**APPLIED GENERAL TOPOLOGY**

# The canonical partial metric and the uniform convexity on normed spaces

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ABSTRACT. In this paper we introduce the notion of canonical partial metric associated to a norm to study geometric properties of normed spaces. In particular, we characterize strict convexity and uniform convexity of normed spaces in terms of the canonical partial metric defined by its norm.
We prove that these geometric properties can be considered, in this sense, as topological properties that appear when we compare the natural metric topology of the space with the non translation invariant topology induced by the canonical partial metric in the normed space.

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## 1. INTRODUCTION

S. G. Matthews introduced in [6] the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks, and obtained, among other results, a nice relationship between partial metric spaces and the so-called weightable quasi-metric spaces. Partial metrics were also used in the context of the complexity analysis of algorithms and programs (see [6, 8, 7]). The domain of words which appears in a natural way by modelling the streams of information in G. Kahn's model of parallel computation provides a well-known example of partial metric space (see [4] and [6]). Other motivations for exploring partial metrics can be found in [6].

In this paper we present an application of the theory of partial metrics to the framework of the normed space theory. Other applications in this direction can be found in [9]. Let $(X, \|.\|)$ be a normed space. Our aim is to show that it

is possible to define a partial metric $p_{\|.\|}$ in $X$ in a canonical way that implicitly contains the information about the *convexity properties* of the space. However, the topology that defines this partial metric is not standard at all, since it fails to satisfy some of the basic properties that topologies on topological linear spaces use to have. In particular, we obtain a topology $\tau_p$ with the following properties.

(1) $\tau_p$ is not $T_1$.
(2) $\tau_p$ is not translation invariant.
(3) $\tau_p$ coincides with the norm topology when restricted to the unit sphere of $X$.

We present these results in three sections. Section 2 is devoted to define and prove the basic properties of the canonical partial metric. In Section 3 we characterize when $(X, \|.\|)$ is strictly convex in terms of the natural base of neighborhoods of $(X, \tau_p)$, providing also several examples. Finally, in Section 4 we present the results concerning uniform convexity of $(X, \|.\|)$ and the characterization of this property in terms of a particular class of neighborhoods of the elements of the unit sphere of $(X, \|.\|)$.

In what follows we introduce the basic definitions and results on partial metrics. In this direction, our main references are [6, 7, 8]. Each partial metric defines a quasi-metric that generates the same topology. Therefore, we start by recalling several definitions on quasi-metrics. Our basic references for quasi-metric spaces are [3] and [4], and for general topological questions [2].

Let us recall that a *quasi-pseudo-metric* on a set $X$ is a nonnegative real valued function $d$ on $X \times X$ such that for all $x, y, z \in X$,

(i) $d(x, x) = 0$;
(ii) $d(x, y) \leq d(x, z) + d(z, y)$.

By a *quasi-metric* on a set $X$ we mean a quasi-pseudo-metric $d$ on $X$ that satisfies also the following condition.

(iii) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$.

A *quasi-metric space* is a pair $(X, d)$ such that $X$ is a (nonempty) set and $d$ is a quasi-metric on $X$.

Each quasi-metric $d$ on $X$ generates a $T_0$-topology $\mathcal{T}(d)$ on $X$ which has as a base the family of open $d$-balls $\{B_d(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

We will write $\mathbb{R}^+$ for the set of nonnegative real numbers. A *partial pseudo-metric* on a (nonempty) set $X$ is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$,

(i) $x = y \Rightarrow p(x, x) = p(x, y) = p(y, y)$;
(ii) $p(x, x) \leq p(x, y)$;
(iii) $p(x, y) = p(y, x)$;
(iv) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

By a *partial metric* on a set $X$ we mean a partial pseudo-metric $p$ on $X$ that satisfies the following condition,

(i)' $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$.

A *partial (pseudo-)metric space* is a pair $(X, p)$ such that $X$ is a (nonempty) set and $p$ is a partial (pseudo-)metric on $X$.

