element of P. We have to verify that $U_0^{-\varepsilon_0}\cdots U_n^{-\varepsilon_n}U_n^{\varepsilon_n}\cdots U_0^{\varepsilon_0}\subseteq O$ for arbitrary $\varepsilon_0,\dots,\varepsilon_n =$

± 1. There exist points $x_0 \in Y$, $x_1 \in Y(x_0)$, ..., $x_n \in Y(x_0, \dots, x_{n-1})$ such that $U_0 = U_{x_0}$, $U_1 = U_{x_0,x_1}$, ..., $U_n = U_{x_0,x_1,\dots,x_n}$. By definition, we have

$$\begin{gathered} U_0 \subseteq (x_0 \cdot W) \cap (W \cdot x_0), \\ U_1 \subseteq (x_1 \cdot W_{x_0}) \cap (W_{x_0} \cdot x_1), \\ \dots\dots \\ U_n \subseteq (x_n \cdot W_{x_0,\dots,x_{n-1}}) \cap (W_{x_0,\dots,x_{n-1}} \cdot x_n). \end{gathered}$$

We claim that

$$U_{k+1}^{-\varepsilon_{k+1}} \cdots U_n^{-\varepsilon_n} U_n^{\varepsilon_n} \cdots U_{k+1}^{\varepsilon_{k+1}} \subseteq V_{x_0,\dots,x_k} \tag{3.2}$$

for each $k = 0, 1, \dots, n-1$. Indeed, if $k = n-1$, then

$$U_n^{-1} U_n \subseteq (x_n W_{x_0,\dots,x_{n-1}})^{-1} \cdot x_n W_{x_0,\dots,x_{n-1}} = W^2_{x_0,\dots,x_{n-1}} \subseteq V_{x_0,\dots,x_{n-1}},$$

and, similarly,

$$U_n \cdot U_n^{-1} \subseteq W_{x_0,\dots,x_{n-1}} x_n \cdot (W_{x_0,\dots,x_{n-1}} x_n)^{-1} = W^2_{x_0,\dots,x_{n-1}} \subseteq V_{x_0,\dots,x_{n-1}},$$

giving (3.2) for $k = n-1$. Suppose that (3.2) holds for some $k > 0$. If $\varepsilon_k = 1$, we have $U_k \subseteq x_k \cdot W_{x_0,\dots,x_{k-1}}$, whence

$$\begin{aligned} & U_k^{-1} U_{k+1}^{-\varepsilon_{k+1}} \cdots U_n^{-\varepsilon_n} U_n^{\varepsilon_n} \cdots U_{k+1}^{\varepsilon_{k+1}} U_k \\ \subseteq U_k^{-1} V_{x_0,\dots,x_k} U_k \subseteq W^{-1}_{x_0,\dots,x_{k-1}} & \cdot x_k^{-1} \cdot V_{x_0,\dots,x_k} \cdot x_k \cdot W_{x_0,\dots,x_{k-1}} \\ & \subseteq W^3_{x_0,\dots,x_{k-1}} \\ & \subseteq V_{x_0,\dots,x_{k-1}}. \end{aligned}$$

Similarly, if $\varepsilon_k = -1$, then we use the inclusion $U_k \subseteq W_{x_0,\dots,x_{k-1}} \cdot x_k$ to deduce that

$$U_k U_{k+1}^{-\varepsilon_{k+1}} \cdots U_n^{-\varepsilon_n} U_n^{\varepsilon_n} \cdots U_{k+1}^{\varepsilon_{k+1}} U_k^{-1} \subseteq V_{x_0,\dots,x_{k-1}}.$$

The inclusion (3.2) now follows. Finally, from $U_0 \subseteq x_0 \cdot W$, and using (3.2) with $k = 0$, it follows that

$$\begin{aligned} U_0^{-1} U_1^{-\varepsilon_1} \cdots U_n^{-\varepsilon_n} U_n^{\varepsilon_n} \cdots U_1^{\varepsilon_1} U_0 & \subseteq W^{-1} \cdot x_0^{-1} \cdot V_{x_0} \cdot x_0 \cdot W \\ & \subseteq W^3 \\ & \subseteq O, \end{aligned}$$

and similarly, from $U_0 \subseteq W \cdot x_0$ it follows that

$$U_0 U_1^{-\varepsilon_1} \cdots U_n^{-\varepsilon_n} U_n^{\varepsilon_n} \cdots U_1^{\varepsilon_1} U_0^{-1} \subseteq O.$$

Since W_P is the union of the sets of the form $U_0^{-\varepsilon_0} \cdots U_n^{-\varepsilon_n} U_n^{\varepsilon_n} \cdots U_0^{\varepsilon_0}$, we have proved the inclusion $W_P \subseteq O$. □

Remark 3.4. Given the statement of the lemma just proved, it is worth remarking that the family $\{W_P : P \in T(X, \mathcal{U})\}$ does not in general constitute a base at the identity in the group $F(X, \mathcal{U})$, or indeed in any group topology on the group $F_a(X)$. In fact, for certain $P \in T(X, \mathcal{U})$, one cannot find $Q \in T(X, \mathcal{U})$ with $W_Q \cdot W_Q \subseteq W_P$.

We will shortly show how a rather more elaborate family of sets constructed using the sets W_P do form an open base in $F(X, \mathcal{U})$.

Let us first establish some other properties of the sets W_P.

Lemma 3.5. *Suppose that $P, Q \in T(X,\mathcal{U})$ and $g \in F_a(X)$. Then one can find $R \in T(X,\mathcal{U})$ such that $W_R \subseteq W_P \cap W_Q$ and $g^{-1} \cdot W_R \cdot g \subseteq W_P$.*

Proof. The existence of $R \in T(X,\mathcal{U})$ satisfying $W_R \subseteq W_P \cap W_Q$ is immediate. Hence it suffices to construct a $\mathcal{U}$-covering tree R such that $g^{-1} \cdot W_R \cdot g \subseteq W_P$. The existence of the required tree R is clear if g is the identity e of $F_a(X)$. If $g \neq e$, then it suffices to consider the case when $g \in X \cup X^{-1}$ and then apply induction on the length of g in the general case. So we assume that $g \in X \cup X^{-1}$. Since $\gamma_P = \{U : (U) \in P(0)\}$ is a cover of X, there exist $U_0 \in \gamma_P$ and $\varepsilon_0 = \pm 1$ such that $g \in U_0^{\varepsilon_0}$. Put

$$R = \{(U_1, \dots, U_n) : (U_0, U_1, \dots, U_n) \in P,\ n \geq 1\}.$$

It is easy to see that $R \in T(X,\mathcal{U})$, and we claim that $g^{-1} \cdot W_R \cdot g \subseteq W_P$. Indeed, if $(U_1, \dots, U_n) \in R$ and $\varepsilon_1, \dots, \varepsilon_n = \pm 1$, then

$$g^{-1} \cdot U_1^{-\varepsilon_1} \cdots U_n^{-\varepsilon_n} U_n^{\varepsilon_n} \cdots U_1^{\varepsilon_1} \cdot g \subseteq U_0^{-\varepsilon_0} U_1^{-\varepsilon_1} \cdots U_n^{-\varepsilon_n} U_n^{\varepsilon_n} \cdots U_1^{\varepsilon_1} U_0^{\varepsilon_0} \subseteq W_P.$$

This proves the inclusion $g^{-1} \cdot W_R \cdot g \subseteq W_P$. □

Now we present our description of a neighborhood base at the identity of $F(X,\mathcal{U})$ in terms of $\mathcal{U}$-covering trees. Again, we need some definitions. Let $s = \langle P_n : n \in \mathbb{N} \rangle \in {}^{\mathbb{N}}T(X,\mathcal{U})$ be a sequence of $\mathcal{U}$-covering trees. Then we define a set $O_s \subseteq F_a(X)$ as follows:

$$O_s = \bigcup_{n \in \mathbb{N}} \bigcup_{\pi \in S_n} W_{P_{\pi(1)}} \cdots W_{P_{\pi(n)}}, \tag{3.3}$$

where S_n is the group of permutations of the set $\{1, \dots, n\}$.

