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The character of free topological groups II

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Abstract. A systematic analysis is made of the character of the free and free abelian topological groups on metrizable spaces and compact spaces, and on certain other closely related spaces. In the first case, it is shown that the characters of the free and the free abelian topological groups on X are both equal to the "small cardinal" $\mathfrak{d}$ if X is compact and metrizable, but also, more generally, if X is a non-discrete k_ω-space all of whose compact subsets are metrizable, or if X is a non-discrete Polish space. An example is given of a zero-dimensional separable metric space for which both characters are equal to the cardinal of the continuum. In the case of a compact space X, an explicit formula is derived for the character of the free topological group on X involving no cardinal invariant of X other than its weight; in particular the character is fully determined by the weight in the compact case. This paper is a sequel to a paper by the same authors in which the characters of the free groups were analysed under less restrictive topological assumptions.

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1. Introduction

In a previous paper [11], we investigated the topological character of free and free abelian topological groups. The results obtained were for the free groups on uniform spaces, with applications to the free groups on topological spaces deduced as appropriate. Also, the principal results were obtained without the imposition of strong uniform or topological conditions on the given spaces, though numerous corollaries were derived at various points for metrizable spaces, compact spaces and other classes of spaces.

In this sequel to [11], we specifically investigate the characters of the free and free abelian topological groups on metrizable spaces and on compact spaces, and on certain closely related spaces, obtaining more detailed information in both cases than was available in [11].

In the metrizable case, we show that the equality $\chi(A(X)) = \chi(F(X)) = \mathfrak{d}$ holds if X is a compact metrizable space (as was already observed in [11]), but also if X is a non-discrete k_ω-space all compact subsets of which are metrizable (Theorem 2.9), or if X is a non-discrete Polish space (Corollary 2.12). On the other hand, there exists a zero-dimensional separable metric space X such that $\chi(A(X)) = \chi(F(X)) = \mathfrak{c}$ (Example 2.18). If X is a metrizable space in which the subset of all non-isolated points is compact and non-empty, then $\chi(A(X)) = \mathfrak{d}$ (Theorem 2.7), but under the same hypotheses the character $\chi(F(X))$ may be arbitrarily large (Example 2.8).

In the case of a compact space X, our main result gives an explicit formula for $\chi(F(X))$ involving no cardinal invariant of X other than the weight (Theorem 3.5), showing in particular that the character is fully determined by the weight in the compact case. If the weight $w(X)$ of X is at least $\mathfrak{c}$, then our result implies that $\chi(A(X)) = \chi(F(X)) = w(X)^{\aleph_0}$.

Our notation and terminology here are as in [11]. From time to time, results from [11] will be used here, and again the reader is referred to the source for these, though on occasion we quote them here for convenience.

2. Free groups on metrizable spaces

As usual, $w(X,\mathcal{U})$ denotes the weight of the Hausdorff uniform space $(X,\mathcal{U})$, where by the weight we mean in all cases the least cardinal of a base of $\mathcal{U}$, so that $w(X,\mathcal{U}) = \aleph_0$ implies in particular that the family of all uniform entourages of the diagonal in X^2 does not have a minimal element. (A similar convention applies to our usage of other cardinal invariants.)

The principal results of [11] on the characters of the free groups on (pseudo) metrizable spaces are the following three (see Theorem 2.21 and Corollaries 2.22 and 3.16, respectively).

Theorem 2.1. *Let $(X,\mathcal{U})$ be an arbitrary uniform space with $w(X,\mathcal{U}) = \aleph_0$. Then $\chi(A(X,\mathcal{U})) = \mathfrak{d}$.*

Corollary 2.2. *If X is an infinite compact metrizable space, then $\chi(A(X)) = \mathfrak{d}$.*

Following [11], we call a space X *ω-narrow* (equivalently, *pseudo-ω_1-compact*) if every locally finite family of open sets in X is countable. It is clear that all Lindelöf and all separable spaces are ω-narrow.

Corollary 2.3. $\chi(A(X)) = \chi(F(X))$ *for every ω-narrow space X.*

From Corollaries 2.2 and 2.3, we have:

Corollary 2.4. *The equalities $\chi(F(X)) = \chi(A(X)) = \mathfrak{d}$ hold for every infinite compact metrizable space X.*

Brief comments were made in [11] about the character of a free topological group when equipped with the Graev topology rather than the free topology. We make one further such observation. Recall that if X is a topological space, then $F_G(X)$ denotes the abstract free group $F_a(X)$ over X topologized with Graev's topology, that is, the finest invariant group topology on $F_a(X)$ coarser than the topology of $F(X)$. From Corollary 2.6 of [11] it follows that $\chi(F_G(X)) = \chi(A(X))$, for every space X. This equality combined with our Corollary 2.3 implies the following result.

Theorem 2.5. *If X is an ω-narrow space, then $\chi(F_G(X)) = \chi(F(X))$.*

A further result from [11] is the following (Theorem 2.23). In it, we use $\chi_\Delta(X)$ to denote the character of the diagonal Δ in $X \times X$.

Theorem 2.6. *If a Tychonoff space X satisfies $\chi_\Delta(X) \le \aleph_0$, then either X and $A(X)$ are discrete or $\chi(A(X)) = \mathfrak{d}$.*

Starting from the above results, we develop here a sequence of new results which give more detailed information on the characters of the free and free abelian topological groups on metrizable spaces and certain spaces closely related to metrizable spaces.

Theorem 2.7. *If X is a metrizable space and the set X' of all non-isolated points of X is compact and non-empty, then $\chi(A(X)) = \mathfrak{d}$.*

Proof. We claim that $\chi_\Delta(X) = \aleph_0$. Indeed, let d be a metric on X which induces the topology of X. For every $x \in X$ and $\varepsilon > 0$, denote by $B(x, \varepsilon)$ the open ball with center at x and radius ε with respect to d. If $n \in \mathbb{N}$, we put

$$U_n = \Delta \,\cup\, \bigcup\{B(x, 1/n) \times B(x, 1/n) : x \in X'\},$$

where Δ is the diagonal in $X \times X$. It is easy to see that the sets U_n form a base at the diagonal Δ in $X \times X$, which proves our claim. Now the desired conclusion follows from Theorem 2.6. □

It is interesting to note that the non-abelian analog of Theorem 2.7 fails, as the next example shows.

