Applied General Topology
© Universidad Politécnica de Valencia
Volume 6, No. 2, 2005
pp. 217-228

On semi-Lipschitz functions with values in a quasi-normed linear space

José Manuel Sánchez-Álvarez*

Abstract. In a recent paper, S. Romaguera and M. Sanchis discussed several properties of semi-Lipschitz real valued functions. In this paper we analyze the structure of the space of semi-Lipschitz functions that are valued in a quasi-normed linear space. Our approach is motivated, in part, by the fact that this structure can be applied to study some processes in the theory of complexity spaces.

2000 AMS Classification: 54E25, 54E50, 54C35.

Keywords: Semi-Lipschitz function, normed cone, bicomplete space, quasi-distance, right K-complete.

1. Introduction and preliminaries

Motivated, in part, by some problems from computer science and their applications (see for instance [5, 6, 13, 15, 16, 17]), the theories of completeness have received a certain attention in the recent years (see, among other contributions, [1, 2, 3, 9, 10, 18, 19]). These advances have also permitted the development of generalizations, to the nonsymmetric case, of classical mathematical theories: hyperspaces, function spaces, etc.

The complexity quasi-metric space was introduced in [16] to study complexity analysis of programs. Recently, it was introduced in [14] the dual complexity space. Several quasi-metric properties of the complexity space were obtained via the analysis of the dual complexity space. In [15] Romaguera and Schellekens show that the structure of a quasi-normed semilinear space provides a suitable setting to carry out an analysis of the dual complexity space.

This paper is a contribution to the study of semi-Lipschitz functions from a nonsymmetric point of view. We show that this set defined on a quasi-metric space, that are valued in a quasi-normed linear space and that vanish

*The author is supported by a grant FPI from the Spanish Ministry of Education and Science.

at a fixed point can be endowed with the structure of a quasi-normed linear space. We show that this space is bicomplete and we also study other types of completeness.

Throughout this paper the letters $\mathbb{R}^+$ and $\mathbb{N}$ will denote the set of nonnegative real numbers and the set of positive integers numbers, respectively. Our basic reference for quasi-metric spaces is [4].

A quasi-metric on a (nonempty) set X is a function $d : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$: (i) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$, and (ii) $d(x, y) \leq d(x, z) + d(z, y)$.

If d can take the value ∞ then it is called a quasi-distance on X.

Given a quasi-metric d on X, the function d^{-1} defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$, is also a quasi-metric on X, called the conjugate of d, and the function d^s defined on $X \times X$ by $d^s(x, y) = d(x, y) \vee d^{-1}(x, y)$, is a metric on X. If d is a quasi-distance, then d^{-1} and d^s are a quasi-distance and a distance on X, respectively.

A quasi-metric space is a pair (X, d) such that X is a (nonempty) set and d is a quasi-metric on X. Each quasi-distance d on X induces a topology $T(d)$ on X which has as a base the family of balls $\{B_d(x, r) : x \in X, r > 0\}$ where $B_d(x, r) = \{y \in X : d(x, y) < r\}$. We remark that the topology $T(d)$ is T_0. Moreover, if condition (i) above is replaced by (i') $d(x, y) = 0 \Leftrightarrow x = y$, then $T(d)$ is a T_1 topology. A quasi-metric d is said to be bicomplete if d^s is a complete metric.

For more information about quasi-metric spaces see [4] and [8].

Following [7], a cone is a triple $(X, +, \cdot)$ such that $(X, +)$ is an abelian semigroup with neutral element 0 and $\cdot$ is a function from $\mathbb{R}^+ \times X$ into X which satisfies for all $a, b \in \mathbb{R}^+$ and $x, y \in X$:
(i) $a \cdot (b \cdot x) = (ab) \cdot x$, (ii) $(a+b) \cdot x = (a \cdot x) + (b \cdot x)$, (iii) $a \cdot (x+y) = (a \cdot x) + (a \cdot y)$ and (iv) $1 \cdot x = x$.

A quasi-norm on a cone $(X, +, \cdot)$ is a function $\| \cdot \| : X \to \mathbb{R}^+$ such that for all $x, y \in X$ and $r \in \mathbb{R}^+$: (i) $x = \mathbf{0}$ if and only if there is $-x \in X$ and $\|x\| = 0 = \| - x\|$, (ii) $\|r \cdot x\| = r\|x\|$, and (iii) $\|x + y\| \leq \|x\| + \|y\|$.

If the quasi-norm q satisfies: (i') $\|x\| = 0$ if and only if $x = 0$, then q is called a norm on the cone $(X, +, \cdot)$.

A (quasi-)normed cone is a pair $(X, \| \cdot \|)$ such that X is a cone and $\| \cdot \|$ is a (quasi-)norm on X.

If $(X, +, \cdot)$ is a linear space and $\| \cdot \|$ is a quasi-norm on X, then the pair $(X, \| \cdot \|)$ is called a quasi-normed linear space. Note that in this case, the function $\| \cdot \|^{-1} : X \to \mathbb{R}^+$ given by $\|x\|^{-1} = \| - x\|$ is also a quasi-norm on X and the function $\| \cdot \|^s : X \to \mathbb{R}^+$ given by $\|x\|^s = \|x\| \vee \|x\|^{-1}$ is a norm on X.

2. On the structure of the set of semi-Lipschitz functions

Let (X, d), (Y, q) be a quasi-metric space and a quasi-normed space respectively. A function $f : X \longrightarrow Y$ is said to be a semi-Lipschitz function if there exists $k \geq 0$ such that $q(f(x) - f(y)) \leq kd(x, y)$ for all $x, y \in X$. The number k is called a semi-Lipschitz constant for f.