Theorem 3.6. *The family $\Sigma = \{O_s : s \in {}^{\mathbb{N}}T(X,\mathcal{U})\}$ forms a base at the identity e of the group $F(X,\mathcal{U})$.*

Proof. Our argument is close to that of [21, Th. 1.1]. It suffices to verify that the following assertions are valid:

(a) Σ is a base for a group topology $\mathcal{T}^*$ on $F_a(X)$;
(b) $\mathcal{T}^*$ is finer than the topology $\mathcal{T}$ of the group $F(X,\mathcal{U})$;
(c) the two-sided uniformity $\mathcal{V}$ of the group $G = (F_a(X), \mathcal{T}^*)$ induces on X a uniformity coarser than $\mathcal{U}$.

Since $F(X,\mathcal{U})$ carries the finest group topology whose two-sided uniformity induces on X the uniformity $\mathcal{U}$, from (a)–(c) it follows that $\mathcal{T}^* = \mathcal{T}$. Let us start with (a).

(a) To verify that Σ is a base at e for a group topology on $F_a(X)$, it suffices to show that Σ has the following four properties:

(1) for every $U, V \in \Sigma$ there exists $W \in \Sigma$ with $W \subseteq U \cap V$;
(2) for every $U \in \Sigma$ there exists $V \in \Sigma$ with $V^{-1} \cdot V \subseteq U$;
(3) for every $U \in \Sigma$ and $g \in U$ there exists $V \in \Sigma$ with $V \cdot g \subseteq U$;
(4) for every $U \in \Sigma$ and $g \in F_a(X)$ there is $V \in \Sigma$ such that $g^{-1} \cdot V \cdot g \subseteq U$.

We only check (2), (3) and (4), since (1) is immediate from Lemma 3.5. Note that by definition, the set W_P is symmetric for each $P \in T(X,\mathcal{U})$, and so is O_s for each $s \in {}^{\mathbb{N}}T(X,\mathcal{U})$. Let $U \in \Sigma$ be arbitrary. Then $U = O_s$ for some $s \in {}^{\mathbb{N}}T(X,\mathcal{U})$, say $s = \langle P_n : n \in \mathbb{N}\rangle$.

Let us check (2). By Lemma 3.5, we can find a sequence $t = \langle Q_n : n \in \mathbb{N}\rangle \in {}^{\mathbb{N}}T(X,\mathcal{U})$ such that $W_{Q_n} \subseteq W_{P_{2n-1}} \cap W_{P_{2n}}$ for each $n \in \mathbb{N}$. We claim that $O_t^{-1} \cdot O_t \subseteq O_s$. Thus, we take $m, n \in \mathbb{N}$ and $\pi \in S_m$ and $\varrho \in S_n$, and show that

$$W_{Q_{\pi(1)}} \cdots W_{Q_{\pi(m)}} W_{Q_{\varrho(1)}} \cdots W_{Q_{\varrho(n)}} \subseteq O_s.$$

In fact, however, it suffices to assume that $m = n$ here, since each W_{Q_p} contains e. Thus, let $n \in \mathbb{N}$ and $\pi, \varrho \in S_n$. Define $\sigma \in S_{2n}$ by $\sigma(i) = 2\pi(i)$ if $1 \le i \le n$ and $\sigma(i) = 2\varrho(i-n) - 1$ if $n < i \le 2n$. Then from our definition of σ and t it follows that

$$\begin{aligned} & W_{Q_{\pi(1)}} \cdots W_{Q_{\pi(n)}} W_{Q_{\varrho(1)}} \cdots W_{Q_{\varrho(n)}} \\ \subseteq\ & W_{P_{\sigma(1)}} \cdots W_{P_{\sigma(n)}} W_{P_{\sigma(n+1)}} \cdots W_{P_{\sigma(2n)}} \subseteq O_s. \end{aligned}$$

This along with (3.3) implies that $O_t^{-1} \cdot O_t = O_t \cdot O_t \subseteq O_s$, as claimed, and the set $V = O_t \in \Sigma$ is as required.

To verify (3), take an arbitrary $g \in U = O_s$. Then $g \in W_{P_{\pi(1)}} \cdots W_{P_{\pi(k)}}$ for some $k \in \mathbb{N}$ and $\pi \in S_k$. Put $Q_n = P_{n+k}$ for each $n \in \mathbb{N}$ and consider $t = \langle Q_n : n \in \mathbb{N}\rangle$. Then $t \in {}^{\mathbb{N}}T(X,\mathcal{U})$ and the set $V = O_t$ satisfies $V \cdot g \subseteq U$. Indeed, for $n \in \mathbb{N}$ and $\sigma \in S_n$, define $\varrho \in S_{n+k}$ by $\varrho(i) = \sigma(i) + k$ if $i \le n$ and $\varrho(i) = \pi(i-n)$ if $n < i \le n+k$. Then

$$\begin{aligned} W_{Q_{\sigma(1)}} \cdots W_{Q_{\sigma(n)}} \cdot g &\subseteq W_{Q_{\sigma(1)}} \cdots W_{Q_{\sigma(n)}} W_{P_{\pi(1)}} \cdots W_{P_{\pi(k)}} \\ &= W_{P_{\varrho(1)}} \cdots W_{P_{\varrho(n+k)}} \\ &\subseteq O_s, \end{aligned}$$

so that $O_t \cdot g \subseteq O_s$ or, equivalently, $V \cdot g \subseteq U$.

The verification of (4) is similar. Let g be an arbitrary element of $F_a(X)$. By Lemma 3.5, for every $n \in \mathbb{N}$ there exists $Q_n \in T(X,\mathcal{U})$ such that $g^{-1} \cdot W_{Q_n} \cdot g \subseteq W_{P_n}$. Put $t = \langle Q_n : n \in \mathbb{N}\rangle$. Then $t \in {}^{\mathbb{N}}T(X,\mathcal{U})$ and $g^{-1} \cdot O_t \cdot g \subseteq O_s$. Indeed, if $n \in \mathbb{N}$ and $\pi \in S_n$, then we have

$$\begin{aligned} g^{-1} \cdot W_{Q_{\pi(1)}} \cdots W_{Q_{\pi(n)}} \cdot g &= (g^{-1} \cdot W_{Q_{\pi(1)}} \cdot g) \cdots (g^{-1} \cdot W_{Q_{\pi(n)}} \cdot g) \\ &\subseteq W_{P_{\pi(1)}} \cdots W_{P_{\pi(n)}} \\ &\subseteq O_s. \end{aligned}$$

This implies that $g^{-1} \cdot O_t \cdot g \subseteq O_s$, and so the set $V = O_t \in \Sigma$ is as required.