Example 2.8. The character of the free topological group $F(X)$ on a metrizable space X with a single non-isolated point can be arbitrarily large (while the character of $A(X)$ is equal to $\mathfrak{d}$ by Theorem 2.7).

Indeed, let $X = C \oplus D$ be the topological sum of a non-trivial convergent sequence C with limit point $x_0 \in C$ and a discrete space D of an infinite cardinality τ. For every $a \in D$, put $C_a = a^{-1}x_0^{-1}Ca$, and consider the subspace $Y = \bigcup_{a\in D} C_a$ of $F(X)$. As is shown in [1], Y is homeomorphic to the Fréchet–Urysohn fan $V(\tau)$ of cardinality τ with vertex at the identity e of the group $F(X)$. A straightforward diagonal argument shows that $\tau < \chi(e, Y) \le \chi(F(X))$. □

It turns out that Corollary 2.4 remains valid in a more general case. Let us say that a space X with a k_ω-decomposition $X = \bigcup_{n\in\omega} X_n$ is a *km_ω-space* if each X_n is metrizable. Equivalently, a km_ω-space is a k_ω-space all compact subsets of which are metrizable.

Theorem 2.9. *Let X be a non-discrete km_ω-space. Then $\chi(A(X)) = \mathfrak{d} = \chi(F(X))$.*

Proof. By assumption, there exists a k_ω-decomposition $X = \bigcup_{n\in\omega} X_n$, where each X_n is compact and metrizable. Clearly X_n is non-discrete for some $n \in \omega$, for otherwise X would be discrete. Therefore, X contains infinite compact subsets (convergent sequences), so Corollary 2.18 of [11] and our Corollary 2.3 together imply that $\mathfrak{d} \le \chi(A(X)) = \chi(F(X))$. It remains to verify that $\chi(F(X)) \le \mathfrak{d}$.

Denote by Y the one-point compactification of the topological sum $X' = \oplus_{n\in\omega} X'_n$, where $X'_n = X_n \times \{n\}$ for each $n \in \omega$. It is easy to see that the infinite compact space Y is metrizable, and so we have $\chi(F(Y)) = \mathfrak{d}$ by Corollary 2.4. Choose an element $a \in Y$ and put $Z = \bigcup_{n\in\omega} a^n \cdot X'_n \subseteq F(Y)$. Then $Z \cap F_{n+1}(Y) = \bigcup_{k=0}^{n} a^k \cdot X'_k$, so the intersection $Z \cap F_{n+1}(Y)$ is closed in $F_{n+1}(Y)$ for each $n \in \omega$. By Graev's theorem in [4], $\bigcup_{n\in\omega} F_n(Y)$ is a k_ω-decomposition of the group $F(Y)$, so Z is closed in $F(Y)$. Therefore, $F(Y)$ contains a subgroup topologically isomorphic to $F(Z)$ (see [8, Th. 1]), and hence $\chi(F(Z)) \le \chi(F(Y))$. Let $f\colon X' \to X$ be the mapping defined by $f(y, n) = y$ for all $y \in X_n$, $n \in \omega$. Since $X = \bigcup_{n\in\omega} X_n$ is a k_ω-decomposition of X, the mapping f is quotient. Clearly, $f(X') = X$. Define a mapping $g\colon X' \to Z$ by $g(x) = a^n x$ for each $x \in X'_n$, $n \in \omega$. It is easy to see that g is a homeomorphism, so that the mapping $h = f \circ g^{-1}\colon Z \to X$ is a quotient. Hence the extension of h to a homomorphism $\widehat{h}\colon F(Z) \to F(X)$ is continuous and open. We therefore conclude that $\chi(F(X)) \le \chi(F(Z)) \le \chi(F(Y)) = \mathfrak{d}$. □

By Corollary 2.3, $\chi(F(X)) = \chi(A(X))$ for each separable metrizable space X. Our next task is to calculate the values $\chi(A(\mathbb{Q})) = \chi(F(\mathbb{Q}))$ and $\chi(A(\mathbb{R}^\omega)) = \chi(F(\mathbb{R}^\omega))$. Here we show that all these cardinals are equal to $\mathfrak{d}$. This will follow from a more general result: If X is a non-discrete separable metrizable space which is absolutely G_δ, F_σ or $G_{\delta\sigma}$, then $\chi(A(X)) = \chi(F(X)) = \mathfrak{d}$ (see Theorem 2.11). In particular, $\chi(A(X)) = \chi(F(X)) = \mathfrak{d}$ for every non-discrete Polish space X.

Lemma 2.10. *Let X be a non-discrete separable metrizable space such that $X \times \omega \cong X$. Then $\chi(A(X)) = \chi(F(X)) = \chi_\Delta(X)$.*

Proof. Since X is Lindelöf and $X \times \omega \cong X$, Corollary 3.21 of [11] gives us $\chi(F(X)) = \chi(A(X)) = \chi_\Delta(X \times \omega) = \chi_\Delta(X)$, as required. □

We recall that a separable metrizable space X is called *absolutely* G_δ, F_σ or $G_{\delta\sigma}$ if X is of type G_δ, F_σ or $G_{\delta\sigma}$, respectively, in some (equivalently, every) metrizable compactification of X.

Theorem 2.11. *Let X be a non-discrete separable metrizable space. If X is absolutely G_δ, F_σ or $G_{\delta\sigma}$, then $\chi(A(X)) = \chi(F(X)) = \mathfrak{d}$.*

Proof. Since $\chi(F(X)) = \chi(A(X))$ by Corollary 2.3, it suffices to verify that $\chi(A(X)) = \mathfrak{d}$. If X is absolutely G_δ, then it is a perfect image of a closed subspace K of the irrationals $\mathbb{P} \cong \mathbb{N}^\omega$, by (C) on page 144 of [2]. Let $f\colon K \to X$ be the corresponding perfect mapping. Then f extends to a continuous open homomorphism $\widehat{f}\colon A(K) \to A(X)$, so that $\chi(A(X)) \le \chi(A(K))$. Since K is closed in the separable metrizable space $\mathbb{P}$, every continuous (pseudo)metric on K extends to a continuous (pseudo)metric on $\mathbb{P}$, and Theorem 1.2.9 of [10] and Lemma 4 of [14] imply that $A(K)$ is topologically isomorphic to a subgroup of $A(\mathbb{P})$. Hence $\chi(A(K)) \le \chi(A(\mathbb{P}))$. Note that $\mathbb{P} \cong \mathbb{P} \times \omega$. Since $\mathbb{P}$ is absolutely G_δ, Theorem 8.13 of [2] implies that $\chi_\Delta(\mathbb{P}) = \mathfrak{d}$, and hence $\chi(A(\mathbb{P})) = \mathfrak{d}$ by Lemma 2.10. Since X is non-discrete (hence contains infinite compact subsets), from Corollary 2.18 of [11] it follows that

$$\mathfrak{d} \le \chi(A(X)) \le \chi(A(K)) \le \chi(A(\mathbb{P})) = \mathfrak{d}.$$