A function f on a quasi-metric space (X, d) with values in a quasi-normed linear space (Y, q) is called $\leq_{(d,q)}$-increasing if $q(f(x) - f(y)) = 0$ whenever $d(x, y) = 0$. By $Y^X_{(d,q)}$ we shall denote the set of all $\leq_{(d,q)}$-increasing functions from (X, d) to (Y, q).

It is clear that if (X, d) is a T_1 quasi-metric space, then every function from X to Y is $\leq_{(d,q)}$ -increasing.

If for each $f, g \in Y^X_{(d,q)}$ and $a \in \mathbb{R}^+$ we define $f + g$ and af in the usual way, then it is a routine to show that $(Y^X_{(d,q)}, +, \cdot)$ is a cone.

Example 2.1. Let $X = \mathbb{Z}_3$. Let d be the quasi-metric on X given by

$$d(x,y) = \begin{cases} 1 & \text{if } x > y, \\ 0 & \text{if } x \leq y. \end{cases}$$

Let $Y = \mathbb{R}$, $q(x) = x \vee 0$ and take f such that $f(0) = 0, f(1) = 1$ and $f(-1) = -2$. It is easy to see that $f \in Y^X_{(d,q)}$ but $-f \notin Y^X_{(d,q)}$. Thus $Y^X_{(d,q)}$ is not a linear space.

A simple but interesting example of a semi-Lipschitz function is the following:

Example 2.2. Let $(\mathbb{N}, d)$ be a quasi-metric space where:

$$d(x,y) = \begin{cases} 1 & \text{if } y > x, \\ 0 & \text{if } y \leq x. \end{cases}$$

Then, the dual complexity space, is the quasi-normed space $(\mathbb{B}^*, q)$, with $\mathbb{B}^* = \{f : \omega \to \mathbb{R} / \sum\limits_{n=0}^{\infty} 2^{-n}(f(n) \vee 0) < \infty\}$ and $q(f) = \sum\limits_{n=0}^{\infty} 2^{-n}(f(n) \vee 0)$.

Let now $F : (\mathbb{N}, d) \to (\mathbb{B}^*, q)$ be the function defined by: $F(0) = 0$, $F(n) = f_n$ such that $n < m$ implies $f_n > f_m$, where the order is given by $f_n > f_m$ if and only if $f_n(x) > f_m(x)$ for all $x \in \omega$.

Clearly F is a semi-Lipschitz function.

Given a quasi-metric space (X, d) and quasi-normed space (Y, q), fix $x_0 \in X$ and put

$$\mathcal{SL}_0(d,q) = \{f \in Y^X_{(d,q)} : \sup_{d(x,y)\neq 0} \frac{q(f(x) - f(y))}{d(x,y)} < \infty \ , f(x_0) = 0\}.$$

Then $\mathcal{SL}_0(d, q)$ is exactly the set of all semi-Lipschitz functions that vanishes at x_0, and it is clear that $(\mathcal{SL}_0(d, q), +, \cdot)$ is a subcone of $(Y^X_{(d,q)}, +, \cdot)$.

Now let $\rho_{(d,q)} : \mathcal{SL}_0(d, q) \times \mathcal{SL}_0(d, q) \longrightarrow [0, \infty]$ defined by

$$\rho_{(d,q)}(f,g) = \sup_{d(x,y)\neq 0} \frac{q((f-g)(x) - (f-g)(y))}{d(x,y)}$$

for all $f, g \in \mathcal{SL}_0(d, q)$. Then $\rho_{(d,q)}$ is a quasi-distance on $\mathcal{SL}_0(d, q)$. However $\rho_{(d,q)}$ is not a quasi-metric in general, as Example 1.1 of [11] shows.

Furthermore, it is clear that for each $f, g, h \in \mathcal{SL}_0(d,q)$ and each $r > 0$, $\rho_{(d,q)}(f+h, g+h) = \rho_{(d,q)}(f,g)$ and $\rho_{(d,q)}(rf, rg) = r\rho_{(d,q)}(f,g)$ i.e., $\rho_{(d,q)}$ is an invariant quasi-distance. Moreover, it is easy to check that $\rho_{(d,q)}(f, \mathbf{0}) = 0$ if and only if $f = \mathbf{0}$, where by $\mathbf{0}$ we denote the function that vanishes at every $x \in X$.

Moreover, we can see that, by example 2.1, there exists $f \in \mathcal{SL}_0(d,q)$ such that $\rho_{(d,q)}(\mathbf{0}, f) = 0$ but $f \neq \mathbf{0}$.

Consequently, the nonnegative function $\|\cdot\|_{(d,q)}$ defined on $\mathcal{SL}_0(d,q)$ by $\|f\|_{(d,q)} = \rho_{(d,q)}(f, \mathbf{0})$ is a norm on $\mathcal{SL}_0(d,q)$. Therefore $(\mathcal{SL}_0(d,q), \|\cdot\|_{(d,q)})$ is a normed cone.

Example 1.1 of [11] provides an instance of a T_1 quasi-metric space (X,d) such that $(\mathcal{SL}_0(d,q), +)$ is not a group for some $x_0 \in X$. This example suggests the question of characterizing when $(\mathcal{SL}_0(d,q), +)$ is a group. In order to give an answer to this question note that if x_0 is a fixed point in the quasi-metric space (X,d), then the set

$$\mathcal{SL}_0(d^{-1}, q) = \{f \in \ Y^X_{(d^{-1},q)} : \sup_{d(y,x)\neq 0} \frac{q(f(x)-f(y))}{d(y,x)} < \infty \ , f(x_0) = 0\}$$

has also a structure of a cone and $(\mathcal{SL}_0(d^{-1},q), \|\cdot\|_{(d^{-1},q)})$ is a normed cone, where $\|f\|_{(d^{-1},q)} = \rho_{(d^{-1},q)}(f, \mathbf{0})$, i.e.,

$$\|f\|_{(d^{-1},q)} = \sup_{d(y,x)\neq 0} \frac{q(f(x)-f(y))}{d(y,x)}$$

for all $f \in \mathcal{SL}_0(d^{-1},q)$.