We conclude, therefore, that Σ is a base for a group topology $\mathcal{T}^*$ on $F_a(X)$. This proves (a).

(b) Let O be an arbitrary neighborhood of e in $F(X,\mathcal{U})$. Choose a sequence $\langle V_n : n \in \omega\rangle$ of open symmetric neighborhoods of e in $F(X,\mathcal{U})$ such that $V_0 \subseteq O$ and $V_{n+1}^3 \subseteq V_n$ for each $n \in \omega$. By Lemma 3.3, for every $n \in \mathbb{N}$ there exists $P_n \in T(X,\mathcal{U})$ such that $W_{P_n} \subseteq V_n$. Note that if $n \in \mathbb{N}$ and $\pi \in S_n$, then $V_{\pi(1)} \cdots V_{\pi(n)} \subseteq V_0$ by Lemma 1.3 of [21]. This immediately implies that

$$W_{P_{\pi(1)}} \cdots W_{P_{\pi(n)}} \subseteq V_{\pi(1)} \cdots V_{\pi(n)} \subseteq V_0,$$

whence it follows that $O_s \subseteq V_0 \subseteq O$, where $s = \langle P_n : n \in \mathbb{N}\rangle$. This proves that the topology $\mathcal{T}^*$ generated by the family Σ is finer than $\mathcal{T}$.

(c) Let $V = O_s$, where $s = \langle P_n : n \in \mathbb{N}\rangle \in {}^{\mathbb{N}}T(X,\mathcal{U})$ is arbitrary. Put

$$W = \bigcup_{(U) \in P_1(0)} U \times U.$$

Then $W \in \mathcal{U}$, and from the definition of W_{P_1} it follows that

$$W \subseteq \{(x,y) \in X \times X : x^{-1}y \in W_{P_1},\ xy^{-1} \in W_{P_1}\}.$$

Since $W_{P_1} \subseteq O_s$, we conclude that

$$W \subseteq \{(x,y) \in X \times X : x^{-1}y \in O_s,\ xy^{-1} \in O_s\}.$$

Therefore, the restriction of the two-sided uniformity of the group $(F_a(X), \mathcal{T}^*)$ to the set $X \subseteq F_a(X)$ is coarser than $\mathcal{U}$. The proof is complete. □

Suppose that $(X,\mathcal{U})$ is an $\aleph_0$-narrow uniform space. Roughly speaking, Theorem 3.6 and Lemma 3.3 show that one has to use only countably many elements of $\mathcal{U}$ to produce a basic neighborhood of the identity in $F(X,\mathcal{U})$. We use this fact as well as the next two lemmas in the proof of Theorem 3.10.

For any uniform space $(X,\mathcal{U})$, we denote by $\mathcal{C}(\mathcal{U})$ the family of all $\mathcal{U}$-uniform covers of X, and for $\gamma, \lambda \in \mathcal{C}(\mathcal{U})$, we write $\gamma \prec \lambda$ provided that γ refines λ.

Lemma 3.7. *If $(X,\mathcal{U})$ is a uniform space, then each of the partially ordered sets $(\mathcal{U}, \subseteq)$ and $(\mathcal{C}(\mathcal{U}), \prec)$ admits an order-preserving mapping onto a dense subset of the other, and hence $d(\mathcal{U}, \subseteq) = d(\mathcal{C}(\mathcal{U}), \prec)$.*

Proof. For each $U \in \mathcal{U}$, we set $\gamma_U = \{B(x,U) : x \in X\}$, where we recall that $B(x,U)$ denotes the set $\{y \in X : (x,y) \in U\}$ for each $x \in X$. Clearly, γ_U is a $\mathcal{U}$-uniform cover of X. It is easy to see that the mapping $U \mapsto \gamma_U$ of $(\mathcal{U}, \subseteq)$ to $(\mathcal{C}(\mathcal{U}), \prec)$ is order-preserving, and the set $\{\gamma_U : U \in \mathcal{U}\}$ is obviously dense in $(\mathcal{C}(\mathcal{U}), \prec)$.

Conversely, for every $\gamma \in \mathcal{C}(\mathcal{U})$, put $W_\gamma = \bigcup\{V \times V : V \in \gamma\}$. It is clear that the mapping $\gamma \mapsto W_\gamma$ of $(\mathcal{C}(\mathcal{U}), \prec)$ to $(\mathcal{U}, \subseteq)$ is order-preserving. To show that $\{W_\gamma : \gamma \in \mathcal{C}(\mathcal{U})\}$ is dense in $(\mathcal{U}, \subseteq)$, let $U \in \mathcal{U}$ and take $V \in \mathcal{U}$ which is symmetric and satisfies $2V \subseteq U$. Then

$$W_{\gamma_V} = \bigcup_{x \in X} B(x,V) \times B(x,V),$$

and it is easy to check that $W_{\gamma_V} \subseteq U$, as required.

The equality $d(\mathcal{U}, \subseteq) = d(\mathcal{C}(\mathcal{U}), \prec)$ now follows from (a version of) Lemma 1.1. □

Suppose that $s = \langle \gamma_n : n \in \omega \rangle$ and $t = \langle \lambda_n : n \in \omega \rangle$ are two sequences of $\mathcal{U}$-uniform covers of X, that is, that $s, t \in {}^{\omega}\mathcal{C}(\mathcal{U})$. We write $s \prec t$ if $\gamma_n \prec \lambda_n$ for each $n \in \omega$. This defines the partially ordered set $({}^{\omega}\mathcal{C}(\mathcal{U}), \prec)$, and the next result follows directly from Lemma 3.7 and Theorem 2.10.

Corollary 3.8. *$d({}^{\omega}\mathcal{C}(\mathcal{U}), \prec) = d({}^{\omega}\mathcal{U}, \leq) = D(\mathcal{P}(X, \mathcal{U}), \leq)$ for every uniform space $(X, \mathcal{U})$.*

Our next step is to show that the difference between the characters of the groups $A(X, \mathcal{U})$ and $F(X, \mathcal{U})$ cannot be too big for any $\aleph_0$-narrow uniform space $(X, \mathcal{U})$ (see Theorem 3.10). First, we deal with the special case when the uniform space has a countable base. Technically, this is the most difficult part of the work.

Lemma 3.9. *Let an $\aleph_0$-narrow uniform space $(X, \mathcal{U})$ satisfy $w(X, \mathcal{U}) \leq \aleph_0$. Then $\chi(F(X, \mathcal{U})) \leq \mathfrak{d}$.*

Proof. (I) Let $\{U_n : n \in \omega\}$ be a countable base for the uniformity $\mathcal{U}$. Since $(X, \mathcal{U})$ is $\aleph_0$-narrow, we can choose a sequence $\Gamma = \langle \gamma_n : n \in \omega \rangle$ of countable $\mathcal{U}$-uniform covers of X such that $\gamma_{n+1} \prec \gamma_n$ and $\bigcup\{V \times V : V \in \gamma_n\} \subseteq U_n$ for each $n \in \omega$. Note that the space $(X, \mathcal{U})$ is pseudometrizable (and hence paracompact, using the term without the assumption of separation which most authors include in the definition of paracompactness), and so each cover γ_n can be additionally chosen to be locally finite. We can further assume that if $U \in \gamma_m$ and $V \in \gamma_n$ where $m > n$, then U does not properly contain V, that is, that $U \supseteq V$ implies $U = V$.