Similarly, if X is absolutely F_σ or $G_{\delta\sigma}$, then so is the product $X \times \omega$, and [2, Th. 8.13] implies that $\chi_\Delta(X \times \omega) = \mathfrak{d}$. Since $X \times \omega \times \omega \cong X \times \omega$, we can apply Lemma 2.10 to conclude that $\chi(A(X \times \omega)) = \mathfrak{d}$. Clearly, X is a continuous open image of $X \times \omega$, which immediately implies that $\chi(A(X)) \le \chi(A(X \times \omega)) = \mathfrak{d}$. Since X is non-discrete, an application of Corollary 2.18 of [11] finishes the proof. □

The above theorem implies, in particular, that $\chi(F(\mathbb{Q})) = \chi(F(K \times \mathbb{Q})) = \mathfrak{d}$ for every compact metrizable space K. Since complete separable metrizable ($\equiv$ Polish) spaces are absolutely G_δ, we obtain the following.

Corollary 2.12. *If X is a non-discrete Polish space, then $\mathfrak{d} = \chi(A(X)) = \chi(F(X))$.*

We therefore have, for example, $\chi(F(\mathbb{P})) = \chi(F(\mathbb{R}^\omega)) = \mathfrak{d}$. However, our results leave the following open problems.

Problem 2.13. *Does the inequality $\chi(F(X)) \le \mathfrak{d}$ hold for any absolutely Borel* (*analytic*) *separable metrizable space X?*

Problem 2.14. *Let X be an ω-narrow space such that $w(X,\mathcal{U}) \le \mathfrak{d}$, where $\mathcal{U}$ is the fine uniformity of X. Is then $\chi(F(X)) \le \mathfrak{d}$? What if $X \times \omega \cong X$?*

Since the group $A(X)$ on a separable metrizable space X is separable, its character does not exceed $\mathfrak{c}$. On the other hand, $\chi(A(X)) \ge \mathfrak{d}$ for a non-discrete metrizable space X by Corollary 2.18 of [11]. Our aim is to show that there exists in ZFC a separable metrizable space X satisfying $\chi(A(X)) = \chi(F(X)) = \mathfrak{c}$. First, we study the character of X in its Čech–Stone compactification βX, and relate it with the character of the diagonal Δ_X in the product $X \times X$. The straightforward proof of the next lemma is left to the reader.

Lemma 2.15. *Let bX be an arbitrary compactification of a Tychonoff space X. Then $\chi(X, bX) = \chi(X, \beta X)$.*

The following result generalizes Corollary 15 of [12].

Lemma 2.16. *If a space X is paracompact, then $\chi(X, \beta X) \le \chi_\Delta(X)$.*

Proof. Let $\mathcal{B}$ be a base for the diagonal Δ_X in X^2 such that $|\mathcal{B}| = \chi_\Delta(X)$. For every $U \in \mathcal{B}$, choose an open cover $\gamma = \gamma_U$ of X such that

$$\bigcup\{V \times V : V \in \gamma\} \subseteq U.$$

For every open set V in X, put $\widetilde{V} = \beta X \setminus cl_{\beta X}(X \setminus V)$. It is clear that $\widetilde{V}$ is open in βX and $\widetilde{V} \cap X = V$. In particular, V is dense in $\widetilde{V}$, and hence $\widetilde{V} \subseteq cl_{\beta X}(V)$. If $U \in \mathcal{B}$, consider the family $\widetilde{\gamma}_U = \{\widetilde{V} : V \in \gamma_U\}$ and the set $W_U = \bigcup \widetilde{\gamma}_U$. Then W_U is open in βX and $X \subseteq W_U$ for each $U \in \mathcal{B}$. We claim that the family $\lambda = \{W_U : U \in \mathcal{B}\}$ is a base for X in βX.

Let W be an arbitrary open neighborhood of X in βX. Put $F = \beta X \setminus W$ and consider the closed subset $P = X \times F$ of $X \times \beta X$. Denote by $\Delta_{\beta X}$ the diagonal in $(\beta X)^2$. Evidently, $\Delta_X = (X \times \beta X) \cap \Delta_{\beta X}$ is closed in $X \times \beta X$ and $P \cap \Delta_X = \varnothing$. Since the product $X \times \beta X$ is normal (see [3, Th. 5.1.38]), we can find disjoint open sets O and O' in $X \times \beta X$ such that $\Delta_X \subseteq O$ and $P \subseteq O'$. Then there exists $U \in \mathcal{B}$ such that $U \subseteq O \cap (X \times X)$. Take an arbitrary element $V \in \gamma_U$ and pick a point $x \in V$. Since $V \times V \subseteq U \subseteq O$, we have $\{x\} \times V \subseteq O$, and hence $\{x\} \times cl_{\beta X}(V) \subseteq cl_{X \times \beta X}(O)$. By our choice, the sets O and O' are disjoint and $P = X \times F \subseteq O'$. Therefore, $cl_{\beta X}(V) \cap F = \varnothing$. Since $\widetilde{V} \subseteq cl_{\beta X}(V)$ and $F = \beta X \setminus W$, we conclude that $\widetilde{V} \subseteq W$. This inclusion holds for each $V \in \gamma_U$, so $W_U = \bigcup \widetilde{\gamma}_U \subseteq W$. This proves our claim.