Proposition 2.3. *Let (X,d), (Y,q) be a quasi-metric space and a quasi-normed space respectively. Then* $f \in \mathcal{SL}_0(d,q)$ *if and only if* $-f \in \mathcal{SL}_0(d^{-1},q)$.

Proof. Let $f \in \mathcal{SL}_0(d,q)$ then there exists $k \in \mathbb{R}^+$ such that $q(f(x)-f(y)) \leq \ kd(x,y)$ for all $x, y \in X$. We change x by y hence $q(f(y) - f(x)) \leq \ kd(y,x)$ and $q(-f(x)-(-f(y))) \leq \ kd^{-1}(x,y)$ then $-f \in \mathcal{SL}_0(d^{-1},q)$. The converse is analogous. □

Corollary 2.4. *Let (X,d), (Y,q) be a quasi-metric space and a quasi-normed space respectively.Then* $(\mathcal{SL}_0(d,q) \cap \mathcal{SL}_0(d^{-1},q), +, \cdot)$ *is a linear space.*

Proof. It follows from Proposition 2.3 that $f \in \mathcal{SL}_0(d,q) \cap \mathcal{SL}_0(d^{-1},q)$ if and only if $-f \in \mathcal{SL}_0(d,q) \cap \mathcal{SL}_0(d^{-1},q)$. □

Remark 2.5. Note that for each $f \in \mathcal{SL}_0(d,q)$, $\|f\|_{(d,q)} = \|-f\|_{(d^{-1},q)}$. Thus the normed cones $(\mathcal{SL}_0(d,q), \|\cdot\|_{(d,q)})$ and $(\mathcal{SL}_0(d^{-1},q), \|\cdot\|_{(d^{-1},q)})$ are isometrically isomorphic by the bijective map $\phi : \mathcal{SL}_0(d,q) \longrightarrow \mathcal{SL}_0(d^{-1},q)$ defined by $\phi(f) = -f$.

Furthermore, we have

$$\mathcal{SL}_0(d,q) \cap \mathcal{SL}_0(d^{-1},q) = \{f \in Y^X_{(d,q)} \cap \ Y^X_{(d^{-1},q)} :$$

$$\sup_{d(x,y)\neq 0} \frac{q(f(x)-f(y)) \vee q(f(y)-f(x))}{d(x,y)} < \infty, f(x_0)=0\}.$$

Hence $(\mathcal{SL}_0(d,q) \cap \mathcal{SL}_0(d^{-1},q), \|\cdot\|_B)$ is a normed linear space, where $\|\cdot\|_B$ is the norm defined by

$$\|f\|_B = \sup_{d(x,y)\neq 0} \frac{q(f(x)-f(y)) \vee q(f(y)-f(x))}{d(x,y)},$$

for all $f \in \mathcal{SL}_0(d,q) \cap \mathcal{SL}_0(d^{-1},q)$. Observe that $\|\cdot\|_B = \|\cdot\|_{(d,q)} \vee \|\cdot\|_{(d^{-1},q)}$ on $\mathcal{SL}_0(d,q) \cap \mathcal{SL}_0(d^{-1},q)$.

The next result, whose proof is very easy, is a characterization that will be useful.

Proposition 2.6. *$f \in \mathcal{SL}_0(d,q) \cap \mathcal{SL}_0(d^{-1},q)$ if and only if $f(x_0)=0$ and there exists $k \geq 0$ such that $q^s(f(x)-f(y)) \leq kd(x,y)$.*

Remark 2.7. It is straightforward to see that $f:(X,d) \longrightarrow (Y,q)$ belongs to $Y^X_{(d,q)} \cap Y^X_{(d^{-1},q)}$ if and only if $f(x)=f(y)$ whenever $d(x,y)=0$.

Example 2.8. Let $(X,d),(Y,q)$ be a quasi-metric and a quasi-normed space such that there is $x_0 \in X$ satisfying $d(x,x_0) \wedge d(x_0,x)=0$ for all $x \in X$. Then $\mathcal{SL}_0(d,q) \cap \mathcal{SL}_0(d^{-1},q)=\{\mathbf{0}\}$.

Example 2.9. Let $X=[0,1]$ and let d be the quasi-metric on X given by $d(x,y)=y-x$ if $x \leq y$ and $d(x,y)=1$ otherwise. Clearly $T(d)$ is the restriction of the Sorgenfrey topology to $[0,1]$. Let (Y,q) be a quasi-normed space and put $x_0=0$. Then, a function $f: X \longrightarrow Y$ satisfies $f \in \mathcal{SL}_0(d,q) \cap \mathcal{SL}_0(d^{-1},q)$ if and only if there is $k \geq 0$ such that $q(f(x)-f(y)) \vee q(f(y)-f(x)) \leq k(d(x,y) \wedge d(y,x))$ for all $x,y \in X$.

Theorem 2.10. *Let (X,d), (Y,q) be a quasi-metric and a quasi-normed space respectively. Then the following assertions are equivalent:*

(1) $\mathcal{SL}_0(d,q)=\mathcal{SL}_0(d^{-1},q)$.
(2) $\mathcal{SL}_0(d,q)$ *is a group.*
(3) $\mathcal{SL}_0(d^{-1},q)$ *is a group.*
(4) $\mathcal{SL}_0(d,q) \subset \mathcal{SL}_0(d^{-1},q)$.
(5) $\mathcal{SL}_0(d^{-1},q) \subset \mathcal{SL}_0(d,q)$.