Each partial metric $p$ on $X$ defines a quasi-metric $d_p$ on $X$ by means of the formula,

$$d_p(x, y) := p(x, y) - p(x, x), \qquad x, y \in X,$$

and the topology given by $p$ is the one generated by $d_p$. Consequently, each partial metric generates a $T_0$-topology $\mathcal{T}(p)$ on $X$ and is the one given by the base $\mathcal{B} = \{V_{\varepsilon, p}(x) : x \in X, \varepsilon > 0\}$, where

$$V_{\varepsilon, p}(x) = \{y \in X : p(x, y) < \varepsilon + p(x, x)\} \quad \text{for all } x \in X \quad \text{and } \varepsilon > 0.$$

The following lemma is a direct consequence of Lemma 2.2 in [6] and gives information about the sets $V_{\varepsilon, p}(x)$ that will be useful in this paper.

**Lemma 1.1.** *Let $(X, p)$ be a partial metric space and $x \in X$. Then $\mathcal{B} = \{V_{\varepsilon, p}(x) : \varepsilon > 0\}$ is a base of open neighborhoods of $x$ for the topology $\mathcal{T}(p)$.*

Let us finish this section by introducing several definitions of Functional Analysis. The basic references for these definitions are [10, 11, 5]. We use standard Banach space notation. If $(X, \|.\|)$ is a normed space, we denote by $S_X$ the corresponding unit sphere $\{x \in X : \|x\| = 1\}$. If $x \in X$ and $\varepsilon > 0$, we denote by $B_{\varepsilon, \|.\|}(x)$ the set

$$B_{\varepsilon, \|.\|}(x) := \{y \in X : \|x - y\| < \varepsilon\}.$$

If $x, y \in X$, we denote by $[x, y]$ the segment $\{z = \theta x + (1 - \theta) y : 0 \leq \theta \leq 1\}$. If $x \in X$, $\langle x \rangle$ denotes the linear span of $x$ into the linear space $X$. Although we present the results of the paper for normed spaces, completeness is not necessary. Therefore, our main results are valid for normed spaces. The same holds for the following definitions.

**Definition 1.2.** *Let $(X, \|.\|)$ be a normed space. We say that the norm $\|.\|$ (equivalently, the normed space $(X, \|.\|)$) is strictly convex if for every pair of norm one elements $x, y \in X$, $\|\frac{x+y}{2}\| = 1$ implies $x = y$.*

It can be proved that this definition is equivalent to the following one. The norm $\|.\|$ (equivalently, the normed space $(X, \|.\|)$) is strictly convex if for every $x, y \in X$, if $x \neq 0$ and $\|x + y\| = \|x\| + \|y\|$, then $y \in \langle x \rangle$. The proof of this equivalence can be found in Proposition 1, Part 3, Ch.I of [1]. Throughout the paper we will use both definitions.

**Definition 1.3.** *Let $(X, \|.\|)$ be a normed space. The norm $\|.\|$ (equivalently, the normed space $(X, \|.\|)$) is uniformly convex if for every $\varepsilon > 0$ there is a $\delta > 0$ -only depending on $\varepsilon$- such that if $\|x\| = 1 = \|y\|$ and $\|x - y\| \geq \varepsilon$, then $\|\frac{x+y}{2}\| \leq 1 - \delta$, for every $x, y \in X$.*

We will use the following notation. Let $n \in N$ and consider the corresponding n-dimensional linear space $\mathbb{R}^n$. Let $x = (x_1, ..., x_n) \in \mathbb{R}^n$. We will write $\|x\|_2$ for

$$\|x\|_2 := \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}},$$

and $\|x\|_1$ for

$$\|x\|_1 := \sum_{i=1}^{n} |x_i|.$$

## 2. Partial metrics in normed spaces

In this section we construct the canonical partial metric $p_{\|.\|}$ associated to the norm of a normed space $(X, \|.\|)$, and we prove the main results that will be used in the following two sections.

The relations between the elements of the base of neighborhoods given by Lemma 1.1 of the topological space space $(X, \tau_{p_{\|.\|}})$ and the translation invariant topology associated to the norm $\|.\|$ gives the characterization of the convexity properties of the normed space $(X, \|.\|)$.