(II) We define an order $\leq_t$ on the collection $T(X, \mathcal{U})$ of $\mathcal{U}$-covering trees by specifying that for $P, Q \in T(X, \mathcal{U})$, we have $P \leq_t Q$ if $W_P \supseteq W_Q$ (see equation (3.1)). Clearly, $\leq_t$ is a quasi-order on $T(X, \mathcal{U})$, though not a partial order. Denote by $T(\Gamma) \subseteq T(X, \mathcal{U})$ the set of all $\mathcal{U}$-covering trees P with the property that $\gamma_P \in \Gamma$ and $\gamma_P(x) \in \Gamma$ for each $x \in P$ (see (iii) and (iv) of Definition 3.2). Also, set $\gamma^* = \bigcup_{n \in \omega} \gamma_n$. Then our definition of the sequence $\Gamma = \langle \gamma_n : n \in \omega \rangle$ implies that every $V \in \gamma^*$ is contained only in finitely many distinct elements of γ^*.

Claim 1. *$T(\Gamma)$ is a dominating subset of $(T(X, \mathcal{U}), \leq_t)$.*

For subsets $U_1, \ldots, U_{n-1}, U_n$ of X, we write

$$\widehat{\pi}(U_1, \ldots, U_{n-1}, U_n) = (U_1, \ldots, U_{n-1})$$

if $n \geq 2$, and we write

$$\pi(U_1, \ldots, U_{n-1}, U_n) = U_n$$

if $n \geq 1$.

To prove the claim, let $P \in T(X, \mathcal{U})$. We construct a tree Q as follows. Since γ_P is a $\mathcal{U}$-uniform cover of X, there is $\delta \in \Gamma$ such that $\delta \prec \gamma_P$. We set $Q(0) = \{x \in (\gamma^*)^1 : \pi(x) \in \delta\}$. Since $\delta \prec \gamma_P$, we can choose a function $p_0 \colon Q(0) \to P(0)$ such that $\pi(x) \subseteq \pi(p_0(x))$ for all $x \in Q(0)$. Now for each $x \in Q(0)$, we pick $\delta_x \in \Gamma$ such $\delta_x \prec \gamma_P(p_0(x))$. Then we put $Q(1) = \{y \in$

$(\gamma^*)^2 : x = \widehat{\pi}(y) \in Q(0),\ \pi(y) \in \delta_x\}$. Since $\delta_x \prec \gamma_P(p_0(x))$ for each $x \in Q(0)$, we can choose a function $p_1 : Q(1) \to P(1)$ such that $\widehat{\pi}(p_1(y)) = p_0(\widehat{\pi}(y))$ and $\pi(y) \subseteq \pi(p_1(y))$ for all $y \in Q(1)$. Now for each $y \in Q(1)$, we pick $\delta_y \in \Gamma$ such $\delta_y \prec \gamma_P(p_1(y))$, and then put $Q(2) = \{z \in (\gamma^*)^3 : y = \widehat{\pi}(z) \in Q(1),\ \pi(z) \in \delta_y\}$. Continuing in this way, we finally obtain the tree $Q = \bigcup_{n\in\omega} Q(n)$, ordered according to (ii) of Definition 3.2. We clearly have $Q \in T(\Gamma)$. Also, if $u = (U_0, \dots, U_n) \in Q(n)$, then, by construction, $p_n(u) \in P(n)$, and if we write $p_n(u) = (V_0, \dots, V_n)$, say, then we have $U_0 \subseteq V_0, \dots, U_n \subseteq V_n$, so that each product in the union of the form (3.1) defining W_Q is contained in some product in the corresponding union defining W_P, giving $W_Q \subseteq W_P$, and hence $P \leq_t Q$, as required to prove the claim.

(III) If we define

$$\mathcal{E} = \{(V_0, \dots, V_n) : V_0, \dots, V_n \in \gamma^*,\ n \in \omega\},$$

then it is clear that $|\mathcal{E}| \leq \aleph_0$. Let $\mathcal{F} = {}^{\mathcal{E}}\omega$ be the family of all mappings from $\mathcal{E}$ to ω. Define a partial order $\leq_e$ on $\mathcal{E}$ as follows: for $p = (U_0, \dots, U_k), q = (V_0, \dots, V_l) \in \mathcal{E}$, we define $p \leq_e q$ if and only if $k \leq l$ and $U_i \supseteq V_i$ for $i = 0, \dots, k$. Now we define a "strong" partial order $\leq_s$ on $\mathcal{F} = {}^{\mathcal{E}}\omega$, as follows (the name and notation distinguish the relation from another that we will define later on the same set). For $f, g \in \mathcal{F}$, we write $f \leq_s g$ if $f = g$ or if $p \leq_e q$ in $\mathcal{E}$ always implies $f(p) \leq g(q)$ in ω. It is straightforward to verify that $\leq_s$ is indeed a partial order. We also define a partial order $\ll_s$ on $\omega \times \mathcal{F}$ coordinate-wise, using the usual order $\leq$ on ω and the order $\leq_s$ on $\mathcal{F}$.

We define a function $\Pi : \omega \times \mathcal{F} \to T(X, \mathcal{U})$. For a pair $(m, f) \in \omega \times \mathcal{F}$, the $\mathcal{U}$-covering tree $P = \Pi(m, f)$ is defined as follows. Define the initial level of P by $P(0) = \{(U) : U \in \gamma_m\}$. Suppose that we have defined $P(n) \subseteq (\gamma^*)^{n+1}$ for some $n \in \omega$. Given an element $p = (U_0, \dots, U_n) \in P(n)$, put

$$p^+ = \{(U_0, \dots, U_n, V) : V \in \gamma_{f(p)}\}$$

and then set

$$P(n+1) = \bigcup\{p^+ : p \in P(n)\}.$$

To finish the construction, we put $P = \bigcup_{n\in\omega} P(n)$ and define the partial order $\leq$ on P as in Definition 3.2. Note that the tree $P = \Pi(m, f)$ is in fact an element of $T(\Gamma) \subseteq T(X, \mathcal{U})$.

Claim 2. *The mapping* Π *is order-preserving as a mapping from* $(\omega \times \mathcal{F}, \ll_s)$ *to* $(T(X, \mathcal{U}), \leq_t)$.