Proof. (1) $\Rightarrow$ (2) By corollary 2.4 $(\mathcal{SL}_0(d,q) \cap \mathcal{SL}_0(d^{-1},q),+,\cdot)$ is a linear space. If $\mathcal{SL}_0(d,q)=\mathcal{SL}_0(d^{-1},q)$ then $(\mathcal{SL}_0(d,q),+)$ is a group.

(2) $\Rightarrow$ (3) Let $f \in \mathcal{SL}_0(d^{-1},q)$. By proposition 2.3 $-f \in \mathcal{SL}_0(d,q)$, since $\mathcal{SL}_0(d,q)$ is a group, $f \in \mathcal{SL}_0(d,q)$, by proposition 2.3 $-f \in \mathcal{SL}_0(d^{-1},q)$.

(3) $\Rightarrow$ (4) The proof is similar to the proof of (2) $\Rightarrow$ (3).

(4) $\Rightarrow$ (5) Let $f \in \mathcal{SL}_0(d^{-1},q)$. Then $-f \in \mathcal{SL}_0(d,q) \subset \mathcal{SL}_0(d^{-1},q)$ hence $-f \in \mathcal{SL}_0(d^{-1},q)$.Thus $f \in \mathcal{SL}_0(d,q)$.

(5) $\Rightarrow$ (1) is the same that (4) $\Rightarrow$ (5). □

Proposition 2.11. *Let (X,d), (Y,q) be a quasi-metric and a quasi-normed space respectively. If there exists $x_0 \in X$ such that $\mathcal{SL}_0(d,q)=\mathcal{SL}_0(d^{-1},q)$, then $\mathcal{SL}_1(d,q)=\mathcal{SL}_1(d^{-1},q)$ for each $x_1 \in X$.*

Proof. Let $f \in \mathcal{SL}_1(d,q)$. Define a function g on X by $g(x) = f(x) - f(x_0)$ for all $x \in X$. It easy to check that $g \in \mathcal{SL}_0(d,q)$. Thus, $g \in \mathcal{SL}_0(d^{-1},q)$. Since $g(x) - g(y) = f(x) - f(y)$ for all $x, y \in X$ we obtain that $f \in \mathcal{SL}_1(d^{-1},q)$. □

3. Completeness properties

In this section, we discuss the completeness properties of the semi-Lipschitz function space.

The following result allows us to prove that if $(Y,\ q)$ is a biBanach space then $(\mathcal{SL}_0(d,q) \cap \mathcal{SL}_0(d^{-1},q), \|\cdot\|_B)$ is a Banach space.

Theorem 3.1. *Let (X,d), (Y,q) be a quasi-metric and a quasi-normed bi-complete space respectively. Then* $(\mathcal{SL}_0(d,q) \cap \mathcal{SL}_0(d^{-1},q), \|\cdot\|_B)$ *is a Banach space.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in $(\mathcal{SL}_0(d,q) \cap \mathcal{SL}_0(d^{-1},q), \|\cdot\|_B)$. Then, given $\varepsilon \geq 0$ there is $n_0 \in \mathbb{N}$ such that

$$(*) \qquad \sup_{d(x,y)\neq 0} \frac{q^s((f_n - f_m)(x) - (f_n - f_m)(y))}{d(x,y)} < \varepsilon$$

for all $n, m \geq n_0$.

If $x = x_0$ then $f_n(x) = 0$ for all $n \in \mathbb{N}$.

Let $x \neq x_0$. We consider the following cases:

Case 1. $d(x,x_0) \neq 0$. Then, we deduce from $(*)$ that given $\frac{\varepsilon}{d(x,x_0)}$ there exists $n_0' \in \mathbb{N}$ such that if $n, m \geq n_0'$ then $q^s(f_n(x) - f_m(x)) < \varepsilon$. Therefore, $f_n(x)$ is a Cauchy sequence in (Y, q^s).

Case 2. $d(x,x_0) = 0$. Then from remark 2.7 $f_n(x) = f_n(x_0)$ and $f_m(x) = f_m(x_0)$. Therefore $q^s(f_n(x) - f_m(x)) = 0 < \varepsilon$.

Consequently, $f_n(x)$ is a Cauchy sequence in (Y, q^s) and $\{f_n(x)\}$ converges to an element $f(x)$ in (Y, q^s) for all $x \in X$. Moreover, $\{f_n\}$ converges to f in $\mathcal{SL}_0(d,q) \cap \mathcal{SL}_0(d^{-1},q)$. Indeed, given ε, since $\{f_n(x)\}$ converges to $f(x)$ for all $x \in X$, for each x, y there exists n' such that if $m' \geq n'$ then

$$\frac{q^s(f(x) - f_m'(x) - (f(y) - f_m'(y)))}{d(x,y)} < \frac{\varepsilon}{2}.$$

Since $\{f_n\}$ is a Cauchy sequence, we can also find n_0 such that if $n, m \geq n_0$ then

$$\frac{q^s(f_n(x) - f_m(x) - (f_n(y) - f_m(y)))}{d(x,y)} < \frac{\varepsilon}{2}$$

for all $x, y \in X$. Thus we have

$$\varepsilon > \frac{q^s(f(x) - f_m'(x) - (f(y) - f_m'(y)))}{d(x,y)} \geq$$

$$\frac{q^s(f(x) - f_n(x) - (f(y) - f_n(y)))}{d(x,y)} - \frac{q^s(f_m'(x) - f_n(x) - (f_m'(y) - f_n(y)))}{d(x,y)}$$

and hence

$$\frac{q^s(f(x) - f_n(x) - (f(y) - f_n(y)))}{d(x,y)} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since n_0 is independent of x, y, we obtain

$$\sup_{d(x,y)\neq 0} \frac{q^s(f(x) - f_n(x) - (f(y) - f_n(y)))}{d(x,y)} < \varepsilon,$$

for all $n \geq n_0$. □

Corollary 3.2. *Let $\{f_n\}$ be a Cauchy sequence in $(\mathcal{SL}_0(d,q) \cap (\mathcal{SL}_0(d^{-1},q), \|\cdot\|_B)$. Then there exists a convergent sequence $\{k_n\}$ in $(\mathbb{R}, T_u)$ such that k_n is a semi-Lipschitz constant for f_n, where T_u is the usual topology.*

