Applied General Topology
© Universidad Politécnica de Valencia
Volume 6, No. 2, 2005
pp. 143-148

A note on locally ν-bounded spaces

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Abstract. In this paper, on the family $\mathcal{O}(Y)$ of all open subsets of a space Y (actually on a complete lattice) we define the so called strong ν-Scott topology, denoted by τ^s_ν, where ν is an infinite cardinal. This topology defines on the set $C(Y,Z)$ of all continuous functions on the space Y to a space Z a topology t^s_ν. The topology t^s_ν, is always larger than or equal to the strong Isbell topology (see [8]). We study the topology t^s_ν in the case where Y is a locally ν-bounded space.

2000 AMS Classification: 54C35

Keywords: Strong Scott topology, Strong Isbell topology, Function space, Admissible topology.

1. Basic Notions

Let X be a space and G a map of X into $C(Y,Z)$. By $\widetilde{G}$ we denote the map of $X \times Y$ to Z such that $\widetilde{G}(x,y) = G(x)(y)$ for every $(x,y) \in X \times Y$.

A topology t on $C(Y,Z)$ is called *admissible* if for every space X, the continuity of a map $G : X \to C_t(Y,Z)$ implies that of the map $\widetilde{G} : X \times Y \to Z$. Equivalently, a topology t on $C(Y,Z)$ is admissible if the *evaluation map* $e : C_t(Y,Z) \times Y \to Z$ defined by relation $e(f,y) = f(y)$, $(f,y) \in C(Y,Z) \times Y$, is continuous (see [1]).

Let L be a poset. The *Scott topology* τ_ω (see, for example, [5]) is the family of all subsets $I\!H$ of L such that:

(α) $I\!H = \uparrow I\!H$, where $\uparrow I\!H = \{y \in L : (\exists x \in I\!H)\ x \leq y\}$, and

(β) for every directed subset D of L with $\sup D \in I\!H$, $D \cap I\!H \neq \varnothing$.

Below, we consider the poset $\mathcal{O}(Y)$ of all open subsets of the space Y on which the inclusion is considered as the order.

The *Isbell topology* t_ω on $C(Y,Z)$ (see, for example, [8], [11] and [9]) is the topology for which the family of all sets of the form

$$(I\!H, U) = \{f \in C(Y,Z) : f^{-1}(U) \in I\!H\},$$

where $U \in \mathcal{O}(Z)$ and $I\!H \in \tau_\omega$, constitute a subbasis for this topology.

The notion of a *bounded subset* was introduced in [3] and the notion of a *locally bounded space* in [7]. Some generalizations of locally bounded spaces are given in [10]. The notion of the *strong Scott topology* (defined on a complete lattice) was given in [8]. This topology determines on the set $C(Y, Z)$ a topology called the *strong Isbell topology* (see [8]). It is proved that a space Y is locally bounded if and only if the strong Isbell topology on $C(Y, \mathbf{2})$, where $\mathbf{2}$ is the Sierpinski space, is admissible. In the case, where Y is locally bounded and Z is an arbitrary space, it is proved that the strong Isbell topology on $C(Y, Z)$ is admissible.

In this paper we denote by ν a fixed infinite cardinal.

A subset D of a poset L is called *ν-directed* if every subset of D with cardinality less than ν has an upper bound in D (see [4]).

Suppose that L is a complete lattice. We say that x is *ν-way below* y and write $x <<_\nu y$ (see [4]) if for every ν-directed subset D of L the relation $y \leq \sup D$ implies the existence of $d \in D$ with $x \leq d$.

In particular, for two elements U and V of the complete lattice $\mathcal{O}(Y)$ we have: $U <<_\nu V$ if for every open cover $\{W_i : i \in I\}$ of V there is a subcollection $\{W_i : i \in J \subseteq I\}$ of this cover such that $|J| < \nu$ and $U \subseteq \cup\{W_i : i \in J\}$. It is clear that if $U \subseteq V <<_\nu Y$, then $U <<_\nu Y$.

