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# A generalized coincidence point index

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Abstract. The paper is devoted to build for some pairs of continuous single-valued maps a coincidence point index. The class of pairs $(f, g)$ satisfies the condition that $f$ induces an epimorphism of the Čech homology groups with compact supports and coefficients in the field of rational numbers $Q$. Using this concept one defines for a class of multi-valued mappings a fixed point degree. The main theorem states that if the general coincidence point index is different from $\{0\}$, then the pair $(f, g)$ admits at least a coincidence point. The results may be considered as a generalization of the above Eilenberg-Montgomery theorems [12], they include also, known fixed-point and coincidence-point theorems for single-valued maps and multi-valued transformations.


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## 1. Introduction

Let $f,\, g : X \longrightarrow Y$ be two continuous single valued maps of Hausdorff topological spaces. The coincidence problem, which is a generalization of the fixed point problem, is concerned with conditions which guarantees the existence of a solution for the equation $f(x) = g(x)$. A such point $x \in X$ is called a coincidence point of the pair of maps $(f, g)$. The study of this problem has been treated first in 1946 by Eilenberg-Montgomery [12]. Note that the Eilenberg-Montgomery theorem is a natural generalization of the Lefschetz fixed point theorem, it implies also, the fixed point theorems of Kakutani [21] and Wallace [30]. Topological invariants for different classes of pairs of maps have been studied by many authors [9], [14], [15], [20], [22], [23], [27] and others. The purpose of this note is to describe a generalized coincidence point index for a new class

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of pairs of continuous maps $(f, g)$ which satisfy the condition that $f$ induces a $r$-homomorphism [3], [4] for homology with compact carries. Moreover, one gives several applications of the general coincidence point index in fixed point theory for multi-valued mappings.

One uses the Dold's fundamental class around a compact of a finite euclidean space $E^n$ [10], $H$ denotes the $\check{C}$ech homology functor with compact carries and coefficients in the field of rational numbers $Q$, from the category $Top_{(2)}$ of Hausdorff topological pairs and continuous maps to the category $L_g$ of graded vector spaces over the set of rational numbers $Q$ and linear maps of degree zero [13], [18], [29].

## 2. Maps $n$-decomposing.

Let $G_1$ and $G_2$ be two additive abelian groups, $\tau : G_1 \longrightarrow G_2$ be a homomorphism.

**Definition 2.1** ([3]). *A homomorphism $\tau$ is a called a $r$-homomorphism if $\tau$ admits a right-inverse homomorphism.*

The definition signifies, since $\tau : G_1 \longrightarrow G_2$ is a $r$-homomorphism then there exists a homomorphism $\sigma : G_2 \longrightarrow G_1$ such that $\tau \circ \sigma = Id_{G_2}$, where $Id_{G_2}$ is the automorphism identity on $G_2$.

*The following properties are satisfied.*

**Proposition 2.2.** *A homomorphism $\tau : G_1 \longrightarrow G_2$ is a $r$-homomorphism if and only if the following conditions are satisfied :*

(1) *$\tau$ is an epimorphism;*
(2) *$G_1 = Ker\,\tau \oplus \mathcal{G}$ , where $\mathcal{G}$ is a subgroup of $G_1$.*

**Proposition 2.3.** *If $G_1$ and $G_2$ are two modules over a field $K$ and if $\tau : G_1 \longrightarrow G_2$ is an epimorphism then $\tau$ is a $r$-homomorphism.*

**Proposition 2.4** ([3]). *Let $\tau_1 : G_1 \longrightarrow G_2$ and $\tau_2 : G_2 \longrightarrow G_3$ be two $r$-homomorphisms then their composition $\tau = \tau_2 \circ \tau_1 : G_1 \longrightarrow G_3$ is also a $r$-homomorphism.*

*The notion of $r$-homomorphisms has been introduced by Borsuk and Kosinsk* [3], [4].

Let $(X, A)$ and $(Y, B)$ be two objects of the category $Top_{(2)}$ of Hausdorff topological pairs and continuous maps and $f : (X, A) \longrightarrow (Y, B)$ be a morphism from the Hausdorff pair $(X, A)$ into an other Hausdorff pair $(Y, B)$.

Let $H$ be the $\check{C}$ech homology functor with compact carries and coefficients in the field of rational numbers $Q$, from the category $Top_{(2)}$ of Hausdorff topological pairs and continuous maps to the category $L_g$ of graded vector spaces over the set of rational numbers $Q$ and linear maps of degree zero [13], [18], [29].

**Definition 2.5.** *A continuous single-valued map $f : (X, A) \longrightarrow (Y, B)$ is said to be $n$-decomposing in the rank $n \geq 0$ on the Hausdorff pair $(Y, B)$ if the homomorphism $f_* : H_n(X, A) \to H_n(Y, B)$ induced by $f$, is a $r$-homomorphism.*

The set of the right-inverse homomorphisms of $f_*$ on $(Y, B)$ will be denoted by $\Omega(f_*; Y, B)$.

*The following propositions and corollaries, prove that the class of $n$-decomposing maps is vast.*

**Definition 2.6** ([3])**.** *A continuous single-valued map $f : (X, A) \longrightarrow (Y, B)$ is called a $r$-map if $f$ admits a continuous right inverse.*

**Proposition 2.7.** *Let $f : (X, A) \longrightarrow (Y, B)$ be a single-valued map which is a $r$-map, then $f$ is $n$-decomposing on $(Y, B)$ for every rank $n \geq 0$.*

**Corollary 2.8.** *A retraction $r$ of a pair $(X, A)$ onto $(X', A')$ is $n$-decomposing on the retract $(X', A')$ of $(X, A)$.*

**Definition 2.9** ([3])**.** *A continuous single-valued map $f : (X, A) \longrightarrow (Y, B)$ is said to be a $h$-map if there exists a continuous single-valued $g : (Y, B) \longrightarrow (X, A)$ such that their composition $f \circ g$ and the identity map $Id_{(Y,B)} : (Y, B) \longrightarrow (Y, B)$ are homotopic.*