**Definition 2.1.** *Let $(X, \|.\|)$ be a normed space. We define the nonnegative function $p_{\|.\|} : X \to \mathbb{R}$ by the formula*

$$p_{\|.\|}(x, y) := \|x - y\| + \|x\| + \|y\|, \qquad x, y \in X.$$

A related construction has been done at [6]

**Definition 2.2.** *Let $\tau$ be a topology on a linear space $X$. We say that a norm $\|.\|$ is 0-compatible with the topology $\tau$ if the balls*

$$B_{\varepsilon, \|.\|}(0) = \{x \in X : \|x\| < \varepsilon\}, \qquad \varepsilon > 0,$$

*define a base of neighborhoods of $0$ for the topology $\tau$.*

**Proposition 2.3.** *If $(X, \|.\|)$ is a normed space, the function $p_{\|.\|}$ is a partial metric that satisfies*
   1) $p_{\|.\|}(x + y, 0) \le p_{\|.\|}(x, 0) + p_{\|.\|}(y, 0)$,
   2) $p_{\|.\|}(\lambda x, \lambda y) = |\lambda| p_{\|.\|}(x, y)$ *for every $x, y \in X$ and $\lambda \in \mathbb{R}$,*
   3) $p_{\|.\|}(x, y) = 0$ *if and only if $x = y = 0$,*
   4) *The norm $\|.\|$ is 0-compatible with the topology $\tau_{p_{\|.\|}}$.*

*Proof.* The following calculations show that $p_{\|.\|}$ is a partial metric. To prove the condition $(i)'$ in the definition of partial metric (Section 1), let $x, y \in X$ such that $p_{\|.\|}(x, x) = p_{\|.\|}(x, y) = p_{\|.\|}(y, y)$, hence

$$2\|x\| = \|x - y\| + \|x\| + \|y\| = 2\|y\|$$

so that $\|x - y\| + \|x\| - \|y\| = 0$ and $\|x - y\| - \|x\| + \|y\| = 0$. This clearly implies $\|x - y\| = 0$, and then $x = y$. Now suppose that $x = y$; the equalities above gives directly $p_{\|.\|}(x, x) = p_{\|.\|}(x, y) = p_{\|.\|}(y, y)$, since $\|x - y\| = 0$.

Now let us show that $p_{\|.\|}(x, x) \le p_{\|.\|}(x, y)$ for every $x, y \in X$. But this is a direct consequence of the triangular inequality for the norm $\|.\|$, since

$$p_{\|.\|}(x, x) = 2\|x\| \le \|x - y\| + \|x\| + \|y\| = p_{\|.\|}(x, y).$$

The definition of $p_{\|.\|}$ clearly gives $p_{\|.\|}(x, y) = p_{\|.\|}(y, x)$ for every $x, y \in X$.

To see the last condition of partial metric, consider $x, y, z \in X$. Then

$$p_{\|.\|}(x, y) + p_{\|.\|}(z, z) = \|x - y\| + \|x\| + \|y\| + 2\|z\|$$

$$\leq \|x-z\| + \|x\| + \|z\| + \|z-y\| + \|y\| + \|z\| = p_{\|.\|}(x,z) + p_{\|.\|}(y,z).$$

Now let us show 1). If $x, y \in X$, then

$$p_{\|.\|}(x+y,0) = \|x+y\| + \|x+y\| \leq 2\|x\| + 2\|y\| = p_{\|.\|}(x,0) + p_{\|.\|}(y,0).$$

Condition 2) is a consequence of the homogeneity of the norm.

$$p_{\|.\|}(\lambda x, \lambda y) = \|\lambda x - \lambda y\| + \|\lambda x\| + \|\lambda y\| = |\lambda|(\|x-y\| + \|x\| + \|y\|) = |\lambda| p_{\|.\|}(x,y)$$

for every $x, y \in X$ and $\lambda \in \mathbb{R}$. 3) is also given directly by the definition. If $x \in X$, obviously $p_{\|.\|}(x,y) = \|x-y\| + \|x\| + \|y\| = 0$ if and only if $x = y = 0$.