2. Other notions

Definition 2.1. *A subset B of Y is called ν-bounded if every open cover of Y contains a cover of B of cardinality less than that of ν. (For the related notion of an (m, n)-bounded subset see* [6]*.)*

A space is called locally ν-bounded if it has a basis for the open subsets consisting of ν-bounded sets. (For the related notion of a local $\mathcal{P}$-space see [10]*.)*

Definition 2.2. *Let $(L, \leq)$ be a fixed complete lattice and 1 the maximal element of L. By τ_ν^s we denote the family of all subsets $I\!H$ of L such that:*

(α) $I\!H = \uparrow I\!H$, where $\uparrow I\!H = \{y \in L : (\exists x \in I\!H)\ x \leq y\}$, and

(β) for every ν-directed subset D of L with $\sup D = 1$ we have $D \cap I\!H \neq \varnothing$.

It is clear that, the family τ_ν^s is a T_0 topology on L called the strong ν-Scott topology.

In the case, where $L = \mathcal{O}(Y)$, a subset $I\!H$ of $\mathcal{O}(Y)$ belongs to the strong ν-Scott topology if the following properties are true:

Property (α). The conditions $U \in I\!H$, $V \in \mathcal{O}(Y)$, and $U \subseteq V$ imply $V \in I\!H$.

Property (β). For every open cover $\{U_i : i \in I\}$ of Y there exists a subset J of I of cardinality less than ν such that $\cup\{U_i : i \in J\} \in I\!H$.

Remark 2.3. If μ is an infinite cardinal such that $\mu \leq \nu$, then $\tau_\omega^s \subseteq \tau_\mu^s \subseteq \tau_\nu^s$, where ω is the first infinite cardinal.

Definition 2.4. *Let L be a complete lattice. An element $x \in L$ is called ν-bounded if $x <<_\nu 1$.*

The lattice L is called weakly ν-continuous if for all $x \in L$

$$x = \sup\{u \in L : u \leq x \text{ and } u <<_\nu 1\}.$$

In the case, where $L = \mathcal{O}(Y)$, a set $U \in \mathcal{O}(Y)$ is ν-bounded if $U <<_\nu Y$.

Notation. We denote by t^s_ν the topology on the set $C(Y, Z)$ for which the sets of the form:

$$(I\!H, U) = \{f \in C(Y, Z) : f^{-1}(U) \in I\!H\},$$

where $U \in \mathcal{O}(Z)$ and $I\!H \in \tau^s_\nu$, compose a subbasis.

Obviously, if $\omega \leq \mu \leq \nu$, then $t^s_\omega \subseteq t^s_\mu \subseteq t^s_\nu$.

Remark 2.5. For $\nu = \omega$ the notions of an ω-bounded subset, a locally ω-bounded space, and a weakly ω-continuous lattice coincide with the notions of a bounded subset, a locally bounded space, and a weakly continuous lattice, respectively.

Also, the topologies τ^s_ω and t^s_ω coincide with the strong Scott topology and the strong Isbell topology, respectively.

3. The results

Proposition 3.1. *If Y is locally ν-bounded, then the topology t^s_ν on $C(Y, Z)$ is admissible.*

Proof. It is sufficient to prove that the evaluation map

$$e : C_{t^s_\nu}(Y, Z) \times Y \to Z$$

is continuous.

Let $(f, y) \in C_{t^s_\nu}(Y, Z) \times Y$, $W \in \mathcal{O}(Z)$, and $e(f, y) = f(y) \in W$. We need to prove that there exist $I\!H \in \tau^s_\nu$, $U \in \mathcal{O}(Z)$, and an open neighborhood V of y in Y such that $f \in (I\!H, U)$ and

$$e((I\!H, U) \times V) \subseteq W.$$

Since Y is locally ν-bounded and $y \in f^{-1}(W)$ there exists an open ν-bounded set V such that:

$$y \in V \subseteq f^{-1}(W).$$

We consider the set

$$I\!H = \{P \in \mathcal{O}(Y) : V \subseteq P\}$$

and prove that $I\!H \in \tau^s_\nu$, that is $I\!H$ satisfies Properties (α) and (β).

Property (α) is clear.