**Proposition 2.10.** *If $f : (X, A) \longrightarrow (Y, B)$ is a $h$-map, then $f$ is $n$-decomposing on $(Y, B)$ for every $n \geq 0$.*

**Corollary 2.11.** *A lower retraction $r : (X, A) \longrightarrow (X', A')$ is $n$-decomposing on each lower retract $(X', A')$ of $(X, A)$.*

**Proposition 2.12.** *Let $f : (X, A) \longrightarrow (Y, B)$ be a continuous single-valued map. If there exists a continuous single-valued map $g : (Z, C) \longrightarrow (X, A)$ such that their composition $f \circ g$ is $n$-decomposing on $(Y, B)$, then $f$ is also $n$-decomposing on $(Y, B)$.*

**Corollary 2.13.** *Let $f : (X, A) \longrightarrow (Y, B)$ be a continuous single-valued map and $(Z, C) \subseteq (X, A)$. If the restriction of $f$ on $(Z, C)$ is $n$-decomposing on $(Y, B)$, then $f$ is also $n$-decomposing on $(Y, B)$.*

**Proposition 2.14.** *Let $f : (X, A) \longrightarrow (Y, B)$ be a $n$-decomposing on $(Y, B)$ and $g : (Y, B) \longrightarrow (Z, C)$ be a $n$-decomposing on $(Z, C)$, then their composition $g \circ f$ is $n$-decomposing on $(Z, C)$.*

**Definition 2.15** ([5])**.** *A space $X$ is $Q$-acyclic provided: (i) $X$ is non-empty, (ii) $H_q(X) = 0$ for all $q \geqslant 1$ and (iii) $H_0(X) \approx Q$.*

**Proposition 2.16.** *Let $f : (X, A) \longrightarrow (Y, B)$ be a continuous single-valued map such that:*

(1) *$f$ is proper and surjective;*
(2) *$f^{-1}(B) = A$;*
(3) *$f^{-1}(y)$ is $Q$-acyclic for every $y \in Y$.*

*Then the map $f$ is $n$-decomposing on $(Y, B)$ for every $n \geq 0$.*

**Proposition 2.17.** *Let $U$ be an open subset of an Euclidean space $E^n$ and $K$ be a compact subset of $U$, then the injection $i : (U, U\backslash K) \longrightarrow (E^n, E^n\backslash K)$ is $n$-decomposing on $(E^n, E^n\backslash K)$.*

## 3. Generalized coincidence point index

Let $U$ be an open subset of an euclidean vector space $E^n$ which has a fixed orientation.

Let $(f, g)$ be a pair of continuous single-valued maps defining as follows:

$$U \overset{f}{\longleftarrow} X \overset{g}{\longrightarrow} E^n \tag{3.1}$$

where $X$ is an arbitrary Hausdorff topological space.

**Definition 3.1.** *An element $x \in X$ is said to be a coincidence point of the pair $(f, g)$ if $f(x) = g(x)$.*

Let $\mathbf{S}(f, g)$ be the set of all coincidence points of the pair $(f, g)$ and $\mathbf{F}(f, g)$ be the subset of $U$ defined as follows:

$$\mathbf{F}(f, g) = \{u \in U \mid u \in g(f^{-1}(u)\}.$$

**Lemma 3.2.** *One has the equality $f(\mathbf{S}(f, g)) = \mathbf{F}(f, g)$.*

*Proof.* The proof is obvious. □

Let $K$ be a compact subset of $U$ which contains $\mathbf{F}(f, g)$. Thus, one obtains the following diagram:

$$(U, U\backslash K) \overset{f}{\longleftarrow} (X, X\backslash f^{-1}(K)) \overset{f-g}{\longrightarrow} (E^n, E^n\backslash \{\theta\}) \tag{3.2}$$

**Definition 3.3.** *A pair of continuous single-valued maps $(f, g)$ as above defined, is called $n$-admissible on $(U, U\backslash K)$ if $f$ is $n$-decomposing on $(U, U\backslash K)$.*

*The set of all $n$-admissible pairs on $(U, U\backslash K)$ is denoted $\mathcal{PD}(U, U\backslash K)$.*

Let $(f, g) \in \mathcal{PD}(U, U \backslash K)$, then if $\sigma \in \Omega(f_*; U, U\backslash K)$ the diagram (3.2) induces the following diagram:

$$\begin{array}{ccccc} H_n(U, U\backslash K) & \overset{f_*}{\longleftarrow} & H_n(X, X\backslash f^{-1}(K)) & \overset{(f-g)_*}{\longrightarrow} & H_n(E^n, E^n\backslash \{\theta\}) \\ & \sigma \searrow & \Updownarrow & & \\ & & H_n(X, X\backslash f^{-1}(K)) & & \end{array} \tag{3.3}$$

Let $O_K \in H_n(U, U\backslash K)$ be the image of 1 under the composite map:

$$Z = H_n(S^n) \longrightarrow H_n(S^n, S^n\backslash K) \approxeq H_n(U, U\backslash K)$$

and $O_{\{\theta\}} \in H_n(E^n, E^n\backslash \{\theta\})$ be the image of 1 under the composition map:

$$Z = H_n(S^n) \longrightarrow H_n(S^n, S^n\backslash \{\theta\}) \approxeq H_n(E^n, E^n\backslash \{\theta\})$$

where $S^n = E^n \cup \{\infty\}$.

The elements $O_K$ and $O_{\{\theta\}}$ are called the fundamental classes around the compacts $K$ and $\{\theta\}$ respectively [9], [10].