Finally, from our definition of λ it follows follows that $|\lambda| \le |\mathcal{B}|$, and hence $\chi(X, \beta X) \le \chi_\Delta(X)$. □

Let X be a paracompact space. Then every open neighborhood of the diagonal Δ_X in X^2 belongs to the fine uniformity $\mathcal{U}$ on X. This implies, in our notation, that $w(X, \mathcal{U}) = \chi_\Delta(X)$. Since the free abelian topological group on X is precisely the free abelian topological group on the uniform space $(X, \mathcal{U})$, Corollary 2.11 of [11] implies the following result.

Corollary 2.17. *Let X be a paracompact space. Then $\chi_\Delta(X) \le \chi(A(X))$.*

By Corollary 2.4, the character of the groups $F(X)$ and $A(X)$ on every infinite compact metrizable space X is equal to the cardinal $\mathfrak{d}$, which is consistently less than $\mathfrak{c}$ (see [2, 15]). The equalities $\chi(F(X)) = \chi(A(X)) = \mathfrak{d}$ remain valid for every non-discrete Polish space X (see Corollary 2.12). In the general case, the situation is different.

Example 2.18. There exists a zero-dimensional separable metric space X such that the groups $F(X)$ and $A(X)$ both have character equal to $\mathfrak{c}$.

Indeed, let X and Y be disjoint Bernstein subsets of the real line $\mathbb{R}$ such that $\mathbb{R} = X \cup Y$ and $|X| = |Y| = \mathfrak{c}$. Then compact subsets of X and Y are at most countable and both X and Y are dense in $\mathbb{R}$. Hence X is a zero-dimensional separable metric space. Clearly, the groups $F(X)$ and $A(X)$ are also separable, so their respective characters do not exceed $\mathfrak{c}$. Since $\chi(A(X)) = \chi(F(X))$, by Corollary 2.3, it suffices to verify that $\chi(A(X)) = \mathfrak{c}$.

Denote by Z the one-point compactification of $\mathbb{R}$. Then Z is also a compactification of X. Therefore, $\chi(X, Z) = \chi(X, \beta X) \leq \chi_\Delta(X)$, by Lemmas 2.15 and 2.16. Since Y is a Bernstein subset of $\mathbb{R}$, we have $|\mathbb{R} \setminus U| \leq \aleph_0$ for every open set U in $\mathbb{R}$ containing X. In addition, the cardinality of Y is equal to $\mathfrak{c}$, so we have

$$\mathfrak{c} \leq \psi(X, \mathbb{R}) = \psi(X, Z) \leq \chi(X, Z) \leq \chi_\Delta(X),$$

where we use ψ to denote the pseudocharacter. This chain of inequalities and Corollary 2.17 enable us to conclude that

$$\mathfrak{c} \leq \chi_\Delta(X) \leq \chi(A(X)) \leq \mathfrak{c}.$$

Thus we have $\chi(F(X)) = \chi(A(X)) = \mathfrak{c}$. □

It may be worth remarking that if $\mathcal{U}'$ is the natural metric uniformity on X inherited from $\mathbb{R}$, then $(X, \mathcal{U}')$ is an ω-narrow uniform space of countably infinite weight, and Theorem 2.1 together with Corollary 2.3 imply that $\chi(F(X, \mathcal{U}')) = \chi(A(X, \mathcal{U}')) = \mathfrak{d}$.

3. Free groups on compact spaces

Let X be a compact Hausdorff space. Then by combining Corollary 2.12 of [11] and our Corollary 2.3, we obtain the inequality

$$w(X) \leq \chi(A(X)) = \chi(F(X)) \leq w(X)^{\aleph_0},$$

which constitutes one of the main facts about the characters of the free groups on compact Hausdorff spaces derived in [11].

In this section, we apply different methods to derive a great deal more detailed information about these characters. As just noted, the difference between $F(X)$ and $A(X)$ mentioned in Example 2.8 disappears in the case when X is compact. In fact, we will show that the character of these groups depends only on the weight of X in the compact case. Our proof of this fact requires several auxiliary results. First, we recall some definitions and notation used in [11].

A pair $(P, \leq)$ is a *quasi-ordered set* if $\leq$ is a reflexive transitive relation on the set P. If $(P, \leq)$ has the additional property of antisymmetry, then it is a *partially ordered set*. A set $D \subseteq P$ is called *dominating* or *cofinal* in the quasi-ordered set $(P, \leq)$ if for every $p \in P$ there exists $q \in D$ such that $p \leq q$. Similarly, a subset E of P is said to be *dense* in $(P, \leq)$ if for every $p \in P$ there exists $q \in E$ with $q \leq p$. The minimal cardinality of a dominating family in $(P, \leq)$ is denoted by $D(P, \leq)$ while we use $d(P, \leq)$ for the minimal cardinality of a dense set in $(P, \leq)$. The notions of dominating and dense sets are dual: if a set S is dense in $(P, \leq)$, then it is dominating in $(P, \geq)$ and vice versa. Therefore, $d(P, \leq) = D(P, \geq)$ and $D(P, \leq) = d(P, \geq)$.

For a space X, denote by $\mathcal{M}_X$ the family of all continuous mappings of X onto separable metrizable spaces. Equivalently, since every separable metrizable space is homeomorphic to a subspace of I^ω, where $I = [0, 1]$, we can consider $\mathcal{M}_X$ as a family of continuous mappings of X to I^ω. If $f\colon X \to Y$ and $g\colon X \to Z$ are elements of $\mathcal{M}_X$, we say that f *refines* g or, in symbols, $f \prec g$ if there exists a continuous mapping $\varphi\colon Y \to Z$ such that $g = \varphi \circ f$. Also, following [11], denote by $\mathcal{P}_X$ the family of all continuous pseudometrics on the space X bounded by 1. For $d_1, d_2 \in \mathcal{P}_X$, we write $d_1 \leq d_2$ if $d_1(x, y) \leq d_2(x, y)$ for all $x, y \in X$. This gives us the quasi-ordered set $(\mathcal{M}_X, \prec)$ and the partially ordered set $(\mathcal{P}_X, \leq)$.