Property (β). Let $\{U_i : i \in I\}$ be an open cover of Y. Since V is ν-bounded there exists a subset J of I of cardinality less than of ν such that $V \subseteq \bigcup\{U_i : i \in J\}$. By the definition of $I\!H$ we have $\bigcup\{U_i : i \in J\} \in I\!H$. Since $V \subseteq f^{-1}(W)$ we have $f^{-1}(W) \in I\!H$ and therefore $f \in (I\!H, W)$. Thus, the subset $(I\!H, W) \times V$ is a neighborhood of (f, y) in $C_{\tau^s_\nu}(Y, Z) \times Y$.

Now, we prove that $e((I\!H, W) \times V) \subseteq W$. Let $(g, z) \in (I\!H, W) \times V$. Then $g^{-1}(W) \in I\!H$, $z \in V$, and $V \subseteq g^{-1}(W)$. Therefore $e((g, z)) = g(z) \in W$.

Thus, the map e is continuous which means that t^s_ν is admissible. □

Proposition 3.2. *For the space Y the following statements are equivalent:*

(1) *Y is locally ν-bounded.*
(2) *For every space Z the evaluation map $e : C_{t^s_\nu}(Y, Z) \times Y \to Z$ is continuous.*
(3) *The evaluation map $e : C_{t^s_\nu}(Y, \mathbf{2}) \times Y \to \mathbf{2}$ is continuous.*
(4) *For every open neighborhood V of a point y of Y there is an open set $I\!\!H \in \tau^s_\nu$ such that $V \in I\!\!H$ and the set $\cap\{P : P \in I\!\!H\}$ is a neighborhood of y in Y.*
(5) *The lattice $\mathcal{O}(Y)$ is weakly ν-continuous.*

Proof. (1) $\Longrightarrow$ (2) Follows by Proposition 3.1.

(2) $\Longrightarrow$ (3) It is obvious.

(3) $\Longrightarrow$ (4) Let V be an open neighborhood of y in Y. Consider the sets $\mathcal{O}(Y)$ and $C(Y, \mathbf{2})$. We identify each element U of $\mathcal{O}(Y)$ with the element f_U of $C(Y, \mathbf{2})$ for which $f_U(U) \subseteq \{0\}$ and $f_U(Y \setminus U) \subseteq \{1\}$. Then, each topology on one of the above sets can be considered as a topology on the other. In this case $t^s_\nu = \tau^s_\nu$ and the map $e : \mathcal{O}(Y) \times Y \to \mathbf{2}$ is continuous. Since $e(V, y) = e(f_V, y) = f_V(y) = 0$, the continuity of e implies that for the open neighborhood $\{0\}$ of $e(V, y)$ in $\mathbf{2}$ there exist an open neighborhood $I\!\!H \in \tau^s_\nu$ of V in $\mathcal{O}(Y)$ and an open neighborhood V' of y in Y such that $e(I\!\!H \times V') \subseteq \{0\}$.

Obviously, $V \in I\!\!H$. We need to prove that the relation

$$V' \subseteq \cap\{P : P \in I\!\!H\}$$

is true. Indeed, in the opposite case, there exist $z \in V'$ and $P \in I\!\!H$ such that $z \notin P$. Then, $e(P, z) = e(f_P, z) = f_P(z) = 1$ which contradicts the fact that $e(I\!\!H \times V') \subseteq \{0\}$. Thus, the set $\cap\{P : P \in I\!\!H\}$ is a neighborhood of y in Y.

(4) $\Longrightarrow$ (5) Let V be an open subset of Y. It suffices to prove that for every $y \in V$ there exists an open ν-bounded neighborhood U of y such that $U \subseteq V$. By assumption there exists a set $I\!\!H \in \tau^s_\nu$ such that $V \in I\!\!H$ and $\cap\{P : P \in I\!\!H\} \equiv Q$ is a neighborhood of y in Y. We prove that the set Q is ν-bounded. Let $\{U_i : i \in I\}$ be an open cover of Y. Since $I\!\!H \in \tau^s_\nu$, by the definition of τ^s_ν there exists a subset J of I of cardinality less than of ν such that $\cup\{U_i : i \in J\} \in I\!\!H$ and therefore $Q \subseteq \cup\{U_i : i \in J\}$, which means that Q is ν-bounded. The required open neighborhood of y is an open subset U of Y such that $y \in U \subseteq Q$.