**Definition 3.4.** *Let $(f,g)$ be a $n$-admissible pair on $(U,U\backslash K)$. The generalized coincidence point index of $(f,g)$ relatively $\sigma \in \Omega(f_*;U,U\backslash K)$ is defined as being the rational number $I_\sigma(f,g)$ which verifies the equality $(f-g)_* \circ \sigma(O_K) = I_\sigma(f,g)\cdot O_{\{\theta\}}$.*

**Definition 3.5.** *Let $(f,g)$ be a $n$-admissible pair on $(U,U\backslash K)$. The generalized coincidence point index of $(f,g)$ is defined as being the set of rational numbers $I(f,g) = \{I_\sigma(f,g) \mid \sigma \in \Omega(f_*;U,U\backslash K)\}$.*

**Proposition 3.6.** *If the single-valued map $f:(X,X\backslash f^{-1}(K)) \longrightarrow (U,U\backslash K)$ verifies the conditions of the proposition (2.16), then the pair $(f,g)$ is $n$-admissible on $(U,U\backslash K)$ and $I(f,g) = \{I_{(f_*)^{-1}}(f,g)\}$.*

*Proof.* The single-valued map $f:(X,X\backslash f^{-1}(K)) \longrightarrow (U,U\backslash K)$ induces an isomorphism $f_*: H_n(X,X\backslash f^{-1}(K)) \longrightarrow H_n(U,U\backslash K)$ therefore $\Omega(f_*;U,U\backslash K) = \{(f_*)^{-1}\}$. □

**Proposition 3.7.** *If $\mathbf{F}(f,g) = \varnothing$, then $I(f,g) = \{0\}$.*

*Proof.* Suppose that $\mathbf{F}(f,g) = \varnothing$ then using lemma (3.2) one deduces that $\mathbf{S}(f,g) = \varnothing$. This equality means that $f(x) \neq g(x)$ for each $x \in X$. Therefore for every $\sigma \in \Omega(f_*;U,U\backslash K)$ we have the following commutative diagram:

$$\begin{array}{ccccc} H_n(U,U\backslash K) & \xrightarrow{\sigma} & H_n(X,X\backslash f^{-1}(K)) & \xrightarrow{(f-g)_*} & H_n(E^n,E^n\backslash\{\theta\}) \\ & & \overline{(f-g)}_* \searrow & & \Updownarrow \\ & & & & H_n(E^n\backslash\{\theta\},E^n\backslash\{\theta\}) \end{array}$$

where $\overline{(f-g)} = f-g$. One concludes the proof remarking that $\overline{(f-g)}_*$ is the trivial homomorphism. □

**Corollary 3.8.** *If $I(f,g) \neq \{0\}$, then the pair $(f,g)$ admits at least a coincidence point.*

*Proof.* This is a consequence of lemma (3.2). □

Let $g: U \longrightarrow E^n$ be a continuous single-valued map defined from an open subset $U$ of an Euclidean vector space $E^n$ and $K$ be a compact subset of $U$ which contains $Fix(g) = \{x \in U \mid x = g(x)\}$. The fixed point index of $g$ defined in [9] is the rational $I_g$ which verifies the equality:

$$(i-g)_{n*}(O_K) = I_g \cdot O_{\{\theta\}},$$

where $i: U \longrightarrow E^n$ is the natural injection.

**Proposition 3.9.** *The generalized coincidence point index of the pair $(i,g)$ is defined and equal to the fixed point index of $g$.*

*Proof.* First note that $\mathbf{F}(i,g) = Fix(g) = \{x \in U \mid x = g(x)\}$. Let $K$ be a compact subset of $E^n$ which contains $\mathbf{F}(i,g) = Fix(g)$. So, one has the diagram:

$$H_n(E^n, E^n\backslash K) \overset{i_*}{\longleftarrow} H_n\ (U, U\backslash K) \overset{(i-g)_*}{\longrightarrow} H_n(E^n, E^n\backslash \{\theta\}).$$

Therefore, $I(i,g) \cdot O_{\{\theta\}} = (i-g)_* \circ i_*^{-1}(O_K) = (i-g)_*(O_K) = I_g \cdot O_{\{\theta\}}$. □

**Corollary 3.10.** *If $I(i,g) \neq \{0\}$ then $g$ admits at least a fixed point.*

Let $(f,g)$ and $(f_1, g_1)$ be two pairs of continuous single-valued maps defining as follows:

$$U \overset{f}{\longleftarrow} X \overset{g}{\longrightarrow} E^n \tag{3.4}$$

and

$$V \overset{f_1}{\longleftarrow} X_1 \overset{g_1}{\longrightarrow} E^n, \tag{3.5}$$

where $U$ and $V$ are two open subsets of $E^n$ and $X$ and $X_1$ are two Hausdorff topological spaces.

Let $K$ and $K_1$ be two compact subsets of $E^n$ which contain $\mathbf{F}(f,g)$ and $F(f_1,g_1)$ respectively and such that $K \subset K_1 \subset V \subset \overline{V} \subset U$.

For instance, one obtains the following diagrams:

$$(U, U\backslash K) \overset{f}{\longleftarrow} (X, X\backslash f^{-1}(K)) \overset{f-g}{\longrightarrow} (E^n, E^n\backslash\{\theta\}) \tag{3.6}$$

and

$$(V, V\backslash K_1) \overset{f_1}{\longleftarrow} (X_1, X_1\backslash f_1^{-1}(K_1)) \overset{f_1-g_1}{\longrightarrow} (E^n, E^n\backslash\{\theta\}). \tag{3.7}$$

**Proposition 3.11.** *Under the above hypotheses, assume that $h : (X_1, X_1\backslash f_1^{-1}(K_1)) \longrightarrow (X, X\backslash f^{-1}(K))$ is a continuous single-valued map such that the following diagram is commutative:*

$$\begin{array}{ccccc} (V, V\backslash K_1) & \overset{f_1}{\longleftarrow} & (X_1, X_1\backslash f_1^{-1}(K_1)) & \overset{f_1-g_1}{\longrightarrow} & (E^n, E^n\backslash\{\theta\}) \\ i\downarrow & & \downarrow h & & \Updownarrow \\ (U, U\backslash K) & \overset{f}{\longleftarrow} & (X, X\backslash f^{-1}(K)) & \overset{f-g}{\longrightarrow} & (E^n, E^n\backslash\{\theta\}) \end{array}$$