Lemma 3.1. *The equality $D(\mathcal{P}_X, \leq) = \mathfrak{d} \cdot d(\mathcal{M}_X, \prec)$ is valid for every infinite compact Hausdorff space X.*

Proof. By Corollary 2.4 of [11] and our Corollary 2.2, we have $D(\mathcal{P}_Y, \leq) = \chi(A(Y)) \leq \mathfrak{d}$ for every compact metrizable space Y. Let d be a continuous pseudometric on a given compact space X, with $d \leq 1$. There exists a continuous mapping $f\colon X \to Y$ onto a compact metrizable space Y and a continuous metric ϱ on Y such that $d(x, y) = \varrho(f(x), f(y))$ for all $x, y \in X$. Since $D(\mathcal{P}_Y, \leq) \leq \mathfrak{d}$, we can find a dominating family D_f in $(\mathcal{P}_Y, \leq)$ satisfying $|D_f| \leq \mathfrak{d}$. For every $\kappa \in D_f$, define a continuous pseudometric $\widetilde{\kappa}$ on X by $\widetilde{\kappa}(x, y) = \kappa(f(x), f(y))$ for all $x, y \in X$. Then $\widetilde{D}_f = \{\widetilde{\kappa} : \kappa \in D_f\} \subseteq \mathcal{P}_X$ for each $f \in \mathcal{M}_X$. Let $\mathcal{N}$ be a dense subset of $(\mathcal{M}_X, \prec)$ satisfying $|\mathcal{N}| = d(\mathcal{M}_X, \prec)$. It is easy to see that the family $\mathcal{D} = \bigcup\{\widetilde{D}_f : f \in \mathcal{N}\}$ is dominating in $(\mathcal{P}_X, \leq)$, so that $D(\mathcal{P}_X, \leq) \leq |\mathcal{D}| \leq \mathfrak{d} \cdot |\mathcal{N}| = \mathfrak{d} \cdot d(\mathcal{M}_X, \prec)$.

Conversely, let $\mathcal{D}$ be a dominating family in $\mathcal{P}_X$ such that $|\mathcal{D}| = D(\mathcal{P}_X, \leq)$. Since X is compact, for every $d \in \mathcal{D}$ we can find a continuous mapping $f = f_d$ of X onto a compact metrizable space Y and a continuous metric ϱ on Y such that $d(x, y) = \varrho(f(x), f(y))$ for all $x, y \in X$. Then the set $\{f_d : d \in \mathcal{D}\}$ is dense in $(\mathcal{M}_X, \prec)$. Indeed, let $g \in \mathcal{M}_X$ be arbitrary. Then the image $Z = g(X)$ is a compact metrizable space. Choose a metric $\varrho \in \mathcal{P}_Z$ which generates the topology of Z and define a continuous pseudometric $\widetilde{\varrho}$ on X by $\widetilde{\varrho}(x, y) = \varrho(g(x), g(y))$ for all $x, y \in X$. Clearly $\widetilde{\varrho} \in \mathcal{P}_X$, so there exists $d \in \mathcal{D}$ such that $\widetilde{\varrho} \leq d$. An easy verification shows that $f_d \prec g$, and hence the family $\{f_d : d \in \mathcal{D}\}$ is dense in $(\mathcal{M}_X, \prec)$. We conclude therefore that $d(\mathcal{M}_X, \prec) \leq |\mathcal{D}| = D(\mathcal{P}_X, \leq)$.

Finally, since $\mathfrak{d} \leq \chi(A(X)) = D(\mathcal{P}_X, \leq)$ by Corollaries 2.4 and 2.18 of [11], we apply the inequalities just proved to deduce that

$$D(\mathcal{P}_X, \leq) \leq \mathfrak{d} \cdot d(\mathcal{M}_X, \prec) \leq \mathfrak{d} \cdot D(\mathcal{P}_X, \leq) = D(\mathcal{P}_X, \leq),$$

which implies the required equality. □

Now we present a theorem which summarizes several results established earlier.

Theorem 3.2. *$\mathfrak{d} \le \chi(A(X)) = \chi(F(X)) = D(\mathcal{P}_X, \le) = \mathfrak{d} \cdot d(\mathcal{M}_X, \prec)$ for every infinite compact space X.*

Proof. Combining Corollaries 2.4 and 2.18 of [11] and Lemma 3.1 of the present paper, we obtain

$$\mathfrak{d} \le \chi(A(X)) = D(\mathcal{P}_X, \le) = \mathfrak{d} \cdot d(\mathcal{M}_X, \prec).$$

Since $\chi(A(X)) = \chi(F(X))$ by Corollary 2.3, this proves the theorem. □

On occasion, the exact calculation of $D(\mathcal{P}_X, \le)$ or $d(\mathcal{M}_X, \prec)$ for a compact space X can be a non-trivial task. In Theorem 3.5, we give an explicit value for the character of the groups $F(X)$ and $A(X)$ on an infinite compact space X which avoids any reference to the quasi-ordered sets $(\mathcal{P}_X, \le)$ or $(\mathcal{M}_X, \prec)$. Nevertheless, our proof of Theorem 3.5 will involve the set $(\mathcal{M}_X, \prec)$ in an essential way, as well as the family $CZ(X)$ of all cozero-sets in X. As usual, we denote by $[CZ(X)]^{\le\omega}$ the collection of all countable subfamilies of $CZ(X)$. Given $\gamma, \lambda \in [CZ(X)]^{\le\omega}$, we write $\gamma \ll \lambda$ if every element of λ is the union of a subfamily of γ. This gives rise to the quasi-ordered set $([CZ(X)]^{\le\omega}, \ll)$.

Lemma 3.3. *If X is compact Hausdorff, then $d([CZ(X)]^{\le\omega}, \ll) = d(\mathcal{M}_X, \prec)$.*

Proof. If $w(X) \le \aleph_0$, then X is metrizable, so that $d([CZ(X)]^{\le\omega}, \ll) = 1 = d(\mathcal{M}_X, \prec)$. Suppose therefore that $w(X) > \aleph_0$.

For every $U \in CZ(X)$, fix a continuous function $f_U \colon X \to I$ such that $X \setminus U = f_U^{-1}(0)$. Given a countably infinite subfamily $\gamma = \{U_n : n \in \omega\}$ of $CZ(X)$, we consider the corresponding diagonal product $f_\gamma = \triangle_{n\in\omega} f_{U_n} \colon X \to I^\omega$. We also consider the analogously defined diagonal product into I^k for some $k \in \mathbb{N}$ corresponding to any given finite subfamily of $CZ(X)$. This correspondence defines a mapping Φ from $[CZ(X)]^{\le\omega}$ to $\mathcal{M}_X$, where Φ is defined by $\Phi(\gamma) = f_\gamma$, if we agree to restrict the range space of f_γ to $f_\gamma(X)$. Choose a dense set Γ in $([CZ(X)]^{\le\omega}, \ll)$ of the minimal cardinality. We claim that the image $\Phi(\Gamma)$ is dense in the quasi-ordered set $(\mathcal{M}_X, \prec)$.

