DISTRIBUTIONALLY CHAOTIC FAMILIES OF OPERATORS ON FRÉCHET SPACES

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Abstract. The existence of distributional chaos and distributional irregular vectors has been recently considered in the study of linear dynamics of operators and C_0-semigroups. In this paper we extend some previous results on both notions to sequences of operators, C_0-semigroups, C-regularized semigroups, and α-times integrated semigroups on Fréchet spaces. We also add a study of rescaled distributionally chaotic C_0-semigroups. Some examples are provided to illustrate all these results.

1. Introduction

Among the different topological notions that describe the dynamics of linear operators, in the last years the one of distributional chaos has been widely studied. Schweizer and Smítal introduced distributional chaos for interval maps in [55]. Its goal was to extend the definition of chaos in the sense of Li and Yorke in order that it was equivalent to positive topological entropy. Distributional chaos was firstly considered in the setting of linear operators when studying a quantum harmonic oscillator [30, 54]. Later, a systematic study of distributional chaos for backward shifts operators was initiated in [49], providing later an example of a backward shift operator with a full scrambled set in [50].

The extension of distributional chaos to C_0-semigroups was done in [10], where it was studied for the translation C_0-semigroup on weighted spaces of integrable functions, and in [1], where some sufficient criteria in terms of the infinitesimal generator were provided. This has permitted to find distributional chaos in phenomena described by drift-diffusion equations with inward streaming [29], such as the ones modeled by an Ornstein-Uhlenbeck operator [21], or by forward and backward control processes [9], see also [7].

Devaney chaos can appear in linear spaces provided that there exists a vector with dense orbit (hypercyclicity) and a dense set of periodic points. Devaney and distributional chaos are closely tied for C_0-semigroups: If chaos in the sense of Devaney is obtained as an application of the Desch-Schappacher-Webb criterion [29] to the infinitesimal generator of the C_0-semigroup, then distributional chaos can be also obtained, see [13, Cor. 31] and [8, Rem. 3.8]. The study of mean-ergodic C_0-semigroups, and its connection with linear dynamics, on Fréchet spaces has been recently considered in [2]. Those results were later improved in [34].

On the one hand, the question whether all the non-trivial operators of a C_0-semigroup inherit its dynamical behaviour has been considered in linear dynamics with different success: for hypercyclicity, frequent hypercyclicity, and distributional chaos the answer is affirmative [1, 23], however it is negative in the case of Devaney chaos [11].

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In this line, it was studied in [25] if a C_0-semigroup holds the Hypercyclicity Criterion, then this dynamics is inherited by arbitrary sequences of its operators. Such analysis was inspired by the extension of this criterion to the case of a hypercyclic sequence of commuting operators with dense range [20], see also [14, 16]. However, some results for distributionally chaotic sequences of operators have not been stated yet.

On the other hand, Devaney chaos was considered for unbounded differentiation operators in [19] and for C_0-semigroups of unbounded operators in [27]. We will complement these results with the study of the notion of distributional chaos not only for C_0-semigroups, but also for integrated semigroups of linear operators.

In this paper we consider the study of distributional chaos on Fréchet spaces. We extend some previous results in [1, 18] to the case when dealing with operators between two different Fréchet spaces (Section 2). We also present the extensions of some results to the case of C_0-semigroups between Fréchet spaces and we show some results related to rescaled distributionally chaotic C_0-semigroups defined on Fréchet spaces (Section 3). Later on, the study of distributional chaos for α-times integrated C-semigroups (Section 4). Along all the paper we present some examples that illustrate the results.

2. Preliminaries

2.1. **Notation.** In this section we present some definitions and preliminary results that will be needed to follow further discussions in the paper. Unless it is explicitly mentioned, we will assume that X is an infinite-dimensional separable Fréchet space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and we denote its topological dual by X^*.

The topology of X will be induced by the fundamental system $(p_n)_{n\in\mathbb{N}}$ of increasing seminorms that yields the translation invariant metric $d : X \times X \to \mathbb{R}_0^+$ defined by

$$d(x,y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x-y)}{1+p_n(x-y)}, \quad \text{for all } x, y \in X. \tag{1}$$

This metric satisfies the following properties:

$$d(x+u, y+v) \leq d(x,y) + d(u,v), \quad x,\ y,\ u,\ v \in X, \tag{2}$$

$$d(cx, cy) \leq (|c|+1)d(x,y),\ c \in \mathbb{K},\ x,\ y \in X, \tag{3}$$

and

$$d(\alpha x, \beta x) \geq \frac{|\alpha-\beta|}{1+|\alpha-\beta|} d(0,x), \quad x \in X,\ \alpha,\ \beta \in \mathbb{K}. \tag{4}$$

Given an arbitrary $\varepsilon > 0$ we define the ball of center 0 and radius ε as $B_d(0,\epsilon) := \{x \in X : d(x,0) < \varepsilon\}$. In the case that $(X, \|\cdot\|)$ is a Banach space, then we will assume that the distance between two elements $x, y \in X$ is given by $d(x,y) := \|x-y\|$. By $L(X)$ we denote the space of all continuous linear mappings from X into X. Let $\mathcal{B}$ be a fundamental family of bounded subsets of X. For every $n \in \mathbb{N}$, $B \in \mathcal{B}$ let us define the continuous seminorm $p_{n,B}(T) := \sup_{x\in B} p_n(Tx)$ on $L(X)$. Then the system of continuous seminorms $(p_{n,B})_{(n,B)\in\mathbb{N}\times\mathcal{B}}$ induces a Hausdorff locally convex topology on $L(X)$.