Theorem 3.3. *Let (Y, q) be a bi-Banach space.*

(1) *(X, d) is a metric space.*
(2) *$\mathcal{SL}_0(d,q) = \mathcal{SL}_0(d^{-1},q)$, $\|\cdot\|_{(d,q)} = \|\cdot\|_{(d^{-1},q)}$.*
(3) *$(\mathcal{SL}_0(d,q), \|\cdot\|_{(d,q)})$ is a Banach space.*

Then: (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3)

Proof. (1) $\Rightarrow$ (2)
Trivial.
(2) $\Rightarrow$ (3)
Trivial.
(3) $\Rightarrow$ (2)
Suppose that $(\mathcal{SL}_0(d,q), \|\cdot\|_{(d,q)})$ is a Banach space. Then $\mathcal{SL}_0(d,q)$ is a group, so $\mathcal{SL}_0(d,q) = \mathcal{SL}_0(d^{-1},q)$. Moreover $\|\cdot\|_{(d,q)}$ is a norm on $\mathcal{SL}_0(d,q)$, so that $\|f\|_{(d,q)} = \|-f\|_{(d,q)}$ for all $f \in \mathcal{SL}_0(d,q)$. Since $-f \in \mathcal{SL}_0(d,q)$ it follows that $\|-f\|_{(d,q)} = \|f\|_{(d^{-1},q)}$. We conclude that $\|\cdot\|_{(d,q)} = \|\cdot\|_{(d^{-1},q)}$ on $(\mathcal{SL}_0(d,q)$. □

To see that in general (3) $\Rightarrow$ (1) is not true we take $Y = \mathbf{0}$ for a quasi-metric space that is not a metric space.

Corollary 3.4. *If q is a bicomplete quasi-norm on Y then $\rho_{(d,q)}$ is a bicomplete quasi-metric in $\mathcal{SL}_0(d,q) \cap \mathcal{SL}_0(d^{-1},q)$.*

The following result allow us to prove that if (Y, q) is a bicomplete space then $(\mathcal{SL}_0(d,q), \rho_{(d,q)})$ is a bicomplete space:

Theorem 3.5. *Let (X, d), (Y, q) be a quasi-metric and a quasi-normed bicomplete spaces respectively. Then $(\mathcal{SL}_0(d,q), \rho_{(d,q)})$ is a bicomplete space.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in $(\mathcal{SL}_0(d,q), \rho_{(d,q)})$. Then, given $\varepsilon \geq 0$ there is $n_0 \in \mathbb{N}$ such that

$$(*) \qquad \sup_{d(x,y)\neq 0} \frac{q((f_n - f_m)(x) - (f_n - f_m)(y))}{d(x,y)} < \varepsilon$$

for all $n, m \geq n_0$.

If $x = x_0$ then $f_n(x) = 0$ for all $n \in \mathbb{N}$.

Let $x \neq x_0$.We consider the following cases.

Case 1. $d(x, x_0) \neq 0$. Then, we deduce from $(*)$ that given $\frac{\varepsilon}{d(x,x_0)}$ there exists $n_0' \in \mathbb{N}$ such that if $n, m \geq n_0'$ then $q(f_n(x) - f_m(x)) < \varepsilon$ and if we change n and m $q(f_m(x) - f_n(x)) < \varepsilon$. Therefore, $f_n(x)$ is a Cauchy sequence in (Y, q^s).

Case 2. $d(x, x_0) = 0$. Then $d(x_0, x) \neq 0$ so $q(f_m(x) - f_n(x)) < \varepsilon$ and $q(f_n(x) - f_m(x)) < \varepsilon$.

Consequently, $f_n(x)$ is a Cauchy sequence in (Y, q^s), thus $\{f_n(x)\}$ converges in (Y, q^s) and we define f such that $\{f_n(x)\} \longrightarrow f(x)$ in (Y, q). Moreover, $\{f_n\}$ converges to f in $(\mathcal{SL}_0(d, q), \rho_{(d,q)})$.

Indeed, given ε, since $f_n(x)$ converges to $f(x)$ for all $x \in X$, for each x, y there exists n' such that if $m' \geq n'$ then

$$\frac{q^s(f(x) - f_{m'}(x) - (f(y) - f_{m'}(y)))}{d(x,y)} < \frac{\varepsilon}{2}$$

and since $\{f_n\}$ is a Cauchy sequence, we can also find n_0 such that if $m', n \geq n_0$ then

$$\frac{q^s(f_{m'}(x) - f_n(x) - (f_{m'}(y) - f_n(y)))}{d(x,y)} < \frac{\varepsilon}{2}$$

for all $x, y \in X$.