Indeed, let g be a continuous mapping of X onto a second countable space Y. Choose a countable base $\mathcal{B}$ for Y and put $\lambda = \{g^{-1}(V) : V \in \mathcal{B}\}$. Then $\lambda \in [CZ(X)]^{\le\omega}$, so we can find $\gamma \in \Gamma$ with $\gamma \ll \lambda$. Let us show that $f_\gamma = \Phi(\gamma)$ satisfies $f_\gamma \prec g$. Suppose that $x, y \in X$ and $g(x) \ne g(y)$. Then $g(x) \in V \not\ni g(y)$ for some $V \in \mathcal{B}$. Since $x \in g^{-1}(V) \in \lambda$ and $\gamma \ll \lambda$, there exists $U \in \gamma$ such that $x \in U \subseteq g^{-1}(V)$. Then $y \notin U$. Clearly $f_U(x) \ne f_U(y)$, and from $U \in \gamma$ it follows that $f_\gamma(x) \ne f_\gamma(y)$. Therefore, $f_\gamma(x) = f_\gamma(y)$ always implies $g(x) = g(y)$. This fact enables us to define a mapping $h \colon f_\gamma(X) \to Y$ such that $g = h \circ f_\gamma$. Since g, f_γ are continuous mappings and f_γ is closed, we conclude that h is also continuous. Hence $f_\gamma \prec g$. This proves our claim, and hence $d(\mathcal{M}_X, \prec) \le |\Phi(\Gamma)| \le |\Gamma| = d([CZ(X)]^{\le\omega}, \ll)$.

Conversely, let $\mathcal{N}$ be a dense set in $(\mathcal{M}_X, \prec)$ of the minimal cardinality. Choose a countable base $\mathcal{B}$ for I^ω and put $\gamma_f = \{f^{-1}(V) : V \in \mathcal{B}\}$ for each $f \in \mathcal{M}_X$. Evidently, $\gamma_f \in [CZ(X)]^{\le\omega}$. Let us verify that the set $\Gamma = \{\gamma_f : f \in \mathcal{N}\}$ is dense in $([CZ(X)]^{\le\omega}, \ll)$. Consider an arbitrary element $\lambda = \{U_n : n \in \omega\}$

of $[CZ(X)]^{\leq\omega}$. As in the first part of the proof, for every $n \in \omega$ take the function $g_n = f_{U_n}$ and put $g = \triangle_{n\in\omega} g_n$. By our choice of $\mathcal{N}$, there exists $f \in \mathcal{N}$ with $f \prec g$. Then $f \prec g_n$, and hence $U_n = f^{-1}f(U_n)$ for each $n \in \omega$. Since f is a closed mapping, the sets $f(U_n)$ are open in $f(X)$. For $n \in \omega$, apply the fact that $\mathcal{B}$ is a base for I^ω to choose a family $\mu_n \subseteq \mathcal{B}$ such that $f(U_n) = f(X) \cap \bigcup \mu_n$. It follows that $U_n = f^{-1}f(U_n) = \bigcup\{f^{-1}(V) : V \in \mu_n\}$, and since $\{f^{-1}(V) : V \in \mu_n\} \subseteq \gamma_f$ for each $n \in \omega$, we conclude that $\gamma_f \ll \lambda$.

This proves that Γ is dense in $([CZ(X)]^{\leq\omega}, \ll)$, whence $d([CZ(X)]^{\leq\omega}, \ll) \leq |\Gamma| \leq |\mathcal{N}| = d(\mathcal{M}_X, \prec)$. The lemma is proved. □

The use of the quasi-ordered set $([CZ(X)]^{\leq\omega}, \ll)$ enables us to calculate the character of the group $F(X)$ on a compact space X in purely set-theoretical terms. The result of this calculation turns out to be somewhat unexpected: $\chi(F(X)) = \chi(F(Y))$ whenever the compact spaces X and Y have the same weight (see Corollary 3.6).

Let τ be an infinite cardinal. A subset $Y = \{x_\alpha : \alpha < \tau\}$ of a space X is called *right-separated* [7] if the set $\{x_\beta : \beta < \alpha\}$ is open in Y for each $\alpha < \tau$. The next fact is well known in the folklore, but is proved here for the reader's convenience.

Lemma 3.4. *If X is compact, then X^2 contains a right-separated subset of cardinality $\tau = w(X)$.*

Proof. Denote by Δ the diagonal in $X \times X$. Then $\chi_\Delta(X) = \chi(\Delta, X^2) = w(X) = \tau$. Let γ be an open cover of $X^2 \setminus \Delta$ such that the closure of each $U \in \gamma$ does not intersect Δ. Since $\psi(\Delta, X^2) = \chi(\Delta, X^2) = \tau$, the set $X^2 \setminus \Delta$ cannot be covered by less than τ elements of γ. Therefore, we can construct by recursion a subset $Y = \{x_\alpha : \alpha < \tau\}$ of $X^2 \setminus \Delta$ and a subfamily $\{U_\alpha : \alpha < \tau\}$ of γ such that $x_\alpha \in U_\alpha$ and $x_\beta \notin U_\alpha$ whenever $\alpha < \beta < \tau$. Then the set Y is as required. □

Theorem 3.5. *If X is an infinite compact space of weight τ, then $\chi(F(X)) = \chi(A(X)) = \mathfrak{d} \cdot D([\tau]^{\leq\omega}, \subseteq)$.*

Proof. Note that $D([\omega]^{\leq\omega}, \subseteq) = 1$, so if $w(X) = \aleph_0$, the required conclusion follows from Corollary 2.4. Hence we assume that $w(X) = \tau > \aleph_0$.