(5) $\Longrightarrow$ (1) Let $y \in Y$ and V be an open neighborhood of y. Since $\mathcal{O}(Y)$ is weakly ν-continuous we have

$$V = \cup\{U \in \mathcal{O}(Y) : U \subseteq V \text{ and } U <<_\nu Y\}$$

and therefore there exists an open ν-bounded subset U of Y such that

$$y \in U \subseteq V.$$

□

Proposition 3.3. *If Y is ν-locally bounded, then the usual compositions operations (see* [2]*)*

i) $T : C_{t_{co}}(X,Y) \times C_{t^s_\nu}(Y,Z) \to C_{t_{co}}(X,Z)$ *and*

ii) $T : C_{t_\omega}(X,Y) \times C_{t^s_\nu}(Y,Z) \to C_{t_\omega}(X,Z)$,

where t_{co} and t_ω is the compact open and the Isbell topology, respectively, are continuous for arbitrary spaces X and Z.

Proof. We prove only the statement ii). The proof of the case i) is similar. Let $(f,g) \in C_{t_\omega}(X,Y) \times C_{t^s_\nu}(Y,Z)$, $I\!H$ a Scott open subset of X, and $U \in \mathcal{O}(Z)$ such that $T(f,g) = g \circ f \in (I\!H, U)$. It suffices to prove that there exist open neighborhoods $I\!H_1$ and $I\!H_2$ of f and g in $C_{t_\omega}(X,Y)$ and $C_{t^s_\nu}(Y,Z)$, respectively, such that

$$T(I\!H_1 \times I\!H_2) \subseteq (I\!H, U).$$

We consider the open set $g^{-1}(U)$ of Y. By locally ν-boundedness of Y, for each point $y \in g^{-1}(U) \in \mathcal{O}(Y)$, there is an open set V_y of Y such that $y \in V_y \subseteq g^{-1}(U)$ and $V_y <<_\nu Y$. Therefore

$$g^{-1}(U) = \cup\{V_y : y \in g^{-1}(U)\}$$

and

$$f^{-1}(g^{-1}(U)) = f^{-1}(\cup\{V_y : y \in g^{-1}(U)\})$$

or

$$(g \circ f)^{-1}(U) = \cup\{f^{-1}(V_y) : y \in g^{-1}(U)\}.$$

Since $g \circ f \in (I\!H, U)$ we have $(g \circ f)^{-1}(U) \in I\!H$ or

$$\cup\{f^{-1}(V_y) : y \in g^{-1}(U)\} \in I\!H.$$

Thus there exists a finite subset J of $g^{-1}(U)$ such that $\cup\{f^{-1}(V_y) : y \in J\} \in I\!H$. Let $V = \cup\{V_y : y \in J\}$. Then $f^{-1}(V) \in I\!H$ and V is a ν-bounded open set of Y.

The set

$$I\!H(V) = \{W \in \mathcal{O}(Y) : V \subseteq W\}$$

is strong ν-Scott open (see the proof of Proposition 3.1). Since

$$V = \cup\{V_y : y \in J \subseteq g^{-1}(U)\}$$

and

$$g^{-1}(U) = \cup\{V_y : y \in g^{-1}(U)\}$$

we have that $V \subseteq g^{-1}(U)$ and therefore $g^{-1}(U) \in I\!H(V)$.

Setting $I\!H_1 = (I\!H, V)$ and $I\!H_2 = (I\!H(V), U)$ we have that the set

$$I\!H_1 \times I\!H_2 = (I\!H, V) \times (I\!H(V), U)$$

is an open neighborhood of (f,g) in $C_{t_\omega}(X,Y) \times C_{t^s_\nu}(Y,Z)$.

Finally, we prove that

$$T((I\!H, V) \times (I\!H(V), U)) \subseteq (I\!H, U).$$

Let $(p,q) \in (I\!H, V) \times (I\!H(V), U)$. Then, $p^{-1}(V) \in I\!H$ and $q^{-1}(U) \in I\!H(V)$. Therefore $V \subseteq q^{-1}(U)$. Thus, $p^{-1}(V) \subseteq p^{-1}(q^{-1}(U)) = (q \circ p)^{-1}(U)$. Since $p^{-1}(V) \in I\!H$, $(q \circ p)^{-1}(U) \in I\!H$, and therefore $q \circ p \in (I\!H, U)$. □

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Received October 2004

Accepted January 2005

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