Thus we have

$$\varepsilon > \frac{q^s(f(x) - f_{m'}(x) - (f(y) - f_{m'}(y)))}{d(x,y)}$$

$$\geq \frac{q^s(f(x) - f_n(x) - (f(y) - f_n(y)))}{d(x,y)} - \frac{q^s(f'_m(x) - f_n(x) - (f'_m(y) - f_n(y)))}{d(x,y)}$$

and hence

$$\frac{q^s(f(x) - f_n(x) - (f(y) - f_n(y)))}{d(x,y)} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since n_0 is independent of x, y

$$\sup_{d(x,y)\neq 0} \frac{q^s(f(x) - f_n(x) - (f(y) - f_n(y)))}{d(x,y)} < \varepsilon,$$

for all $n \geq n_0$. Consequently $(\mathcal{SL}_0(d, q), \rho_{(d,q)})$ is a bicomplete space. □

Corollary 3.6. *Let $\{f_n\}$ be a Cauchy sequence in $(\mathcal{SL}_0(d, q), \|\cdot\|_{(d,q)})$ there exists a convergent sequence $\{k_n\}$ in $(\mathbb{R}, T_u)$ such that k_n is a semi-Lipschitz constant for f_n.*

4. Another completeness properties

In this section, we discuss another completeness properties of the semi-Lipschitz function space.

Let us recall that right K-completeness and left K-completeness constitute very useful extensions of the notion of completeness to the nonsymmetric context.

In fact, they have been successfully applied to different fields from hyperspaces and function spaces to topological algebra and theoretical computer science.

Let (X, d) be a quasi-metric space. A net $\{x_\delta\} \subset X$, $\delta \in \Lambda$, is called left K-Cauchy provided that for each $\varepsilon > 0$ there is δ_0 such that $d(x_{\delta_2}, x_{\delta_1}) < \varepsilon$ for all $\delta_1 \geq \delta_2 \geq \delta_0$, and $\{x_\delta\} \subset X$ is called right K-Cauchy provided that for each $\varepsilon > 0$ there exists δ_0 such that $d(x_{\delta_1}, x_{\delta_2}) < \varepsilon$ for all $\delta_1 \geq \delta_2 \geq \delta_0$.

A quasi-metric ρ is called left K-complete (resp. right K-complete) if each left K-Cauchy net (resp. right K-Cauchy net) converges.

The following result allow us to prove that if (Y, q) is a biBanach and finite dimensional space then $(\mathcal{SL}_0(d, q)\, \rho_{(d,q)})$ is a right K-complete space:

Theorem 4.1. *Let (X, d), (Y, q) be a quasi-metric and a quasi-normed bicomplete finite dimensional space respectively. Then $\rho_{(d,q)}$ is right K-complete.*

Proof. Let $\{f_\delta\}$ be a right K-Cauchy net in $(\mathcal{SL}_0(d, q), \rho_{(d,q)})$. Then, given $\varepsilon \geq 0$ there is δ_0 such that

$$(\star) \qquad \sup_{d(x,y)\neq 0} \frac{q((f_{\delta_1} - f_{\delta_2})(x) - (f_{\delta_1} - f_{\delta_2})(y))}{d(x, y)} < \varepsilon$$

for all $\delta_1 \geq \delta_2 \geq \delta_0$.

Let $x \neq x_0$.We consider the following cases.

Case 1. $d(x, x_0) \neq 0$ and $d(x_0, x) \neq 0$. Then, we deduce from $(\star)$ that given $\frac{\varepsilon}{d(x,x_0)}$ and $\frac{\varepsilon}{d(x_0,x)}$ respectively there exists δ_0' such that if $\delta_1 \geq \delta_2 \geq \delta_0'$ then $q^s(f_{\delta_1}(x) - f_{\delta_2}(x)) < \varepsilon$. Therefore$\{f_\delta(x)\}$ is a Cauchy net in (Y, q).

Case 2. $d(x_0, x) = 0$ and $d(x, x_0) \neq 0$. Then

$$q(f_{\delta_1}(x)) - q(f_{\delta_2}(x)) \leq q(f_{\delta_1}(x) - f_{\delta_2}(x)) \leq \varepsilon d(x, x_0)$$

for all $\delta_1 \geq \delta_2 \geq \delta_0$ and $q(-f_\delta(x)) = 0$, since $q^s(f_{\delta_1}(x)) \leq \varepsilon d(x, x_0) + q(f_{n_0}(x))$ thus $\{f_n(x)\}$ is a bounded net in the finite dimensional space (Y, q), there exists a convergent subnet $\{f_{\delta'}(x)\}$ in (Y, q). Given $\varepsilon > 0$ there exists $\delta_0 \in \Lambda$ such that if $\delta_1 \geq \delta_2' \geq \delta_0$ then

$$\begin{aligned} q^s(f_{\delta_1}(x) - f(x)) &= q^s(f_{\delta_1}(x) - f_{\delta_2'}(x) + f_{\delta_2'}(x)) \\ &\leq q^s(f_{\delta_1}(x) - f_{\delta_2'}(x)) + q^s(f_{\delta_2'}(x) - f(x)). \end{aligned}$$

Now $q^s(f_{\delta_2'}(x) - f(x)) < \frac{\varepsilon}{2}$ because $\{f_{\delta_1'}(x)\}$ is a convergent net. Given $\delta_1' \geq \delta_1$, such that $f_{\delta_1'}$ is in the subnet then

$$\begin{gathered} q(f_{\delta_2'}(x) - f_{\delta_1}(x)) = \\ q(f_{\delta_2'}(x) - f_{\delta_1'}(x) + f_{\delta_1'}(x) - f_{\delta_1}(x)) \leq \\ q(f_{\delta_2'}(x) - f_{\delta_1'}(x)) + q(f_{\delta_1'}(x) - f_{\delta_1}(x)) < \frac{\varepsilon}{2} \end{gathered}$$

since $\{f_{\delta_1'}(x)\}$ converges and $\{f_\delta\}$ is right K-Cauchy. On the other hand $q(f_{\delta_1}(x) - f_{\delta_2'}(x)) < \frac{\varepsilon}{2}$ because $\{f_{\delta_2}\}$ is a right K-Cauchy net. Thus $\{f_\delta(x)\}$ is a convergent net.