*where $i$ is the natural injection. Then if the pair $(f_1, g_1) \in \mathcal{PD}(V, V\backslash K_1)$ one can infer that $(f,g) \in \mathcal{PD}(U, U\setminus K)$ and $I(f_1,g_1) \subset I(f,g)$.*

*Proof.* Of course, $i$ induces an isomorphism $i_* : H_n(V, V\backslash K_1) \longrightarrow H_n(U, U\backslash K)$ which takes $O_{K_1}$ in $O_K$. Moreover, if $\sigma \in \Omega(f_{1*}, V, V\backslash K_1)$, then $h_* \circ \sigma \circ i_*^{-1} \in \Omega(f_*, U, U\backslash K)$. □

Let $(f,g)$ be a pair of continuous single-valued maps such that:

$$U \overset{f}{\longleftarrow} X \overset{g}{\longrightarrow} E^n$$

and $h : X_1 \longrightarrow X$ be a continuous single-valued map defined between two Hausdorff topological spaces $X_1$ and $X$.

**Proposition 3.12.** *If $h : (X_1, X_1\backslash(f \circ h)^{-1}(K)) \longrightarrow (X, X\backslash f^{-1}(K))$ is $n$-decomposing on $(X, X\backslash f^{-1}(K))$ and the pair $(f, g)$ is $n$-admissible on $(U, U\backslash K)$, then $(f \circ h, g \circ h) \in \mathcal{PD}(U, U \setminus K)$ and $I(f \circ h, g \circ h) \subset I(f, g)$.*

*Proof.* Note that $\mathbf{F}(f \circ h, g \circ h) \subseteq \mathbf{F}(f, g) \subseteq K$, the composition $f \circ h$ is $n$-decomposing on $(U, U\backslash K)$ (see proposition 2.14), and one has the following diagram:

$$(U, U\backslash K) \quad \overset{f\circ h}{\longleftarrow} \quad (X_1, X_1\backslash(f \circ h)^{-1}(K)) \quad \overset{f\circ h - g\circ h}{\longrightarrow} \quad (E^n, E^n\backslash \{\theta\})$$

Let $k \in I(f \circ h, g \circ h)$, then there exists $\sigma \in \Omega((f \circ h)_*, U, U\backslash K)$ such that $(f \circ h - g \circ h)_* \circ \sigma(O_K) = k \cdot O_\theta$, therefore $(f - g)_* \circ h_* \circ \sigma(O_K) = k \cdot O_\theta$. Because $h_* \circ \sigma \in \Omega(f_*; U, U\backslash K)$, one deduces that $k \in I(f, g)$. □

**Definition 3.13.** *Two pairs of continuous single-valued maps defined as follows:*

$$U \overset{f_i}{\longleftarrow} X \overset{g_i}{\longrightarrow} E^n,\ i = 0, 1,$$

*are called equivariant on a compact $K \subset E^n$ if there exist:*

(1) *a Hausdorff pair $(X, X\backslash X')$ such that:*

$$(U, U\backslash K) \overset{f_i}{\longleftarrow} (X, X\backslash X') \overset{f_i - g_i}{\longrightarrow} (E^n, E^n\backslash \{\theta\}),\ i = 0, 1,$$

(2) *a pair of continuous maps $(\varphi, \psi)$ $n$-admissible on $(U, U\backslash K)$ such that:*

$$(U, U\backslash K) \quad \overset{\varphi}{\longleftarrow} \quad (X, X\backslash \varphi^{-1}(K)) \quad \overset{\varphi - \psi}{\longrightarrow} \quad (E^n, E^n\backslash \{\theta\})$$

(3) *a single-valued map $h : (X, X\backslash \varphi^{-1}(K)) \longrightarrow (X, X\backslash X')$ $n$-decomposing on $(X, X\backslash X')$ such that the following diagram is commutative:*

$$\begin{array}{lclcl}
(U, U\backslash K) & \overset{f_0}{\longleftarrow} & (X, X\backslash X') & \overset{f_0 - g_0}{\longrightarrow} & (E^n, E^n\backslash \{\theta\}) \\
\Updownarrow & & \uparrow h & & \Updownarrow \\
(U, U\backslash K) & \overset{\varphi}{\longleftarrow} & (X, X\backslash \varphi^{-1}(K)) & \overset{\varphi - \psi}{\longrightarrow} & (E^n, E^n\backslash \{\theta\}) \\
\Updownarrow & & \downarrow h & & \Updownarrow \\
(U, U\backslash K) & \overset{f_1}{\longleftarrow} & (X, X\backslash X') & \overset{f_1 - g_1}{\longrightarrow} & (E^n, E^n\backslash \{\theta\})
\end{array}$$

**Proposition 3.14.** *If $(f_i, g_i)$, $i = 0, 1$ are two equivariant pairs on a compact $K \subset E^n$, then $(f_i, g_i) \in \mathcal{PD}(U, U \setminus K)$, $i = 0, 1$, and $I(f_0, g_0) = I(f_1, g_1)$.*

*Proof.* Assume $(f_0, g_0)$ and $(f_1, g_1)$ are equivariant, then $f_{0*} \circ h_* = \varphi_* = f_{1*} \circ h_*$ therefore $f_{0*}$, $f_{1*}$ are both $n$-decomposing on $(U, U\backslash K)$ and $f_{0*} = f_{1*}$. Moreover, $(f_0 - g_0)_* \circ h_* = (\varphi - \psi)_* = (f_1 - g_1)_* \circ h_*$ so $(f_0 - g_0)_* = (f_1 - g_1)_*$. □