Thus, we suppose that $m, n \in \omega$ and $m \leq n$ and that $f, g \in \mathcal{F}$ and $f \leq_s g$, and we show that $W_{\Pi(n,g)} \subseteq W_{\Pi(m,f)}$. Indeed, by (3.1), $W_{\Pi(n,g)}$ is the union over all k of all the sets of the form

$$V_0^{-\varepsilon_0} \cdot V_1^{-\varepsilon_1} \cdots V_k^{-\varepsilon_k} \cdot V_k^{\varepsilon_k} \cdots V_1^{\varepsilon_1} \cdot V_0^{\varepsilon_0},$$

where

$$V_0 \in \gamma_n,\ V_1 \in \gamma_{g(q_1)},\ V_2 \in \gamma_{g(q_2)},\ \dots,\ V_k \in \gamma_{g(q_k)},$$

where

$$q_1 = (V_0),\ q_2 = (V_0, V_1),\ \dots,\ q_k = (V_0, V_1, \dots, V_{k-1}),$$

and where $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_k = \pm 1$. Fix one such set. Since γ_n refines γ_m, there exists $U_0 \in \gamma_m$ such that $V_0 \subseteq U_0$. Put $p_1 = (U_0)$. Then from $f \leq_s g$ it follows that $f(p_1) \leq g(q_1)$. Hence, $\gamma_{g(q_1)}$ refines $\gamma_{f(p_1)}$, so there exists $U_1 \in \gamma_{f(p_1)}$ such that $V_1 \subseteq U_1$. From $f \leq_s g$ it follows that $f(p_2) \leq g(q_2)$, where $p_2 = (U_0, U_1)$. Again, $\gamma_{g(q_2)}$ refines $\gamma_{f(p_2)}$, so there exists $U_2 \in \gamma_{f(p_2)}$ such that $V_2 \subseteq U_2$. Continuing this way, we finally obtain $U_k \in \gamma_{f(p_k)}$ such that $V_k \subseteq U_k$, where $p_k = (U_0, U_1, \dots, U_{k-1})$. Clearly, $p_k \in \Pi(m, f)$, so that the set $U_0^{-\varepsilon_0} \cdot U_1^{-\varepsilon_1} \cdots U_k^{-\varepsilon_k} \cdot U_k^{\varepsilon_k} \cdots U_1^{\varepsilon_1} \cdot U_0^{\varepsilon_0}$ is a summand in the union of the form (3.1) corresponding to $W_{\Pi(m,f)}$. By construction, we have $V_i \subseteq U_i$ for each $i = 0, \dots, k$, whence it follows that

$$V_0^{-\varepsilon_0} \cdots V_k^{-\varepsilon_k} \cdot V_k^{\varepsilon_k} \cdots V_0^{\varepsilon_0} \subseteq U_0^{-\varepsilon_0} \cdots U_k^{-\varepsilon_k} \cdot U_k^{\varepsilon_k} \cdots U_0^{\varepsilon_0}.$$

We conclude, therefore, that $W_{\Pi(n,g)} \subseteq W_{\Pi(m,f)}$, proving our claim.

We claim next that $\Pi(\omega \times \mathcal{F}) = T(\Gamma)$. That $\Pi(\omega \times \mathcal{F}) \subseteq T(\Gamma)$ is clear. For the reverse inclusion, let $P \in T(\Gamma)$. Now $\gamma_P \in \Gamma$, so that $\gamma_P = \gamma_n \in \Gamma$ for some $n \in \mathbb{N}$. Also, for all $x \in P$, we have $\gamma_P(x) \in \Gamma$, so that $\gamma_P(x) = \gamma_{n_x} \in \Gamma$ for some $n_x \in \mathbb{N}$. Define $f : \mathcal{E} \to \omega$, that is, $f \in \mathcal{F}$, by setting $f(x) = n_x$ for all $x \in P$ and $f(x) = 0$ for all $x \notin P$. Then it is easy to see that $\Pi(n, f) = P$, proving that $\Pi(\omega \times \mathcal{F}) \supseteq T(\Gamma)$, and hence our claim.

We now consider the quasi-ordered sets

$$({}^{\mathbb{N}}(\omega \times \mathcal{F}), \ll_s^*) \quad \text{and} \quad ({}^{\mathbb{N}}T(X, \mathcal{U}), \leq_t^*),$$

where the orders $\ll_s^*$ and $\leq_t^*$ are defined coordinate-wise in terms of $\ll_s$ and $\leq_t$, respectively. We likewise consider the mapping

$$\Pi^* \colon ({}^{\mathbb{N}}(\omega \times \mathcal{F}), \ll_s^*) \to ({}^{\mathbb{N}}T(X, \mathcal{U}), \leq_t^*)$$

defined coordinate-wise in terms of Π. It is immediate from what we have shown above that the mapping Π^* is order-preserving, and maps ${}^{\mathbb{N}}(\omega \times \mathcal{F})$ to a dominating subset of $({}^{\mathbb{N}}T(X, \mathcal{U}), \leq_t^*)$. Thus, we conclude that

$$D({}^{\mathbb{N}}T(X, \mathcal{U}), \leq_t^*) \leq D({}^{\mathbb{N}}(\omega \times \mathcal{F}), \ll_s^*). \tag{3.4}$$

(IV) We wish to obtain a different expression for the right-hand side of (3.4). To this end, we define a "weak" partial order $\leq_w$ on $\mathcal{F} = {}^{\mathcal{E}}\omega$, as follows. For $f, g \in \mathcal{F}$, we write $f \leq_w g$ if $f(p) \leq g(p)$ for each $p \in \mathcal{E}$. It is clear that $\leq_w$ is a partial order. It is also clear that $f \leq_s g$ implies $f \leq_w g$. Now we define $\ll_w$ on $\omega \times \mathcal{F}$ and then $\ll_w^*$ on ${}^{\mathbb{N}}(\omega \times \mathcal{F})$ in exact analogy to the earlier definitions of $\ll_s$ and $\ll_s^*$.

Claim 3. $D({}^{\mathbb{N}}(\omega \times \mathcal{F}), \ll_s^*) = D({}^{\mathbb{N}}(\omega \times \mathcal{F}), \ll_w^*)$.

Indeed, it is clear that $D({}^{\mathbb{N}}(\omega \times \mathcal{F}), \ll_w^*) \leq D({}^{\mathbb{N}}(\omega \times \mathcal{F}), \ll_s^*)$, since every dominating set in $({}^{\mathbb{N}}(\omega \times \mathcal{F}), \ll_s^*)$ remains dominating in $({}^{\mathbb{N}}(\omega \times \mathcal{F}), \ll_w^*)$. Next, we have to verify that $D({}^{\mathbb{N}}(\omega \times \mathcal{F}), \ll_s^*) \leq D({}^{\mathbb{N}}(\omega \times \mathcal{F}), \ll_w^*)$. For $q \in \mathcal{E}$, put

$$\mathcal{E}(q) = \{p \in \mathcal{E} : p \leq_e q\}.$$

From our assumptions about the covers γ_n, it follows that $\mathcal{E}(q)$ is finite for every $q \in \mathcal{E}$. In fact, suppose that there exists $q = (V_0, \dots, V_l) \in \mathcal{E}$ such that $\mathcal{E}(q)$ is infinite. Then for some $k \leq l$, the set

$$\mathcal{E}_k(q) = \{p \in \mathcal{E} : p = (U_0, \dots, U_k),\ p \leq_e q\}$$

is infinite. Then for some $i \leq k$, we can find a sequence of elements

$$p_n = (U_0^{(n)}, \dots, U_i^{(n)}, \dots, U_k^{(n)}) \in \mathcal{E}_k(q)$$

for $n \in \mathbb{N}$ such that the sets $U_i^{(n)}$ are distinct for all n and such that $U_i^{(n)} \supseteq V_i$ for all n. If $n_0 \in \mathbb{N}$ is such that $V_i \in \gamma_{n_0}$, then $U_i^{(n)}$ must properly contain V_i for infinitely many n, and it follows that there is $n_1 \leq n_0$ such that infinitely many of the $U_i^{(n)}$ are in γ_{n_1}, contradicting the local finiteness of γ_{n_1}. This contradiction shows that $\mathcal{E}(q)$ is finite, as claimed. It is clear furthermore that each $\mathcal{E}(q)$ is also non-empty.