Finally, let us show that $\|.\|$ is 0-compatible with $\tau_{p_{\|.\|}}$. It is enough to write explicitly the basic neighborhoods $V_{\varepsilon, p_{\|.\|}}(x)$ for the case $x = 0$.

$$V_{\varepsilon, p_{\|.\|}}(0) = \{x \in X : p_{\|.\|}(x,0) < \varepsilon + p(0,0)\} = \{x \in X : 2\|x\| < \varepsilon\} = B_{\frac{\varepsilon}{2}, \|.\|}(0)$$

$\square$

We will call the function $p_{\|.\|}$ the *canonical partial metric* associated to $\|.\|$. Consider $\varepsilon > 0$ and $x \in X$. The basic neighborhood of $x$, $V_{\varepsilon, \|.\|}(x)$ is given in this case by the particular expression

$$V_{\varepsilon, p_{\|.\|}}(x) := \{y \in X : p_{\|.\|}(x,y) < p_{\|.\|}(x,x) + \varepsilon\} = \{y \in X : \|x-y\| + \|y\| - \|x\| < \varepsilon\}.$$

This description of the neighborhood $V_{\varepsilon, p_{\|.\|}}(x)$ will be useful in the following sections.

## 3. STRICT CONVEXITY AND THE CANONICAL PARTIAL METRIC

Let $(X, \|.\|)$ be a normed space. In this section we characterize when the norm $\|.\|$ is strictly convex in terms of the base of neighborhoods for the topology $\tau_{p_{\|.\|}}$ given by Lemma 1.1 and described at the end of Section 2 for the particular case of the canonical partial metric.

**Lemma 3.1.** *Let $(X, \|.\|)$ be a normed linear space. For every $x \in X$, $[0, x] \subset \cap_{\varepsilon > 0} V_{\varepsilon, p_{\|.\|}}(x)$.*

*Proof.* Let $y \in [0, x]$ and let $\varepsilon > 0$. Then there exists an $\alpha$ such that $0 \leq \alpha \leq 1$ and $y = \alpha x$. Thus,

$$p(x,y) = \|x-y\| + \|y\| + \|x\| = (1-\alpha)\|x\| + \alpha\|x\| + \|x\| = 2\|x\| = p(x,x) < p(x,x) + \varepsilon.$$

This proves the lemma, since implies that $y \in V_{\varepsilon, p_{\|.\|}}(x)$ for every $\varepsilon > 0$. $\square$

Note that the only point that satisfies that the intersection of all its neighborhoods is the same point is 0. As a direct consequence, we obtain that the topology generated by the canonical partial metric only satisfy the separation axiom $T_0$.

**Remark 3.2.** The topology defined by the canonical partial metric in a non trivial normed space is not $T_1$. To prove this, consider $x \in X - \{0\}$ and define $y = \frac{1}{2}x$. Suppose that there is a neighborhood of $x$, $V$ such that $y$ do not belong to $V$. Since $V$ is a neighborhood of $x$, by Lemma 1.1 there exists $\varepsilon > 0$ such that $V_{\varepsilon, p_{\|.\|}}(x) \subset V$. But $y \in V_{\varepsilon, p_{\|.\|}}$ as a consequence of Lemma 3.1, and then $y \in V$, a contradiction.

Let us discuss in what follows the situation for the converse inclusion that the one given in Lemma 3.1. The first one shows that the 2-dimensional Euclidean space satisfies also this inclusion.

**Example 3.3.** Consider the 2-dimensional Euclidean space $\mathbb{R}_2^2 := (\mathbb{R}^2, \|.\|_2)$, and let $x_0 \in \mathbb{R}_2^2$. Then

$$\bigcap_{\varepsilon > 0} V_{\varepsilon, p_{\|.\|}}(x_0) = \bigcap_{\varepsilon > 0} \{y \in \mathbb{R}^2 : \|x_0 - y\|_2 + \|y\|_2 < \varepsilon + \|x_0\|_2\} =$$
$$= \{y : \|x_0 - y\|_2 + \|y\|_2 = \|x_0\|_2\},$$

since the inequality $\|x\| \leq \|x - y\| + \|y\|$ always holds for every $x, y \in X$ and every norm. In terms of the Euclidean distance $d_2$ in $\mathbb{R}_2^2$, the above condition can be written as

$$d_2(y, x_0) + d_2(0, y) = d_2(0, x_0),$$

which only holds when $y = \alpha x_0$ for some $\alpha \in [0, 1]$. Therefore, in this case $\bigcap_{\varepsilon > 0} V_{\varepsilon, p_{\|.\|}}(x_0) = [0, x_0]$.