**Definition 3.15.** *Two pairs $(f_i, g_i)$, $i = 0, 1$ defined as follows:*

$$(U, U\backslash K) \overset{f_i}{\longleftarrow} (X, X\backslash X') \overset{f_i - g_i}{\longrightarrow} (E^n, E^n\backslash\{\theta\}),\ i = 0, 1,$$

*are called homotopic on a compact $K \subset E^n$ if the following conditions are verified:*

(1) *there exists a pair of single-valued maps* $(\varphi, \psi)$ *n-admissible on* $(U, U\backslash K)\times [0,1]$ *such that:*

$$(U, U\backslash K) \times [0,1] \overset{\varphi}{\longleftarrow} (X, X\backslash \varphi^{-1}(K \times [0,1])) \overset{\varphi-\Psi}{\longrightarrow} (E^n, E^n\backslash\{\theta\}),$$

(2) *there exists a single valued map*

$$h : (X, X\backslash X') \longrightarrow (X, X\backslash \varphi^{-1}(K \times [0,1])),$$

*n-decomposing on* $(X, X\backslash \varphi^{-1}(K \times [0,1]))$,

(3) *the following diagram is commutative:*

$$\begin{array}{llll}
(U, U\backslash K) & \overset{f_0}{\longleftarrow} & (X, X\backslash X') & \overset{f_0-g_0}{\longrightarrow} & (E^n, E^n\backslash\{\theta\}) \\
\chi_0 \downarrow & & h\downarrow & & \Updownarrow \\
(U, U\backslash K)\times[0,1] & \overset{\varphi}{\longleftarrow} & (X, X\backslash \varphi^{-1}(K\times[0,1])) & \overset{\varphi-\Psi}{\longrightarrow} & (E^n, E^n\backslash\{\theta\}) \\
\chi_1 \uparrow & & h\uparrow & & \Updownarrow \\
(U, U\backslash K) & \overset{f_1}{\longleftarrow} & (X, X\backslash X') & \overset{f_1-g_1}{\longrightarrow} & (E^n, E^n\backslash\{\theta\})
\end{array}$$

*where* $\chi_i(x) = (x, i)$, *for every* $x \in U$ *and* $i = 0, 1$.

**Proposition 3.16.** *If* $(f_0, g_0)$ *and* $(f_1, g_1)$ *are homotopic on a compact* $K \subset E^n$ *then* $(f_i, g_i) \in \mathcal{PD}(U, U \setminus K)$, $i = 0, 1$ *and* $I(f_0, g_0) = I(f_1, g_1)$.

*Proof.* Of course, $\chi_{0*}$ and $\chi_{1*}$ are both isomorphisms and are equal, so $f_{0_*} = f_{1_*}$. One deduces also that $f_{0_*}$ and $f_{1_*}$ are both $n$-decomposing on $(U, U\backslash K)$. In an other hand, from the commutativity of the diagram one obtains that $(f_0 - g_0)_* \circ h_* = (f_1 - g_1)_* \circ h_* = (\varphi - \psi)_*$ therefore $(f_0 - g_0)_* = (f_0 - g_0)_*$. □

Let $(f, g)$ and $(f', g')$ be two pairs defined by the following way:

$$U \overset{f}{\longleftarrow} X \overset{g}{\longrightarrow} E^n$$

and

$$U' \overset{f'}{\longleftarrow} X \overset{g'}{\longrightarrow} E^m$$

where $U$ and $U'$ are two open subsets of $E^n$ and $E^m$ respectively.

Let $K$ be a compact subset of $E^n$ which contains $\mathbf{F}(f, g)$ and $K'$ be a compact subset $E$ which contains $\mathbf{F}(f', g')$.

**Proposition 3.17.** *If the pairs* $(f, g)$ *and* $(f', g')$ *are n-admissible on* $(U, U\backslash K)$ *and* $(U', U'\backslash K')$ *respectively then the pair* $(f \times f', g \times g')$ *is* $(n + m)$*-admissible on* $(U \times U', U \times U'\backslash K \times K')$ *and* $I(f \times f', g \times g') \supset I(f, g) \cdot I(f', g')$.

*Proof.* One has the following equalities:

$$\mathbf{F}(f \times f', g \times g') = \mathbf{F}(f, g) \times \mathbf{F}(f', g'),$$

$O_{K\times K'} = O_K \times O_{K'} \in H_{n+m}\left[(U, U\backslash K) \times (U', U'\backslash K')\right] = H_{n+m}(U \times U', U \times U'\backslash K \times K')$ and the inclusion:

$$K \times K' \supset \mathbf{F}(f, g) \times \mathbf{F}(f', g').$$

Therefore, if $(\sigma, \sigma') \in \Omega(f_*, U, K) \times \Omega(f'_*, U', K')$ one obtains the equalities: $(f \times f' - g \times g')_* \circ (\sigma \times \sigma')(O_{K\times K'}) =$

$[(f-g)_* \circ \sigma \times (f'-g')_* \circ \sigma']\,(O_K \times O_{K'}) =$
$(f-g)_* \circ \sigma(O_K) \times (f'-g')_* \circ \sigma'(O_K) = [I_\sigma(f,g) \cdot I_{\sigma'}(f',g')]\, O_{\{\theta\}}.$ □

## 4. Generalized fixed point degree of multi-valued mappings

Let $X$ and $Y$ be two Hausdorff topological spaces. A multi-valued mapping taking $X$ to $Y$ is a relation $F$ which associates to each element $x \in X$ a non empty subset $F(x) \subset Y$. Let $K(Y)$ be the collection of all non empty compact subsets of $Y$ and $F : X \longrightarrow K(Y)$ be a multi-valued mapping.

The subset:

$$\Gamma_X(F) = \{(x,y) \in X \times Y \mid y \in F(x)\},$$

of $X \times Y$ is called the graph of the multi-valued mapping $F$ on $X$.

In this case one could define two natural projectors:

$$t_F : \Gamma_X(F) \longrightarrow X$$

and

$$r_F : \Gamma_X(F) \longrightarrow Y$$

such $t_F(x,y) = x,\ r_F(x,y) = y$ for every $(x,y) \in \Gamma_X(F)$.