Now for every $f \in \mathcal{F}$ and $q \in \mathcal{E}$, put

$$\widetilde{f}(q) = \max\{f(p) : p \in \mathcal{E}(q)\},$$

noting that by the argument above the value $\widetilde{f}(q)$ is defined validly. Therefore, we obtain a function $\widetilde{f}\colon \mathcal{E} \to \omega$, that is, $\widetilde{f} \in \mathcal{F}$. It is easy to see that the mapping $f \mapsto \widetilde{f}$ from $(\mathcal{F}, \leq_w)$ to $(\mathcal{F}, \leq_s)$ is order-preserving, and from our definition of the order $\leq_s$ and the function $\widetilde{f}$, it is also easily checked that $f \leq_s \widetilde{f}$ for each $f \in \mathcal{F}$, so that the image of the mapping is a dominating subset of $(\mathcal{F}, \leq_s)$.

For $\varphi \in {}^{\mathbb{N}}(\omega \times \mathcal{F})$, we have $\varphi(n) \in \omega \times \mathcal{F}$ for each $n \in \mathbb{N}$, and we have $\varphi(n) = (\pi_1(\varphi(n)), \pi_2(\varphi(n)))$, where π_1 and π_2 are the projections from $\omega \times \mathcal{F}$ into ω and $\mathcal{F}$, respectively. We then define $\widetilde{\varphi} \in {}^{\mathbb{N}}(\omega \times \mathcal{F})$ by setting $\widetilde{\varphi}(n) = (\pi_1(\varphi(n)), \widetilde{\pi_2(\varphi(n))})$ for each $n \in \mathbb{N}$. Then by the argument above, the mapping $\varphi \mapsto \widetilde{\varphi}$ from $({}^{\mathbb{N}}(\omega \times \mathcal{F}), \ll_w^*)$ to $({}^{\mathbb{N}}(\omega \times \mathcal{F}), \ll_s^*)$ is order-preserving and has as its image a dominating set, proving the inequality $D({}^{\mathbb{N}}(\omega \times \mathcal{F}), \ll_s^*) \leq D({}^{\mathbb{N}}(\omega \times \mathcal{F}), \ll_w^*)$, and hence our claim.

(V) We can now complete the proof of the lemma. Equation (3.3) (immediately preceding Theorem 3.6) defines for us a mapping $s \mapsto O_s$ from ${}^{\mathbb{N}}T(X, \mathcal{U})$ to $\mathcal{N}(e)$, the family of open neighborhoods of the identity e in $F(X, \mathcal{U})$. Further, it is immediate from the relevant definitions that this mapping is order-reversing from $({}^{\mathbb{N}}T(X, \mathcal{U}), \leq_t^*)$ to $(\mathcal{N}(e), \subseteq)$, where $\leq_t^*$ is defined coordinate-wise in terms of $\leq_t$. Moreover, rephrased in this terminology, Theorem 3.6 states that the mapping has a dense subset of $(\mathcal{N}(e), \subseteq)$ as its image. Hence we have

$$d(\mathcal{N}(e), \subseteq) \leq D({}^{\mathbb{N}}T(X, \mathcal{U}), \leq_t^*). \tag{3.5}$$

In addition, we have obvious order-isomorphisms as follows:

$$\begin{aligned} ({}^{\mathbb{N}}(\omega \times \mathcal{F}), \ll_w^*) &\cong ({}^{\mathbb{N}}\omega, \leq) \times ({}^{\mathbb{N}}\mathcal{F}, \leq_w^*) \\ &\cong ({}^{\mathbb{N}}\omega, \leq) \times ({}^{\mathbb{N}\times\mathcal{E}}\omega, \leq) \\ &\cong ({}^{\omega}\omega, \leq) \times ({}^{\omega}\omega, \leq) \\ &\cong ({}^{\omega}\omega, \leq) \qquad (3.6) \end{aligned}$$

(where $\leq_w^*$ is the coordinate-wise extension of $\leq_w$ from $\mathcal{F}$ to ${}^{\mathbb{N}}\mathcal{F}$). Therefore, from (3.5), (3.4), Claim 3 and (3.6), we have

$$\begin{aligned} \chi(F(X,\mathcal{U})) &= d(\mathcal{N}(e), \subseteq) \\ &\leq D({}^{\mathbb{N}}T(X,\mathcal{U}), \leq_t^*) \\ &\leq D({}^{\mathbb{N}}(\omega \times \mathcal{F}), \ll_s^*) \\ &= D({}^{\mathbb{N}}(\omega \times \mathcal{F}), \ll_w^*) \\ &= D({}^{\omega}\omega, \leq) \\ &= \mathfrak{d}. \end{aligned}$$

This finishes the proof. □

The conclusion of the next theorem will be strengthened in Theorem 3.15.

Theorem 3.10. *If* $(X,\mathcal{U})$ *is an* $\aleph_0$*-narrow uniform space, then* $\chi(F(X,\mathcal{U})) \leq \mathfrak{d} \cdot \chi(A(X,\mathcal{U}))$.

Proof. According to Theorem 2.3 and Theorem 2.10, we have

$$\chi(A(X,\mathcal{U})) = D(\mathcal{P}(X,\mathcal{U}), \leq) = d({}^{\omega}\mathcal{U}, \leq), \qquad (3.7)$$

where $\mathcal{P}(X,\mathcal{U})$ is the family of uniformly continuous pseudometrics on $(X,\mathcal{U})$ bounded by 1. Therefore, all we have to verify is that

$$\chi(F(X,\mathcal{U})) \leq \mathfrak{d} \cdot d({}^{\omega}\mathcal{U}, \leq). \qquad (3.8)$$

Denote by $\mathcal{S}$ the family of all sequences $\mathcal{V} = \langle U_n : n \in \omega\rangle \in {}^{\omega}\mathcal{U}$ such that $3U_{n+1} \subseteq U_n$ for each $n \in \omega$. It is clear that $\mathcal{S}$ is dense in $({}^{\omega}\mathcal{U}, \leq)$, whence $d(\mathcal{S}, \leq) = d({}^{\omega}\mathcal{U}, \leq)$. Choose a dense subset $\mathcal{D}$ of $(\mathcal{S}, \leq)$ of the minimal cardinality. Then $\mathcal{D}$ is also dense in $({}^{\omega}\mathcal{U}, \leq)$. Note that the set of terms of the sequence $\mathcal{V}$ is a base for a (non-Hausdorff) uniformity $\widetilde{\mathcal{V}}$ on X for each $\mathcal{V} \in \mathcal{S}$, and hence Lemma 3.9 implies that $\chi(F(X,\widetilde{\mathcal{V}})) \leq \mathfrak{d}$. So, for every $\mathcal{V} \in \mathcal{D}$, we can find a base $\mathcal{B}(\mathcal{V})$ at the identity in $F(X,\widetilde{\mathcal{V}})$ such that $|\mathcal{B}(\mathcal{V})| \leq \mathfrak{d}$. It is clear that the family $\mathcal{B} = \bigcup_{\mathcal{V}\in\mathcal{D}} \mathcal{B}(\mathcal{V})$ satisfies $|\mathcal{B}| \leq \mathfrak{d} \cdot d({}^{\omega}\mathcal{U}, \leq)$, and we claim that $\mathcal{B}$ is a base at the identity in the group $F(X,\mathcal{U})$.