**Example 3.4.** Consider the 2-dimensional space $\mathbb{R}_1^2 := (\mathbb{R}^2, \|.\|_1)$, and the element $x_0 := (1/2, 1/2)$. Then

$$V_{\varepsilon, p_{\|.\|}}(x_0) = \{(y_1, y_2) : \|(1/2 - y_1, 1/2 - y_2)\|_1 + \|(y_1, y_2)\|_1 < \varepsilon + \|(1/2, 1/2)\|_1\},$$

and then $\bigcap_{\varepsilon > 0} V_{\varepsilon, p_{\|.\|}}(x_0) = \{(y_1, y_2) : \|(1/2 - y_1, 1/2 - y_2)\|_1 + \|(y_1, y_2)\|_1 = \|(1/2, 1/2)\|_1\}$. Consider now any element $(a, b) \in [0, 1/2] \times [0, 1/2]$. Then the condition that appears in $\bigcap_{\varepsilon > 0} V_{\varepsilon, p_{\|.\|}}(x_0)$ can be written as

$$|\frac{1}{2} - a| + |\frac{1}{2} - b| + |a| + |b| = \frac{1}{2} - a + \frac{1}{2} - b + a + b = 1$$

that obviously holds for every $(a, b) \in [0, 1/2] \times [0, 1/2]$. Then $[0, 1/2] \times [0, 1/2] \subset \bigcap_{\varepsilon > 0} V_{\varepsilon, p_{\|.\|}}(x_0)$, which implies that $\bigcap_{\varepsilon > 0} V_{\varepsilon, p_{\|.\|}}(x_0)$ is not contained in $[0, x_0]$.

**Lemma 3.5.** *A normed space $(X, \|.\|)$ is strictly convex if and only if for every $x, y \in S_X$, if $\|2x - y\| = 1$ then $x = y$.*

*Proof.* Suppose that $(X, \|.\|)$ is strictly convex, and consider two elements $x, y \in S_X$ such that $\|2x - y\| = 1$. Then

$$\|2x - y\| + \|x + y\| \geq \|3x\| = 3,$$

and so $\|x + y\| \geq 2$. Moreover,

$$2\|x + y\| = \|2x + 2y\| \leq \|2x - y\| + \|3y\| = 4,$$

and then $\|x + y\| \leq 2$. Since $(X, \|.\|)$ is strictly convex, we obtain that $x = y$. Conversely, suppose that the second property in the statement of the lemma holds, and consider two elements $x, y \in S_X$ satisfying $\|x + y\| = 2$. Then we define $z_1 = \frac{x+y}{2}$ and $z_2 = y$, that obviously satisfy $\|z_1\| = \|z_2\| = 1$ and $\|2z_1 - z_2\| = 1$. Thus the property gives $z_1 = z_2$, and then $\frac{x+y}{2} = y$, which clearly implies $x = y$. $\qquad \square$

**Theorem 3.6.** *A normed linear space $(X, \|.\|)$ is strictly convex if and only if $\cap_{\varepsilon>0} V_{\varepsilon, p_{\|.\|}}(x) = [0, x]$.*

*Proof.* First let us prove that $\cap_{\varepsilon>0} V_{\varepsilon, p_{\|.\|}}(x) = [0, x]$ implies that the normed linear space is strictly convex. Suppose that $\|x\| = 1$. Then

$$\cap_{\varepsilon>0} V_{\varepsilon, p_{\|.\|}}(2x) \cap S_X = [0, 2x] \cap S_X = \{x\}.$$

Since

$$\cap_{\varepsilon>0} V_{\varepsilon, p_{\|.\|}}(2x) \cap S_X = \{y : \|y\| = 1, \|2x - y\| = 1\},$$

we obtain the result as a consequence of Lemma 3.5. Conversely, we use the characterization given by Proposition 1 (p. 175 of [1]) of the strict convexity of $(X, \|.\|)$ that we have referred in Section 1. Consider $x \in S_X$ and suppose that $y \in \cap_{\varepsilon>0} V_{\varepsilon, p_{\|.\|}}(x)$ but $y$ is not in $[0, x]$. Then $\|x - y\| + \|y\| = \|x\| = 1$ and in particular $\|y\| \leq 1$ and $\|x - y\| \leq 1$. These two inequalities and the fact that $y$ is not an element of $[0, x]$ clearly imply that $y$ is not a linear combination of $x$. If we write $z_1 = x - y$ and $z_2 = y$, we have $\|z_1\| + \|z_2\| = 1 = \|z_1 + z_2\|$, and then the proposition quoted above gives that there is a $\lambda \neq 0$ such that $z_1 = \lambda z_2$, and then $x - y = \lambda y$. Thus $x = (1 + \lambda)y$, a contradiction. $\square$