Case 3. $d(x_0, x) \neq 0$ and $d(x, x_0) = 0$. Then $q(-f_{\delta_1}(x)) \leq \varepsilon d(x_0, x) + q(-f_{\delta_0}(x))$ and $q(f_\delta(x)) = 0$ respectively, since $\{f_\delta\}$ is a bounded sequence on the finite dimensional space (Y, q).

Consequently $\{f_\delta(x)\}$ is a convergent net in (Y, q^s), and we define f such that $\{f_n(x)\}$ converges to $f(x)$ for each $x \in X$. Let $\{f_\delta\}$ a right K-Cauchy net in $(\mathcal{SL}_0(d,q), \rho_{(d,q)})$.

Let us see that $\{\frac{q(f_\delta(x)-f_\delta(y))}{d(x,y)}\}$ converges to $\frac{q(f(x)-f(y))}{d(x,y)}$ for all $x, y \in X$ such that $d(x,y) \neq 0$.

Since $\{f_\delta\}$ is a right K-Cauchy net, given $\varepsilon > 0$ there exists δ_0 such that if $\delta_1 \geq \delta_2 \geq \delta_0$ then

$$\sup_{d(x,y)\neq 0} \frac{q((f_{\delta_1} - f_{\delta_2})(x) - (f_{\delta_1} - f_{\delta_2})(y))}{d(x,y)} < \frac{\varepsilon}{2}.$$

Since $f_n(x)$ converges to $f(x) \forall x \in X$, for each x, y there exists δ_0' such that if $\delta_1' \geq \delta_0'$ then

$$\frac{q^s(f(x) - f_{\delta_1'}(x) - (f(y) - f_{\delta_1'}(y)))}{d(x,y)} < \frac{\varepsilon}{2}.$$

Thus given $\varepsilon > 0$, for all $\delta' \geq \delta_0$ and for each $x, y \in X : d(x,y) \neq 0$ and we take $\delta_1 \geq (\delta' \vee \delta_0')$

$$\frac{q(f(x) - f_{\delta'}(x) - (f(y) - f_{\delta'}(y)))}{d(x,y)} =$$

$$\frac{q(f(x) - f_{\delta'}(x) - f_{\delta_1}(x) + f_{\delta_1}(x) - (f(y) - f_{\delta'}(y) - f_{\delta_1}(y) + f_{\delta_1}(y)))}{d(x,y)} \leq$$

$$\frac{q(f(x) - f_{\delta_1}(x) - (f(y) + f_{\delta_1}(y)))}{d(x,y)} + \frac{q(f_{\delta_1}(x) - f_{\delta'}(x) - (f_{\delta'}(y) + f_{\delta_1}(y)))}{d(x,y)} < \varepsilon.$$

for all x, y such that $d(x,y) \neq 0$

$$\sup_{d(x,y)\neq 0} \frac{q(f(x) - f_{\delta'}(x) - (f(y) - f_{\delta'}(y)))}{d(x,y)} < \varepsilon,$$

for all $\delta_n' \geq \delta_0$.

□

Corollary 4.2. *Let (X,d), (Y,q) be a quasi-metric and a quasi-normed space respectively.*

Let $\{f_\delta\}$ be a right K-Cauchy in $(\mathcal{SL}_0(d,q), \rho_{(d,q)})$. If for each $x \in X$ $\{f_\delta(x)\}$ converges to $f(x)$ in (Y, q^s) then $\{f_\delta\}$ converges to f in $(\mathcal{SL}_0(d,q), \rho_{(d,q)})$.

Theorem 4.3. *Let (X,d), (Y,q) be a quasi-metric T_1 and a quasi-normed bicomplete space respectively. Then $\rho_{(d,q)}$ is right K-complete.*

Proof. Let $\{f_\delta\}$ be a right K-Cauchy net in $(\mathcal{SL}_0(d,q), \rho_{(d,q)})$. Then, given $\varepsilon \geq 0$ there is δ_0 such that

$$(\lozenge) \qquad \sup_{d(x,y)\neq 0} \frac{q((f_{\delta_1} - f_{\delta_2})(x) - (f_{\delta_1} - f_{\delta_2})(y)))}{d(x,y)} < \varepsilon$$

for all $\delta_1 \geq \delta_2 \geq \delta_0$.

Let $x \neq x_0$.

Since (X,d) is T_1, $d(x,x_0)\neq 0$ and $d(x_0,x)\neq 0$ then, we deduce from $(\Diamond)$ that given $\frac{\varepsilon}{d(x,x_0)}$ and $\frac{\varepsilon}{d(x_0,x)}$ there exists δ'_0 such that if $\delta_1\geq\delta_2\geq\delta'_0$ then $q^s(f_{\delta_1}(x)-f_{\delta_2}(x))<\varepsilon$. Therefore$f_n(x)$ is a Cauchy net in (Y,q) for all $x\in X$. Thus $\{f_\delta(x)\}$ is a convergent net in (Y,q^s) and we define f such that $\{f_n(x)\}$ converges to $f(x)$ for each $x\in X$. Let $\{f_\delta\}$ a right K-Cauchy net in $(\mathcal{SL}_0(d,q),\rho_{(d,q)})$.

Let us see that $\{\frac{q(f_\delta(x)-f_\delta(y))}{d(x,y)}\}$ converges to $\frac{q(f(x)-f(y))}{d(x,y)}$ for all $x,y\in X$ such that $d(x,y)\neq 0$.