For each element $x \in X$ one has the equality $F(x) = r_F(t_F^{-1}(x))$. The quintuple $[X, Y, \Gamma_X(F), t_F, r_F]$ is called the canonical representation of the multi-valued $F : X \longrightarrow K(Y)$.

Let $[X_1, X_2, X_0, f_1, f_2]$ be a quintuple constituted of Hausdorff topological spaces $X_i$, $i = 0,1,2$ and continuous maps $f_j : X_0 \longrightarrow X_i$, $j = 1,2$ and such that $f_1$ is onto and the inverse image of each element $x \in X_1$ is compact, then the equality $F(x) = g \circ f^{-1}(x)$ defines a multi-valued mapping $F : X_1 \longrightarrow K(X_2)$. In this case the quintuple $[X_1, X_2, X_0, f_1, f_2]$ is called a representation of $F : X_1 \longrightarrow K(X_2)$.

Two quintuples $[X_1, X_2, X_0, f_1, f_2]$, $[X_1, X_2, X_0, g_1, g_2]$ are called equivalents if $g_1 \circ f_1^{-1}(x) = F(x) = g_2 \circ f_2^{-1}(x)$ for each $x \in X_1$.

A multi-valued mapping $F : X \longrightarrow K(Y)$ is called upper semi continuous if $F_+^{-1}(V) = \{x \in X \mid F(x) \subset V\}$ is an open subset of $Y$ for every open subset $V$ of $X$.

A multi-valued $G : X \longrightarrow K(Y)$ is said to be a selector of $F : X \longrightarrow K(Y)$ if $G(x) \subseteq F(x)$ for every element $x \in X$.

Let $H$ be the Čech homology functor with compact carries and coefficient in the set of rational numbers $Q$. A multi-valued mapping $F : X \longrightarrow K(Y)$ is called to be $Q$-acyclic provided the image $F(x)$ is $Q$-acyclic for every element $x \in X$, $F$ is said to be compact provided $F(X)$ is contained in a compact subset of $Y$.

M*ore properties on multi-valued mappings can be found in* [24].

Let $F : U \longrightarrow K(E^n)$ be a multi-valued mapping and $K$ be a compact subset of $U \subseteq E^n$. In this case $\mathbf{F}(t_F, r_F) = \{x \in U \mid x \in r_F(t_F^{-1})(x)\} = \{x \in U \mid x \in F(x)\} = Fix(F)$.

**Definition 4.1.** *A multi-valued mapping $F : U \longrightarrow K(E^n)$ is called $n$-admissible on $(U, U\backslash K)$ if the pair $(t_F, r_F)$ of projectors*:

$$U \overset{t_F}{\longleftarrow} \Gamma_U(F) \overset{r_F}{\longrightarrow} E^n$$

*satisfies the following conditions:*

(1) $K \supset Fix(F) = \{x \in U \mid x \in F(x)\}$;
(2) *the pair $(t_F, r_F)$ is $n$-admissible on $(U, U\backslash K)$.*

**Lemma 4.2.** *Let $F : U \longrightarrow K(E^n)$ be a multi-valued mapping $n$-admissible on $(U, U\backslash K)$, then one has the following diagram:*

$$H_n(U, U\backslash K) \overset{(t_F)_*}{\longleftarrow} H_n(\Gamma_U(F), \Gamma_{U\backslash K}(F)) \overset{(t_F - r_F)_*}{\longrightarrow} H_n(E^n, E^n\backslash \{\theta\})$$

*Proof.* The proof is obvious. □

**Definition 4.3.** *The generalized fixed point degree of a $n$-admissible multi-valued mapping $F$ on $(U, U\backslash K)$ is defined as the following set of rational numbers:*

$$\mathcal{I}(F; U, K) = I(t_F, r_F) = \{I_\sigma(t_F, r_F) \mid \sigma \in \Omega((t_F)_*; U, U\backslash K)\}$$

*Let us describe some properties of this generalized fixed point degree.*

**Theorem 4.4.** *If $\mathcal{I}(F; U, K) \neq \{0\}$ then $F$ admits at least a fixed point i.e. a point $x \in U$ such that $x \in F(x)$.*

*Proof.* This is a consequence of corollary (3.8). □

**Definition 4.5.** *A representation $\rho = [U, E^n, Z, f, g]$ of a multi-valued mapping $F : U \longrightarrow K(E^n)$ is called $n$-admissible on $(U, U\backslash K)$ if the pair $(f, g)$ is $n$-admissible on $(U, U\backslash K)$ and $\{x \in U \mid x \in F(x)\} \subseteq K$.*

Let $U$ and $V$ be two open subsets of $E^n$, $K$ and $K_1$ be two compact subsets of $E^n$ such that $K \subset K_1 \subset V \subset \overline{V} \subset U$. If the restriction $\widetilde{F} : V \longrightarrow K(E^n)$ of $F : U \longrightarrow K(E^n)$ defined by the rule $\widetilde{F}(x) = F(x)$ for every $x \in V$ admits a representation $\rho = [V, E^n, Z, f, g]$ $n$-admissible on $(V, V\backslash K_1)$, so one can consider the following diagram:

$$(V, V\backslash K_1) \overset{f}{\longleftarrow} (Z, Z\backslash f^{-1}(K_1)) \overset{f-g}{\longrightarrow} (E^n, E^n\backslash\{\theta\}) \tag{4.8}$$

Let $\Omega(f_*; V, V\backslash K_1)$ be the set of the right inverse homomorphisms of:

$$f_* : H_n(Z, Z\backslash f^{-1}(K_1)) \longrightarrow H_n(V, V\backslash K_1).$$

In this case one can define:

$$\mathcal{I}_\rho(\widetilde{F}; V, K_1) = \{I_\sigma(f, g) \mid \sigma \in \Omega(f_*; V, V\backslash K_1)\}.$$