## 4. Uniform convexity and the canonical partial metric

Strict convexity of the norm can be understood, in a certain sense, as a limit case of the uniform convexity. After the result obtained in the theorem above, we will prove in this section that it is also possible to give a characterization of the uniform convexity of a normed space $(X, \|.\|)$ in terms of a particular class of neighborhoods of the points of $X$ for the topology $\tau_{p_{\|.\|}}$.

Let us fix a normed space $(X, \|.\|)$. Let $x \in S_X$ and $\delta > 0$. We define the set

$$W_{\delta, \|.\|}(x) = \{y \in X : \|x\| + \|y\| \leq \|x + y\| + \delta\}.$$

**Lemma 4.1.** *Let $(X, \|.\|)$ be a linear normed space. For every $x \in X$ and $\varepsilon > 0$, $V_{\varepsilon, p_{\|.\|}}(x) \subset W_{\varepsilon, \|.\|}(x)$. In particular, $W_{\varepsilon, \|.\|}(x)$ is a neighborhood of $x$ for the topology $\tau_{p_{\|.\|}}$.*

*Proof.* Fix $x \in X$ and consider an element $y \in V_{\varepsilon, p_{\|.\|}}(x)$. Then

$$2\|x\| + \|y\| \leq \|x + y\| + \|x - y\| + \|y\| \leq \|x + y\| + \|x\| + \varepsilon,$$

Thus $\|x\| + \|y\| < \|x + y\| + \varepsilon$, and so $y \in W_{\varepsilon, \|.\|}(x)$. $\square$

**Lemma 4.2.** *For every $x \in S_X$ and every $\varepsilon > 0$, $V_{\varepsilon, p_{\|.\|}}(x) \cap S_X = B_{\varepsilon, \|.\|}(x) \cap S_X$.*

*Proof.* If $\|x\| = 1$ and $y \in V_{\varepsilon, p_{\|.\|}}(x) \cap S_X$,

$$\|x - y\| + \|y\| = \|x - y\| + 1 \leq \|x\| + \varepsilon = 1 + \varepsilon.$$

Then $\|x - y\| < \varepsilon$, and thus $y \in B_{\varepsilon, \|.\|}(x)$. The same argument shows the opposite inclusion. $\square$

**Definition 4.3.** *Let $(X, \tau)$ be a topological space. For every element $x \in X$, consider two sets of neighborhoods of $x$ both of them indexed by $\varepsilon \in \mathbb{R}^+$;*

$$\mathtt{V}_x = \{V_\varepsilon(x) : \varepsilon > 0\} \qquad and \qquad \mathtt{W}_x = \{W_\varepsilon(x) : \varepsilon > 0\}.$$

*Consider now the families of neighborhoods*

$$\mathtt{V} = \{\mathtt{V}_x : x \in X\} \qquad and \qquad \mathtt{W} = \{\mathtt{W}_x : x \in X\}.$$

*We say that $\mathtt{V}$ and $\mathtt{W}$ are uniformly equivalent if they verify the following relations:*

(i) *for every $\varepsilon > 0$ there is a $\delta > 0$ (only depending on $\varepsilon$) such that $W_\delta(x) \subset V_\varepsilon(x)$ for all $x \in X$, and*

(ii) *for every $\varepsilon' > 0$ there is a $\delta' > 0$ (only depending on $\varepsilon'$) such that $V_{\varepsilon'}(x) \subset W_{\delta'}(x)$ for all $x \in X$.*