Put $\kappa = D([\tau]^{\leq\omega}, \subseteq)$. First we show that $\chi(F(X)) \leq \mathfrak{d} \cdot \kappa$. Let $2 = \{0, 1\}$ be the discrete doubleton. Since $w(X) = \tau$, we can find a closed subspace Y of the Cantor cube $Z = 2^\tau$ and a continuous onto mapping $f : Y \to X$. Extend f to a continuous homomorphism $\widehat{f} : F(Y) \to F(X)$. Since f is a closed mapping, the homomorphism $\widehat{f}$ is open. Therefore, $\chi(F(X)) \leq \chi(F(Y))$. In addition, Y is compact, so $F(Y)$ is topologically isomorphic to the subgroup $F(Y, Z)$ of $F(Z)$ generated by Y [4, §12], and hence $\chi(F(Y)) \leq \chi(F(Z))$. From Theorem 3.2 it follows that $\chi(F(Z)) = \mathfrak{d} \cdot d(\mathcal{M}_Z, \prec)$, so it suffices to verify that $d(\mathcal{M}_Z, \prec) \leq \kappa$.

For a non-empty $A \subseteq \tau$, denote by π_A the projection of $Z = 2^\tau$ onto 2^A. As is well known, every continuous mapping $h : Z \to M$ to a metrizable space M depends on at most countably many coordinates [9, 6]. In other words, there exists a countable set $A \subseteq \tau$ such that if $x, y \in Z$ and $\pi_A(x) = \pi_A(y)$, then $h(x) = h(y)$. Hence we can define a mapping $g\colon 2^A \to M$ satisfying $g \circ \pi_A = h$. Since π_A is an open mapping, we conclude that g is continuous. Therefore, $\pi_A \prec h$. This means that the family $\{\pi_A : A \in [\tau]^{\leq\omega}\}$ is dense in $(\mathcal{M}_Z, \prec)$. It is clear, further, that if a set $\mathcal{A} \subseteq [\tau]^{\leq\omega}$ is dominating in $([\tau]^{\leq\omega}, \subseteq)$, then the family $\{\pi_A : A \in \mathcal{A}\}$ is dense in $(\mathcal{M}_Z, \prec)$. This proves that $d(\mathcal{M}_Z, \prec) \leq D([\tau]^{\leq\omega}, \subseteq) = \kappa$, so that $\chi(F(X)) \leq \chi(F(Z)) \leq \mathfrak{d} \cdot \kappa$.

To show that $\chi(F(X)) \geq \mathfrak{d}\cdot\kappa$, we argue as follows. The group $F(X)$ contains a closed subspace homeomorphic to X^2, so $F(X)$ also contains a subgroup topologically isomorphic to $F(X^2)$ [8]. Hence $\chi(F(X^2)) \leq \chi(F(X))$. Put $Y = X^2$. Then $\chi(F(Y)) = \mathfrak{d}\cdot d([CZ(Y)]^{\leq\omega}, \ll)$ by Theorem 3.2 and Lemma 3.3. Therefore, all we need to prove is that $d([CZ(Y)]^{\leq\omega}, \ll) \geq \kappa$.

Let $\mathcal{D}$ be a dense set in $([CZ(Y)]^{\leq\omega}, \ll)$ of the minimal cardinality. It follows from Lemma 3.4 that the space $Y = X^2$ contains a right-separated subset $\{x_\alpha : \alpha < \tau\}$. For every $\alpha < \tau$, choose a cozero set U_α in Y such that $x_\alpha \in U_\alpha$ and $x_\beta \notin U_\alpha$ if $\alpha < \beta < \tau$. If U is a non-empty element of $CZ(Y)$, we define α_U as the maximal element of the set $\{\beta < \tau : x_\beta \in U\}$ in the case when it exists, and $\alpha_U = 0$ otherwise. Given a countable subfamily μ of $CZ(Y)$, put $A_\mu = \{\alpha_U : U \in \mu\}$. We claim that the family $\mathcal{A} = \{A_\mu : \mu \in \mathcal{D}\}$ is dominating in $([\tau]^{\leq\omega}, \subseteq)$. Indeed, let A be a countable subset of τ. Then $\gamma = \{U_\alpha : \alpha \in A\}$ is an element of $[CZ(Y)]^{\leq\omega}$, so there exists $\mu \in \mathcal{D}$ such that $\mu \ll \gamma$. By definition of the quasi-order $\ll$, for every $\alpha \in A$ there exists a subfamily $\mu_\alpha \subseteq \mu$ such that $U_\alpha = \bigcup \mu_\alpha$. Hence μ_α contains an element V such that $x_\alpha \in V \subseteq U_\alpha$. In particular, $\alpha = \alpha_V$, so that $A \subseteq A_\mu$. This proves that $\mathcal{A}$ is dominating in $([\tau]^{\leq\omega}, \subseteq)$. Therefore, we have

$$\kappa = D([\tau]^{\leq\omega}, \subseteq) \leq |\mathcal{A}| \leq |\mathcal{D}| = d([CZ(Y)]^{\leq\omega}, \ll).$$

This finishes the proof. □

Corollary 3.6. *If infinite compact spaces X and Y satisfy $w(X) = w(Y)$, then $\chi(A(X)) = \chi(F(X)) = \chi(F(Y)) = \chi(A(Y))$.*

Proof. Let $G(Z)$ be either $A(Z)$ or $F(Z)$, where $Z \in \{X, Y\}$. If $w(X) = w(Y) = \aleph_0$, then $\chi(G(X)) = \mathfrak{d} = \chi(G(Y))$ by Corollary 2.4. If $w(X) = w(Y) = \tau > \aleph_0$, then we apply Theorem 3.5 to conclude that $\chi(G(X)) = \chi(G(Y)) = \mathfrak{d} \cdot D([\tau]^{\leq\omega}, \subseteq)$. □

Finally, one applies Lemma 4.1 of the next section and Theorem 3.5 to deduce the following two corollaries.

Corollary 3.7. *Let X be an infinite compact space satisfying $w(X) < \aleph_\omega$. Then $\chi(A(X)) = \chi(F(X)) = \mathfrak{d} \cdot w(X)$.*

Corollary 3.8. *If a compact space X satisfies $w(X) \geq \mathfrak{c}$, then $\chi(A(X)) = \chi(F(X)) = w(X)^{\aleph_0}$.*

It is worth noting that if X and Y are infinite compact spaces satisfying $w(X) = \aleph_0$ and $w(Y) = \aleph_1$, then nevertheless $\chi(F(X)) = \chi(F(Y)) = \mathfrak{d}$. This follows easily from Corollary 3.7 and the fact that $\mathfrak{d} \geq \aleph_1$.