**Proposition 4.6.** *If a multi-valued mapping $F : U \longrightarrow K(E^n)$ has a restriction $\widetilde{F} : V \longrightarrow K(E^n)$ which admits a representation $\rho = [V, E^n, Z, f, g]$ $n$-admissible on $(V, V\backslash K_1)$ then the multi-valued mapping $F$ is $n$-admissible on $(U, U\backslash K)$ and $\mathcal{I}_\rho(\widetilde{F}; V, K_1) \subset \mathcal{I}(F; U, K)$.*

*Proof.* The proof is a consequence of proposition (3.11) and the following commutative diagram:

$$\begin{array}{lcccc} H_n(V,V\backslash K_1) & \overset{f_*}{\longleftarrow} & H_n(Z,Z\backslash f^{-1}(K_1)) & \overset{(f-g)_*}{\longrightarrow} & H_n(E^n,E^n\backslash\{\theta\}) \\ i_*\downarrow & & \downarrow\alpha_* & & \Updownarrow \\ H_n(U,U\backslash K) & \overset{(t_F)_*}{\longleftarrow} & H_n(\Gamma_U(F),\Gamma_{U\backslash K}(F)) & \overset{(t_F-r_F)_*}{\longrightarrow} & H_n(E^n,E^n\backslash\{\theta\}) \end{array}$$

where $\alpha(z) = (f(z), g(z))$ for each $z \in Z$. □

**Corollary 4.7.** *If a multi-valued mapping $F : U \longrightarrow K(E^n)$ admits a representation $\rho = [V, E^n, Z, f, g]$ n-admissible on $(V, V\backslash K_1)$, then $F$ is n-admissible on $(U, U\backslash K)$ and $\mathcal{I}_\rho(F; V, K_1) \subset \mathcal{I}(F; U, K)$.*

**Proposition 4.8.** *Let $F : U \longrightarrow K(E^n)$ be a multi-valued mapping and $\Phi : U \longrightarrow K(E^n)$ be a selector of $F$, then if $\Phi$ is a multi-valued mapping n-admissible on $(U, U\backslash K)$ the multi-valued mapping $F$ is also n-admissible on $(U, U\backslash K)$ and $\mathcal{I}(\Phi; U, K) \subset \mathcal{I}(F; U, K)$.*

*Proof.* The proof is a consequence of the following commutative diagram:

$$\begin{array}{lcccc} H_n(U,U\backslash K) & \overset{(t_\Phi)_*}{\longleftarrow} & H_n(\Gamma_U(\Phi),\Gamma_{U\backslash K}(\Phi)) & \overset{(t_\Phi-r_\Phi)_*}{\longrightarrow} & H_n(E^n,E^n\backslash\{\theta\}) \\ \Updownarrow & & i_*\downarrow & & \Updownarrow \\ H_n(U,U\backslash K) & \underset{(t_F)_*}{\longleftarrow} & H_n(\Gamma_U(\Phi),\Gamma_{U\backslash K}(\Phi)) & \underset{(t_F-r_F)_*}{\longrightarrow} & H_n(E^n,E^n\backslash\{\theta\}) \end{array}$$

where $i$ is the canonical injection. □

**Definition 4.9.** *A continuous single-valued map $\lambda : [0,1] \times U \times E^n \longrightarrow E^n$ is said to be a distortion of $E^n$ if for each element $x \in U$ the single-valued map $\lambda(0, x, .) : E^n \longrightarrow E^n$ is the map identity.*

**Definition 4.10.** *A multi-valued $F : U \longrightarrow K(E^n)$ n-admissible on $(U, U\backslash K)$ distorts into the multi-valued $G : U \longrightarrow K(E^n)$ if there exists a distortion of $E^n$ such that :*

(1) *$\lambda(1, x, F(x)) = G(x)$ for every $x \in U$;*
(2) *$x \notin \lambda(t, x, F(x))$ for every $t \in [0,1]$ and $x \in (U\backslash K)$.*

**Proposition 4.11.** *If a multi-valued $F : U \longrightarrow K(E^n)$ n-admissible on $(U, U\backslash K)$ distorts into the multi-valued $G : U \longrightarrow K(E^n)$, then $G$ is n–admissible on $(U, U\backslash K)$ and $\mathcal{I}(F; U, K) \subset \mathcal{I}(G; U, K)$.*

*Proof.* Consider $\xi : (\Gamma_U(F), \Gamma_{U\backslash K}(F)) \longrightarrow (\Gamma_U(G), \Gamma_{U\backslash K}(G))$ defined by the rule $\xi(x, u) = (x, \lambda(1, x, u))$ for every $(x, u) \in \Gamma_U(F)$. Form the equality $t_F = t_G \circ \xi$ one deduces that $G$ is $n$-admissible on $(U, U\backslash K)$. In an other hand, the continuous single-valued maps $(t_F - r_F)$, $(t_G - r_G) \circ \xi : (\Gamma_U(F), \Gamma_{U\backslash K}(F)) \longrightarrow (E^n, E^n\backslash\{\theta\})$ are homotopic by the homotopy $h(t, (x, u) = x - \lambda(t, x, u)$ for every $t \in [0, 1]$ and $(x, u) \in \Gamma_U(F)$. Let $\sigma \in \Omega\left((t_F)_*\right)$, then $\xi_* \circ \sigma \in \Omega\left((t_G)_*\right)$ and one has the equalities: $I_\sigma \cdot O_{\{\theta\}} = (t_F - r_F)_* \circ \sigma(O_K) = (t_G - r_G)_* \circ \xi_* \circ \sigma(O_K) = I_{\xi_* \circ \sigma} \cdot O_{\{\theta\}}$, which means that $\mathcal{I}(F; U, K) \subset \mathcal{I}(G; U, K)$. □

Assume that $U$ and $V$ are two open subsets of $E^n$, $K$ and $K_1$ are two compact subsets of $E^n$ such that $K \subset K_1 \subset V \subset \overline{V} \subset U$.