By assumption, the space $(X,\mathcal{U})$ is $\aleph_0$-narrow, so the group $F(X,\mathcal{U})$ is $\aleph_0$-narrow according to [2, Lemma 3.2] or [6]. Hence the topology of $F(X,\mathcal{U})$ is generated by continuous homomorphisms to second countable topological groups (see [6] or [25, Lemma 3.7]). In other words, given a neighborhood U of the identity in $F(X,\mathcal{U})$, one can find a continuous homomorphism $f\colon F(X,\mathcal{U}) \to G$ to a second countable topological group G and an open neighborhood V of the identity in G such that $f^{-1}(V) \subseteq U$. Choose a countable base $\{V_n : n \in \omega\}$

at the identity of G such that $V_0 = V$ and $V_{n+1}^3 \subseteq V_n$ for each $n \in \omega$. For $n = 0, 1, \ldots$, put

$$U_n = \{(x, y) \in X \times X : f(x)^{-1} \cdot f(y) \in V_n,\ f(x) \cdot f(y)^{-1} \in V_n\}. \tag{3.9}$$

Evidently $3U_{n+1} \subseteq U_n$ for each $n \in \omega$, so that $\mathcal{V} = \langle U_n : n \in \omega\rangle \in \mathcal{S}$. Since $\mathcal{D}$ is dense in $(\mathcal{D}, \leq)$, we can assume that $\mathcal{V} \in \mathcal{D}$. Our choice of $\mathcal{V}$ (see (3.9)) guarantees that the restriction of f to X is a uniformly continuous mapping of $(X, \widetilde{\mathcal{V}})$ to $(G, {}^*\mathcal{V}^*)$, where ${}^*\mathcal{V}^*$ is the two-sided uniformity of the group G. Hence the homomorphism $f \colon F(X, \widetilde{\mathcal{V}}) \to G$ remains continuous. Take an element $W \in \mathcal{B}(\mathcal{V})$ such that $f(W) \subseteq V$. Then $W \subseteq f^{-1}(V) \subseteq U$, and hence $\mathcal{B}$ is a base at the identity in $F(X,\mathcal{U})$. This proves (3.8) and the theorem. □

Combining Corollary 2.17 and Theorem 3.10, we obtain the following result, which will be given its final form in Theorem 3.15.

Corollary 3.11. *If an $\aleph_0$-narrow Hausdorff uniform space $(X,\mathcal{U})$ contains an infinite precompact set, then $\chi(A(X,\mathcal{U})) = \chi(F(X,\mathcal{U}))$.*

Now we proceed to show, in Theorem 3.15, that the existence of an infinite precompact set in $(X,\mathcal{U})$ can be omitted in the assumptions of Corollary 3.11. The main additional information we need is given in the following result.

Lemma 3.12. *If $(X,\mathcal{U})$ is an $\aleph_0$-narrow uniform P-space, then the group $F(X,\mathcal{U})$ has a base at the identity consisting of open normal subgroups. In particular, the topology of $F(X,\mathcal{U})$ is generated by Graev's extensions of the uniformly continuous pseudometrics on $(X,\mathcal{U})$.*

Proof. The first assertion follows from the uniform analog of [22, Th. 4]. For the second assertion, suppose that $(X,\mathcal{U})$ is a uniform P-space, and let U be an open neighborhood of the identity e in $F(X,\mathcal{U})$. Then there exists an open normal subgroup V of $F(X,\mathcal{U})$ with $V \subseteq U$. Consider the open cover $\gamma = \{X \cap xV : x \in X\}$ of the space X. It is clear that γ is a $\mathcal{U}$-uniform cover. Since V is a subgroup of $F(X,\mathcal{U})$, the family γ is a partition of X, i.e., every two elements of γ are disjoint or coincide. Define a pseudometric ϱ on X by setting $\varrho(x, y) = 0$ if $x, y \in O$ for some $O \in \gamma$, and $\varrho(x, y) = 1$ otherwise. Clearly, ϱ is uniformly continuous. Let $\widehat{\varrho}$ be Graev's extension of ϱ to the maximal invariant pseudometric on $F_a(X)$ (see [5, Section 3]). Then the set

$$W_\varrho = \{g \in F_a(X) : \widehat{\varrho}(g, e) < 1\}$$

is an open neighborhood of e in $F(X,\mathcal{U})$ by the continuity of $\widehat{\varrho}$ on $F(X,\mathcal{U})$. It remains to show that $W_\varrho \subseteq V$.

From the fact that the pseudometric ϱ is $\{0, 1\}$-valued, it is immediate from Graev's construction that the extension $\widehat{\varrho}$ is integer-valued. We therefore have

$$W_\varrho = \{g \in F_a(X) : \widehat{\varrho}(g, e) = 0\}.$$

(Indeed, it follows that W_ϱ is an open normal subgroup of $F(X,\mathcal{U})$.) Further, Graev's construction shows straightforwardly that for $g \in W_\varrho$, there exist (non-reduced) representations

$$g = x_1^{\varepsilon_1} \cdots x_{2n}^{\varepsilon_{2n}} \quad \text{and} \quad e = y_1^{\varepsilon_1} \cdots y_{2n}^{\varepsilon_{2n}}$$

of g and e, for some $n \in \mathbb{N}$, some $x_1, \ldots x_{2n}, y_1, \ldots y_{2n} \in X$, and $\varepsilon_1, \ldots \varepsilon_{2n} = \pm 1$, such that $\varrho(x_i, y_i) = 0$ for $i = 1, \ldots, 2n$. Now we clearly have $y_i^{\varepsilon_i} y_{i+1}^{\varepsilon_{i+1}} = e$ for some i, from which it follows that $x_i^{\varepsilon_i} x_{i+1}^{\varepsilon_{i+1}} \in V$. Set $g_1 = x_1^{\varepsilon_1} \cdots x_{i-1}^{\varepsilon_{i-1}}$ and $g_2 = x_{i+2}^{\varepsilon_{i+2}} \cdots x_{2n}^{\varepsilon_{2n}}$, and put $\widehat{g} = g_1 g_2$. Note that $\hat{g} \in W_\varrho$. If we assume inductively that $\widehat{g} \in V$, we also have $g_2 g_1 \in V$ by the normality of V, and then we have $g_1^{-1} g g_1 = x_i^{\varepsilon_i} x_{i+1}^{\varepsilon_{i+1}} g_2 g_1 \in V$, from which we have $g \in V$, again by normality. It follows by induction that $W_\varrho \subseteq V$, as required. □

This allows us to extend Theorem 2.13 to the non-abelian case, assuming additionally that our uniform space is $\aleph_0$-narrow.