For the following theorem, we consider the particular families of neighborhoods $\mathtt{V} = \{\mathtt{V}_x : x \in S_X\}$, $\mathtt{W} = \{\mathtt{W}_x : x \in S_X\}$ and $\mathtt{B} = \{\mathtt{B}_x : x \in S_X\}$, given by

$$\mathtt{V}_x = \{V_{\varepsilon, p_{\|.\|}}(x) \cap S_X : \varepsilon > 0\},$$
$$\mathtt{W}_x = \{W_{\varepsilon, \|.\|}(x) \cap S_X : \varepsilon > 0\}$$

and

$$\mathtt{B}_x = \{B_{\varepsilon, \|.\|}(x) \cap S_X\}.$$

**Theorem 4.4.** *Let $(X, \|.\|)$ be a normed space. The following are equivalent.*

(1) *$(X, \|.\|)$ is uniformly convex.*

(2) *For every $\varepsilon > 0$ there is a $\delta > 0$ (only depending on $\varepsilon$) such that $W_{\delta, \|.\|}(x) \cap S_X \subset V_{\varepsilon, p_{\|.\|}}(x)$, for all $x \in S_X$.*

(3) *$\mathtt{W}$ and $\mathtt{V}$ are uniformly equivalent families of neighborhoods.*

(4) *$\mathtt{W}$ and $\mathtt{B}$ are uniformly equivalent families of neighborhoods.*

*Moreover, if any of the statements (1) to (4) holds, then the family $\mathtt{W}_x$ defines a local base in the topological space $(S_X, \tau_{p_{\|.\|}}|_{S_X})$ (equivalently, in the topological space $(S_X, \tau\|.\||_{S_X})$) for every $x \in S_X$.*

*Proof.* Let us prove first that (1) implies (2). Suppose now that $(X, \|.\|)$ is uniformly convex. It follows that for all $x, y \in S_X$ and $\varepsilon$ there is a $\delta > 0$ such that if $\|x + y\| \geq 2 - 2\delta$ it is true that $\|x - y\| < \varepsilon$.

Let us show that $W_{\delta, \|.\|}(x) \cap S_X \subset V_{\varepsilon, p_{\|.\|}}(x)$. To check this, consider $y \in W_{\delta, \|.\|}(x) \cap S_X$. Note that

$$W_{\delta, \|.\|}(x) \cap S_X = \{y \in S_X : \|x\| + \|y\| \leq \|x + y\| + \delta\} = \{y \in S_X : 2 - \delta \leq \|x + y\|\}.$$

Hence

$$2 - 2\delta \leq 2 - \delta \leq \|x + y\|,$$

and since $X$ is uniformly convex this implies that $\|x - y\| < \varepsilon$. Consequently, it follows that

$$\|x - y\| + \|y\| \leq \|x\| + \varepsilon.$$

This completes the proof of (1) implies (2).

To deduce (1) from (2), suppose that for every $\varepsilon > 0$ there is a $\delta > 0$ such that $W_{\delta,\|.\|}(x) \cap S_X \subset V_{\varepsilon,p}(x)$ for every $x \in S_X$.

Consider $\varepsilon > 0$ and $\delta' = \frac{\delta}{2}$. If $y \in W_{\delta,\|.\|}(x) \cap S_X = \{y \in S_X : 2 - \delta \leq \|x + y\|\}$, we have that $y \in V_{\varepsilon_{p_{\|.\|}}}(x)$, and then $\|x - y\| < \varepsilon$. This condition is equivalent to the following one. For every $x, y \in S_X$, if $\|x - y\| \geq \varepsilon$, then $2 - 2\delta' \geq \|x + y\|$. Hence, for such $x, y \in S_X$, we have

$$\|x + y\| < 2 - \delta = 2 - 2\delta',$$

which can be written as $\frac{\|x+y\|}{2} < 1 - \delta'$. This shows that $X$ is uniformly convex.

Let us prove now the equivalence between (2) and (3). As a consequence of Lemma 4.1, we only need to prove that for each $\varepsilon > 0$ there is a $\delta > 0$ (only depending on $\varepsilon$) such that $W_{\delta,\|.\|}(x) \cap S_X \subset V_{\varepsilon,p_{\|.\|}}(x)$ for all $x \in S_X$. But this is what the statement (2) of the theorem assures, so $\mathtt{V}$ and $\mathtt{W}$ are uniformly equivalent. The converse is obvious.