4. The possible values of the character

It is of interest to discover which cardinal values the characters of free and free abelian topological groups can assume. By Corollary 2.16 of [11], we know that the character of the groups $F(X)$ and $A(X)$ on a non-P-space X is at least $\mathfrak{d}$. As usual, we say that X is a P-space if every G_δ-set in X is open. One can enquire whether there exists in ZFC a space X such that $\chi(A(X)) = \aleph_1$ or $\chi(A(X)) = \aleph_2$, etc. We show below that the answer is affirmative. Since the place of the cardinal $\mathfrak{d}$ in the line of alephs is undefined in ZFC, such a space X has necessarily to be a P-space.

We start with a simple auxiliary fact, the very beginning of the pcf theory founded by Shelah [13].

Lemma 4.1. *Let τ be a cardinal. Then*:

(a) $D([\tau]^{\le\omega}, \subseteq) = \tau$ *if* $\tau = \aleph_n$ *for some integer* $n \ge 1$;
(b) *If* $\tau \ge \mathfrak{c}$, *then* $D([\tau]^{\le\omega}, \subseteq) = \tau^\omega$.

Proof. First we note that $\tau \le D([\tau]^{\le\omega}, \subseteq)$ for every $\tau > \aleph_0$, because τ can be partitioned into τ disjoint countably infinite subsets, and these cannot be covered by any collection of fewer than τ countable subsets.

(a) It suffices to show that $D([\aleph_n]^{\le\omega}, \subseteq) \le \aleph_n$. If $n = 1$, then the required dominating family in $([\aleph_1]^{\le\omega}, \subseteq)$ is $\{\alpha : \alpha < \omega_1\}$. Suppose that the lemma holds for some integer $n \ge 1$. By assumption, for every uncountable ordinal $\alpha < \aleph_{n+1}$ there exists a dominating family γ_α in $([\alpha]^{\le\omega}, \subseteq)$ satisfying $|\gamma_\alpha| = |\alpha| \le \aleph_n$. Put $\gamma = \bigcup\{\gamma_\alpha : \omega_1 \le \alpha < \aleph_{n+1}\}$. Then $|\gamma| \le \aleph_{n+1}$, and it is easy to see that γ is dominating in $([\aleph_{n+1}]^{\le\omega}, \subseteq)$. Indeed, if A is a countable subset of $\aleph_{n+1}$, then $A \subseteq \alpha$ for some uncountable $\alpha < \aleph_{n+1}$, and hence there exists $B \in \gamma_\alpha$ with $A \subseteq B$. Since $\gamma_\alpha \subseteq \gamma$, this proves that γ is dominating in $([\aleph_{n+1}]^{\le\omega}, \subseteq)$. Therefore, $D([\aleph_{n+1}]^{\le\omega}, \subseteq) \le \aleph_{n+1}$.

(b) The case $\tau = \mathfrak{c}$ is trivial, so we assume that $\tau > \mathfrak{c}$. Suppose that $\gamma = \{t_i : i \in I\}$ is a dominating subset of $([\tau]^{\le\omega}, \subseteq)$ of the minimal cardinality. It is clear that the number of elements t of $[\tau]^{\le\omega}$ such that $t \subseteq t_i$ for any fixed $i \in I$ is at most $\mathfrak{c}$, and that the cardinality of $[\tau]^{\le\omega}$ is τ^ω. Therefore, we have the inequality $\tau^\omega \le \mathfrak{c} \cdot |I|$. But using the assumption that $\tau > \mathfrak{c}$, we have $\tau^\omega \ge \tau > \mathfrak{c}$, and hence $\mathfrak{c} < \mathfrak{c} \cdot |I|$. It follows that $|I| > \mathfrak{c}$, and therefore that $\mathfrak{c} \cdot |I| = |I|$, from which we have $|I| \ge \tau^\omega$. Since we know already that $|I| \le \tau^\omega$, we finally have $|I| = \tau^\omega$, as required. □

Proposition 4.2. *Let P^* be the one-point Lindelöfication of a discrete space P of cardinality $\tau > \aleph_0$. Then $\chi(F(P^*)) = \chi(A(P^*)) = D([\tau]^{\le\omega}, \subseteq)$.*

Proof. Put $\kappa = D([P]^{\le\omega}, \subseteq) = D([\tau]^{\le\omega}, \subseteq)$. Suppose that $P^* = P \cup \{x^*\}$, where x^* is the unique non-isolated point in P^*. First, we consider the group $F(P^*)$. For every countable subset K of P, denote by U_K the minimal normal subgroup of $F(P^*)$ containing the set $P^* \setminus K$. By Lemma 2.9 of [5], the family $\{U_K : K \in [P]^{\le\omega}\}$ is a base at the identity in $F(P^*)$. Note that if $K, L \in [P]^{\le\omega}$ and $K \subseteq L$, then $U_L \subseteq U_K$. Choose a dominating family γ in $([P]^{\le\omega}, \subseteq)$ with

$|\gamma| = \kappa$. Then $\{U_K : K \in \gamma\}$ is again a base at the identity in $F(P^*)$, and hence $\chi(F(P^*)) \le |\gamma| = \kappa$. In addition, P^* is a subspace of $F(P^*)$, so $\kappa = D([P]^{\le\omega}, \subseteq)) = \chi(x^*, P^*) \le \chi(F(P^*))$. We have thus proved that $\chi(F(P^*)) = \kappa$. Since the Lindelöf space P^* is ω-narrow, Corollary 2.3 implies that $\chi(A(P^*)) = \kappa$. □

Combining Lemma 4.1 and Proposition 4.2, we obtain:

Corollary 4.3. *Let P^* be the one-point Lindelöfication of a discrete space P of cardinality $\tau > \aleph_0$. Then $\chi(F(P^*)) = \chi(A(P^*)) = \tau$ if $\aleph_1 \le \tau < \aleph_\omega$, and $\chi(F(P^*)) = \chi(A(P^*)) = \tau^\omega$ if $\tau \ge \mathfrak{c}$.*

Note that it follows in particular that when the (uncountable) cardinality of a discrete space P is sufficiently small, then it is consistent with ZFC that P^*, the one-point Lindelöfication of P, satisfies $\chi(F(P^*)) = \tau < \tau^\omega$, but that the situation differs markedly if P has sufficiently large cardinality.

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