Theorem 3.13. *If $(X,\mathcal{U})$ is an $\aleph_0$-narrow uniform P-space, then $\chi(F(X,\mathcal{U})) = w(X,\mathcal{U})$.*

Proof. By Lemma 3.12, the topology of $F(X,\mathcal{U})$ is generated by Graev's extensions of the uniformly continuous pseudometrics on $(X,\mathcal{U})$. It follows, therefore, that $\chi(F(X,\mathcal{U})) \le D(\mathcal{P}(X,\mathcal{U}), \le) = w(X,\mathcal{U})$. However, Theorem 2.3 and Lemma 3.1 together imply that $D(\mathcal{P}(X,\mathcal{U}), \le) = \chi(A(X,\mathcal{U})) \le \chi(F(X,\mathcal{U}))$. Combining these inequalities, we obtain the required conclusion. □

In contrast to Theorem 2.13, the assumption of $\aleph_0$-narrowness cannot be removed in Theorem 3.13, as is shown by the following example.

Example 3.14. For every cardinal $\tau > \aleph_1$, there exists a Hausdorff uniform P-space $(X,\mathcal{U})$ such that $w(X,\mathcal{U}) = \aleph_1 < \tau < \chi(F(X,\mathcal{U}))$.

Indeed, let $X = L \oplus D$ be the topological sum of the one-point Lindelöfication L of a discrete space Y with $|Y| = \aleph_1$ and a discrete space D of cardinality $\tau > \aleph_1$. Denote by x_0 the unique non-isolated point of L (and of X). Then a base of open neighborhoods of x_0 in L (and in X) consists of the sets $L \setminus C$, where C is an arbitrary countable subset of Y. Since $|Y| = \aleph_1$, it is easy to see that $\chi(x_0, L) = \chi(x_0, X) = \aleph_1$. Let $\mathcal{U}$ be the fine uniformity of the space X. Then a basic entourage of the diagonal Δ in $X \times X$ has the form

$$U_C = \{(x,y) \in L \times L : x, y \in L \setminus C\} \cup \Delta,$$

where $C \subseteq Y$ is countable. Therefore, $w(X,\mathcal{U}) = \aleph_1$. Let us show that $\chi(F(X,\mathcal{U})) \ge \tau$.

For every $a \in D$, put $L_a = a^{-1} x_0^{-1} L a$, and consider the subspace $Z = \bigcup_{a \in D} L_a$ of $F(X,\mathcal{U}) \cong F(X)$. Apply an argument similar to that in [1, Prop. 3.2] to show that Z is homeomorphic to the fan $V(\tau)$ obtained from the topological sum T of τ copies of L by identifying to a point the set T' of all non-isolated points of T. Since each of the τ distinct spines of the fan $V(\tau)$ is homeomorphic to L, a straightforward diagonal argument implies that $\tau < \chi(e, Z) \le \chi(F(X))$. □

Finally, we have the main result of this section.

Theorem 3.15. *The equality* $\chi(A(X,\mathcal{U})) = \chi(F(X,\mathcal{U}))$ *holds for every* $\aleph_0$-*narrow uniform space* $(X,\mathcal{U})$.

Proof. If $(X,\mathcal{U})$ is a uniform P-space, then the required equality is given by Theorems 2.13 and 3.13, while if $(X,\mathcal{U})$ is not a uniform P-space, then the equality follows from Theorems 2.15 and 3.10 and Lemma 3.1. □

The above theorem has several applications; the following four are immediate.

Corollary 3.16. $\chi(A(X)) = \chi(F(X))$ *for every* $\aleph_0$-*narrow space* X.

Using Corollary 2.22, we have in particular:

Corollary 3.17. *If* X *is an infinite compact metrizable space, then* $\chi(F(X)) = \mathfrak{d}$.

Corollary 3.18. *Suppose that a space* X *contains a dense Lindelöf subspace. Then* $\chi(A(X))=\chi(F(X))$.

Proof. It is easy to see that the space X is $\aleph_0$-narrow. Indeed, let Y be a dense Lindelöf subspace of X. If γ is a discrete family of non-empty open sets in X, then the family $\mu = \{U \cap Y : U \in \gamma\}$ is a discrete family of non-empty open subsets of Y. However, every such family of subsets of Y is countable, so $|\gamma| = |\mu| \leq \aleph_0$. Now the necessary conclusion follows from Corollary 3.16. □

Clearly, every space of countable cellularity is $\aleph_0$-narrow. Therefore, we have the following.

Corollary 3.19. *If a space* X *has countable cellularity, then* $\chi(A(X)) = \chi(F(X))$.

We believe that Corollary 3.21 below compared with Theorem 2.3 or Corollary 2.4 gives a more comprehensive expression for the character of the groups $F(X)$ and $A(X)$ on a Lindelöf space X. It is based on a simple relation between dense subsets of $({}^{\omega}\mathcal{U}_X, \leq)$ and the character of the diagonal in $(X \times \omega)^2$, where $\mathcal{U}_X$ is the fine uniformity of the space X and the set ω carries the discrete topology.

We recall that if $s = \{U_n : n \in \omega\}$ and $t = \{V_n : n \in \omega\}$ are elements of ${}^{\omega}\mathcal{U}$, where $\mathcal{U}$ is a uniformity on some set, then $s \leq t$ means that $U_n \subseteq V_n$ for each $n \in \omega$.

Lemma 3.20. *If* X *is a paracompact topological space and* $\mathcal{U}_X$ *is the fine uniformity on* X, *then* $d({}^{\omega}\mathcal{U}_X, \leq) = \chi_\Delta(X \times \omega)$.

Proof. We can assume that X is not discrete—otherwise, the equality becomes trivial. Denote by $\mathcal{U}_Y$ the fine uniformity on $Y \equiv X \times \omega$. The paracompactness of X implies that every neighborhood of the diagonal Δ_X in X^2 belongs to the fine uniformity $\mathcal{U}_X$, and similarly for the diagonal Δ_Y in Y^2. The family of all neighborhoods of Δ_Y in Y^2 contains a base which can be naturally identified with the family of all sequences $\{U_n : n \in \omega\}$, where $U_n \in \mathcal{U}_X$ for each $n \in \omega$. It is now immediate that $d({}^\omega\mathcal{U}_X, \leq) = \chi_\Delta(X \times \omega)$. □

Finally, Theorem 3.15, Corollary 2.4, Theorem 2.10 (with $\mathcal{U} = \mathcal{U}_X$) and Lemma 3.1 imply the following result.

Corollary 3.21. *Let X be a Lindelöf space. Then $\chi(F(X)) = \chi(A(X)) = \chi_\Delta(X \times \omega)$.*

It may be worth remarking that the conclusion of the corollary holds in particular if X is a k_ω-space or a compact space. In the sequel [14] to this paper, we will investigate the compact case in more depth.

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