**Proposition 4.12.** *Let $F : U \longrightarrow K(E^n)$ be a multi-valued mapping upper semi continuous compact and $Q$-acyclic. If $G : U \longrightarrow K(E^n)$ is a selector of $F$ and $n$-admissible on $(V, V\backslash K_1)$, then $\mathcal{I}(G; V, K_1) = \mathcal{I}(F; U, K) = \{k\}$, where $k$ is the rational number which verifies the equality $(t_F - r_F)_* \circ (t_F)_*^{-1}(O_K) = k \cdot O_{\{\theta\}}$.*

*Proof.* The proof is a consequence of the Vietoris maps theorems [12], proposition (4.8) and the following commutative diagram:

$$\begin{array}{ccccc} H_n(V, V\backslash K_1) & \overset{t_{G*}}{\longleftarrow} & H_n(\Gamma_V(G), \Gamma_{V\backslash K_1}(G)) & \overset{(t_G - r_G)_*}{\longrightarrow} & H_n(E^n, E^n\backslash\{\theta\}) \\ i_* \downarrow & & j_* \downarrow & & \Updownarrow \\ H_n(U, U\backslash K) & \underset{t_{F*}}{\longleftarrow} & H_n(\Gamma_U(F), \Gamma_{U\backslash K}(F)) & \underset{(t_F - r_F)_*}{\longrightarrow} & H_n(E^n, E^n\backslash\{\theta\}) \end{array}$$

where $i : (V, V\backslash K_1) \longrightarrow (U, U\backslash K)$ and $j : (\Gamma_V(G), \Gamma_{V\backslash K_1}(G)) \longrightarrow (\Gamma_U(F), \Gamma_{U\backslash K}(F))$ are the natural injections. □

**Proposition 4.13.** *Let $K$ be a compact $Q$-acyclic subset of $E^n$ and $F : U \longrightarrow K(E^n)$ be a multi-valued mapping such that $F(U) \subset K$, then $F$ is $n$-admissible on $(U, U\backslash K)$ and $\mathcal{I}(F; U, K) = \{1\}$.*

*Proof.* Consider $x_0 \in K$ and let $f : U \longrightarrow K(E^n)$ be the map defined by the rule $f(x) = \{x_0\}$ for each $x \in U$. The quintuple $\rho = [U, E^n, U, Id_U, f]$ is a representation $n$-admissible on $(U, U\backslash K)$ of $f$. Consider the following commutative diagram:

$$\begin{array}{ccccc} H_n(U, U\backslash K) & \overset{(Id_U)_*}{\longleftarrow} & H_n(U, U\backslash K) & \overset{(Id_U - f)_*}{\longrightarrow} & H_n(E^n, E^n\backslash\{\theta\}) \\ & & j_* \downarrow & & \Updownarrow \\ & & H_n(E^n, E^n\backslash\{x_0\}) & \underset{(Id_{E^n} - f)_*}{\longrightarrow} & H_n(E^n, E^n\backslash\{\theta\}) \end{array}$$

where $j_*$ is an isomorphism induced by the natural injection and $(Id_{E^n} - f)_*$ is the isomorphism induced by the homeomorphism $(Id_{E^n} - f) : (E^n, E^n\backslash\{x_0\}) \longrightarrow (E^n, E^n\backslash\{\theta\})$ defined by the rule $(Id_{E^n} - f)(x) = x - x_0$ for every $x \in E^n$. For instance, one deduces that $\mathcal{I}_\rho(f; U, K) = \{1\}$. In an other hand, consider the following commutative diagram:

$$\begin{array}{ccccc} H_n(U, U\backslash K) & \overset{(Id_U)_*}{\longleftarrow} & H_n(U, U\backslash K) & \overset{(Id_U - f)_*}{\longrightarrow} & H_n(E^n, E^n\backslash\{\theta\}) \\ \Updownarrow & & \downarrow \mu_* & & \Updownarrow \\ H_n(U, U\backslash K) & \underset{(t_F)_*}{\longleftarrow} & H_n(\Gamma_U(i - R), \Gamma_{U\backslash K}(i - R)) & \underset{(t_F - r_F)_*}{\longrightarrow} & H_n(E^n, E^n\backslash\{\theta\}) \end{array}$$

where $\mu(x) = (x, f(x))$, for each $x \in U$. The multi-valued mapping $F$ is $n$-admissible on $(U, U\backslash K)$ because $(Id_U)_*$ is an isomorphisms. From the propositions (3.9), (4.7) and the commutativity of the above diagram one infers $\mathcal{I}_\rho(f; U, K) \subset \mathcal{I}(F; U, K)$. The multi-valued mapping $F : U \longrightarrow K(E^n)$ is a selector of the upper semi continuous, compact and $Q$-acyclic multi-valued mapping $G : U \longrightarrow K(E^n)$ defined by the rule $G(x) = K$ for each $x \in U$.

Using the proposition (4.12), one deduces $\mathcal{I}(F;U,K) = \mathcal{I}(G;U,K) = \{k\}$ so $k = 1$. □

**Proposition 4.14.** *Let $C$ be a compact subset of $E^n$ which is a neighborhood retract. Let $F : C \longrightarrow K(C)$ be an upper semi continuous and $Q$-acyclic multi-valued mapping. Then $F$ admits at least a fixed point.*

*Proof.* Consider $U$ an open subset of $E^n$ and let $\rho : U \longrightarrow C$ be a retraction from $U$ into $C$. The multi-valued $G = F \circ \rho : U \longrightarrow K(C) \subset K(E^n)$ is upper semi continuous compact with $Q$-acyclic values, therefore $\mathcal{I}(G;U,C) = \{1\}$. One deduces that $G$ admits in $U$, at least, a fixed point $x \in G(x) = F(\rho(x))$. However, $x \in C$ then $\rho(x) = x$. □

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