It follows easily that (3) if and only if (4), as a consequence of Lemma 4.2.

Finally, we have to check that $\{W_{\delta,\|.\|}(x) \cap S_X : \delta > 0\}$ defines for every $x \in S_X$ a local base for the topological space $(S_X, \tau_{p_{\|.\|}}|_{S_X})$. It is clear, using Lemma 4.2, that this is equivalent to the fact that $\{W_{\delta,\|.\|}(x) \cap S_X : \delta > 0\}$ a local base for the topological space $(S_X, \|.\||_{S_X})$ for each $x \in S_X$.

Obviously $\{W_{\delta,\|.\|}(x) \cap S_X : \delta > 0\} \neq \phi$ since $x \in \{W_{\delta,\|.\|}(x) \cap S_X : \delta > 0\}$. It is also clear that given $W_{\delta_1,\|.\|}(x) \cap S_X$ and $W_{\delta_2,\|.\|}(x) \cap S_X$ we obtain $x \in W_{\delta_3,\|.\|}(x) \cap S_X \subset (W_{\delta_1,\|.\|}(x) \cap S_X) \cap (W_{\delta_2,\|.\|}(x) \cap S_X)$, where $\delta_3 := \min\{\delta_1, \delta_2\}$.

Consider $W_{\delta,\|.\|}(x) \cap S_X$. Since as $\mathtt{V}$ and $\mathtt{W}$ are uniformly equivalent families of neighborhoods, there is $\varepsilon > 0$ such that $V_{\varepsilon,p_{\|.\|}}(x) \cap S_X \subset W_{\delta,\|.\|}(x) \cap S_X$. Since $V_{\varepsilon,p_{\|.\|}}(x) \cap S_X$ is a local base, then there is a $V_{\varepsilon_1,p_{\|.\|}}(x) \cap S_X \subset V_{\varepsilon,p_{\|.\|}}(x) \cap S_X$ such that for all $y \in V_{\varepsilon_1,p_{\|.\|}}(x) \cap S_X$ there is an $\varepsilon_2 > 0$ such that $V_{\varepsilon_2,p_{\|.\|}}(y) \cap S_X \subset V_{\varepsilon,p_{\|.\|}}(x) \cap S_X$.

Given $V_{\varepsilon_1}(x)$ we can find a $\delta_1 > 0$ such that it verifies that $W_{\delta_1,\|.\|}(x) \cap S_X \subset V_{\varepsilon_1,p_{\|.\|}}(x) \cap S_X \subset W_{\delta,\|.\|}(x) \cap S_X$. Now we have to show that for all $y \in W_{\delta_1,\|.\|}(x) \cap S_X$ there is a $\delta_2 > 0$ such that $W_{\delta_2,\|.\|}(y) \cap S_X \subset W_{\delta,\|.\|}(x) \cap S_X$.

We know that for all $y \in W_{\delta_1,\|.\|}(x) \cap S_X \subset V_{\varepsilon_1,p_{\|.\|}}(x) \cap S_X$ there is $V_{\varepsilon_2,p_{\|.\|}}(y) \subset V_{\varepsilon,p_{\|.\|}}(x)$. Then for each $y \in W_{\delta_1,\|.\|}(x)$ there exists $V_{\varepsilon_2,p_{\|.\|}}(y) \subset V_{\varepsilon}(x)$, and given $V_{\varepsilon_2,p_{\|.\|}}(y)$ there is $\delta_2$ such that $W_{\delta_2,\|.\|}(y) \subset V_{\varepsilon_2,p_{\|.\|}}(y) \subset W_{\delta,\|.\|}(x)$. Therefore, we have proved that the family $\{W_{\delta,\|.\|}(x) \cap S_X : \delta > 0\}$ defines for every $x \in S_X$ a local base for the topological space $(S_X, \tau_{p_{\|.\|}}|_{S_X})$. This finishes the proof. $\square$

The ideas exposed in this paper allow to consider the convexity properties of normed spaces as sequential properties, since they can be characterized using the topology defined by the canonical partial metric. This provides a new framework for the study of certain geometric properties of normed spaces and in the particular case of the Banach spaces.

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