Since $\{f_\delta\}$ is a right K-Cauchy net, given $\varepsilon>0$ there exists δ_0 such that if $\delta_1\geq\delta_2\geq\delta_0$ then

$$\sup_{d(x,y)\neq 0}\frac{q((f_{\delta_1}-f_{\delta_2})(x)-(f_{\delta_1}-f_{\delta_2})(y))}{d(x,y)}<\frac{\varepsilon}{2}.$$

Since $f_n(x)$ converges to $f(x)\forall x\in X$, for each x,y there exists δ'_0 such that if $\delta'_1\geq\delta'_0$ then

$$\frac{q^s(f(x)-f_{\delta'_1}(x)-(f(y)-f_{\delta'_1}(y)))}{d(x,y)}<\frac{\varepsilon}{2}.$$

Thus given $\varepsilon>0$, for all $\delta'\geq\delta_0$ and for each $x,y\in X:d(x,y)\neq 0$ and we take $\delta_1\geq(\delta'\vee\delta'_0)$

$$\frac{q(f(x)-f_{\delta'}(x)-(f(y)-f_{\delta'}(y)))}{d(x,y)}=$$

$$\frac{q(f(x)-f_{\delta'}(x)-f_{\delta_1}(x)+f_{\delta_1}(x)-(f(y)-f_{\delta'}(y)-f_{\delta_1}(y)+f_{\delta_1}(y)))}{d(x,y)}\leq$$

$$\frac{q(f(x)-f_{\delta_1}(x)-(f(y)+f_{\delta_1}(y)))}{d(x,y)}+\frac{q(f_{\delta_1}(x)-f_{\delta'}(x)-(f_{\delta'}(y)+f_{\delta_1}(y)))}{d(x,y)}<\varepsilon.$$

for all x,y such that $d(x,y)\neq 0$

$$\sup_{d(x,y)\neq 0}\frac{q(f(x)-f_{\delta'}(x)-(f(y)-f_{\delta'}(y)))}{d(x,y)}<\varepsilon,$$

for all $\delta'_n\geq\delta_0$.

□

References

[1] J. Deák, *A bitopological view of quasi-uniform completeness*, I, Studia Sci. Math. Hungar. **30** (1995), 389–409; II **30** (1995), 411-431; **31** (1996), 385–404.

[2] D. Doitchinov, *On completeness in quasi-metric spaces*, Topology Appl. **30** (1988), 127–148.

[3] P. Flecher and W. Hunsaker, *Completeness using pairs of filters*, Topology Appl. **44** (1992), 149–155.

[4] P. Fletcher and W. F. Lindgren, *On Quasi-Uniform Spaces*, Marcel Dekker, New York, (1982).

[5] L. M. García-Raffi, S. Romaguera and E. A. Sánchez-Pérez, *Sequence spaces and asymmetric norms in the theory of computational complexity*, Math. Comput. Model. **36** (2002), 1–11.

[6] L. M. García-Raffi, S. Romaguera and E. A. Sánchez-Pérez, *The supremum asymmetric norm on sequence algebras: a general framework to measure complexity distances*, Electronic Notes in Theoretical Computer Science **74** (2003). URL:Http://www.elsevier.nl/locate/entcs/volume74.html, 12 pages.

[7] K. Keimel and W. Roth, *Ordered Cones and Approximation*, Springer-Verlag, Berlin (1992).

[8] H. P. A. Künzi, *Nonsymmetric distances and their associated topologies: about the origin of basic ideas in the area of asymmetric topology*, Handbook of the History of General Topology, ed. by C. E. Aull and R. lowen, Vol **3**, Hist. Topol. **3**, Kluwer Acad. Publ., Dordrecht, (2001), 853–968.

[9] H. P. A. Künzi, *Nonsymmetric topology, in Topology with Applications*, Bolyai Soc. Math. Studies **4**, pp. 303–338, Szekszard, Hungary, (1993).

[10] S. Romaguera, *Left K-completeness in quasi-metric spaces*, Math. Nachr. **157** (1992), 15–23.

[11] S. Romaguera and M. Sanchis, *Properties of the normed cone of semi-Lipschitz functions*, Acta Math. Hungar. **108** (1-2) (2005), 55–70.

[12] S. Romaguera and M. Sanchis, *On semi-Lipschitz functions and best approximation in quasi-metric spaces*, J. Approximation Theory **283** (2000), 292–301.

[13] S. Romaguera and M. Sanchis, *Applications of utility functions defined on quasi-metric spaces*, J. Math. Anal. Appl. **283** (2003), 219–235.

[14] S. Romaguera and M. Schellekens, *Quasi-metric properties of complexity spaces*, Topology Appl. **98** (1999), 311–322.

[15] S. Romaguera and M. Schellekens, *Duality and quasi-normability for complexity spaces*, Appl. Gen. Topol. **3** (1) (2002), 91–112.

[16] M. Schellekens, *The Smyth completion: A common foundation for denotational semantics and complexity analysis*, Electron. Notes Comput. Sci. **1** (1995), 1–22.

[17] M. B. Smyth, *Totally bounded spaces and compact ordered spaces as domains of computation*, in Topology and Category Theory in Computer Science, G. M. Reed, A. W. Rosco, and R. F. Wachter, Eds., Clarendon, Oxford (1991), pp. 207–229.

[18] M. B. Smyth, *Completeness of quasi-uniform and syntopological spaces*, J. London Math. Soc. **49** (1994), 385–400.

[19] Ph. Sünderhauf, *Quasi-uniform completeness in terms of Cauchy nets*, Acta Math. Hungar. **69** (1995), 47–54.

Received January 2005

Accepted May 2005

José Manuel Sánchez-Álvarez (jossnclv@doctor.upv.es)
Escuela de Caminos, Departamento de Matemática Aplicada, Universidad Politécnica de Valencia, 46071 Valencia, SPAIN.