2.2. **Distributional chaos for single operators.** A sequence of operators $(T_k)_{k\in\mathbb{N}} \subseteq L(X)$ is said to be *universal* if there exists some $x \in X$ such that $\{T_k x : k \geq 0\}$ is dense in X. When $T_k := T^k$ for some $T \in L(X)$ and for all $k \in \mathbb{N}$, we say that T is *hypercyclic*. In this case, the set $\{T^k(x) : k \geq 0\}$ is known as the orbit of the element x by the operator T. The connections between both notions have been extensively reported on [38]. An element $x \in X$ is a *periodic point* for the operator T if there exists $n_0 \in \mathbb{N}$ such that $T^{n_0}x = x$. An operator T is called *Devaney chaotic* if it is hypercyclic and the set of periodic points is dense in X. For further information on linear dynamics we refer the reader to [12, 39], see also [17]. Another dynamical properties related to C_0-semigroups are frequent hypercyclicity, topological transitivity and mixing (for more details see [39]).

Distributional chaos requires a quite complicated statistical dependence between orbits of a given set, since they have to be proximal but not asymptotic. For this purpose, we recall that the *upper density* of a set $\Gamma \subseteq \mathbb{N}$ is defined by

$$\overline{\text{dens}}(\Gamma) := \limsup_{n\to+\infty} \frac{\text{card}(\Gamma \cap \{0,1,\dots,n-1\})}{n}, \tag{5}$$

where card denotes the usual cardinality of a subset $\Gamma \subseteq \mathbb{N}$.

An operator T on X is said to be *distributionally chaotic* if there exist an uncountable set $S \subseteq X$ and $\sigma > 0$ such that for each $\varepsilon > 0$ and for each pair $x,\ y \in S$ of distinct points we have that

$$\overline{\text{dens}}(\{n \in \mathbb{N} : d(T^n x, T^n y) \geq \sigma\}) = 1 \text{ and} \tag{6}$$

$$\overline{\text{dens}}(\{n \in \mathbb{N} : d(T^n x, T^n y) < \epsilon\}) = 1, \tag{7}$$

see for instance [10, 18, 49]. If, moreover, we can choose S to be dense in X, then T is said to be *densely distributionally chaotic.* The question whether T is distributionally chaotic or not is closely connected with the existence of *distributionally irregular vectors*, i.e., those elements $x \in X$ such that for each $\sigma > 0$ we have that

$$\overline{\text{dens}}(\{n \in \mathbb{N} : d(T^n x, 0) > \sigma\}) = 1, \text{ and} \tag{8}$$

$$\overline{\text{dens}}(\{n \in \mathbb{N} : d(T^n x, 0) < \sigma\}) = 1, \tag{9}$$

see more details in [13, 18].

2.3. **Distributional chaos for C_0-semigroups.** We recall that $(T(t))_{t\geq 0}$, with $T(t) \in L(X)$ for each $t \geq 0$, is a C_0*-semigroup* if $T(0) = I$, $T(t+s) = T(t)T(s)$ and $\lim_{s\to t} T(s)x = T(t)x$ for all $x \in X$ and $t \geq 0$. Given an arbitrary C_0-semigroup $(T(t))_{t\geq 0}$ on X, it can be shown that

$$Ax := \lim_{t\to 0} \frac{T(t)x - x}{t} \tag{10}$$

exists on a dense subspace of X. The set of these x is the domain of A, that it is denoted by $D(A)$. Then A, or rather $(A, D(A))$, is called the *infinitesimal generator of the semigroup* $(T(t))_{t\geq 0}$. Moreover $T(t)(D(A)) \subset D(A)$ with $AT(t)x = T(t)Ax$, for every $t \geq 0$ and $x \in D(A)$. Another important property is provided by the point spectral mapping theorem for C_0-semigroups. If X is a complex Fréchet space then, for every $x \in X$ and $\lambda \in \mathbb{C}$,

$$Ax = \lambda x \Rightarrow T(t)x = e^{\lambda t} x \tag{11}$$

for every $t \geq 0$. Further information on C_0-semigroups can be found in [33, 57] on Banach spaces and in [28, 43] on locally convex spaces.

We recall that a C_0*-semigroup* $(T(t))_{t\geq 0}$ on X is said to be *hypercyclic* if there exists $x \in X$ such that the set $\{T(t)x : t \geq 0\}$ is dense in X. An element $x \in X$ is a *periodic point* for the semigroup if there exists $t > 0$ such that $T(t)x = x$. A C_0-semigroup $(T(t))_{t\geq 0}$ is called *chaotic (in the sense of Devaney)* if it is hypercyclic and the set of periodic points is dense in X, further information in [39, Section 7.2].

In many situations we can obtain the infinitesimal generator of a C_0-semigroup although we do not have the explicit representation of its operators. The classical Desch-Schappacher-Webb criterion permits to state Devaney chaos (and hypercyclicity) of a C_0-semigroup in terms of the abundance of eigenvectors of its infinitesimal generator. [29, The. 3.1]. See also [6, 31, 21, 39].

Theorem 2.1. *Let X be a complex separable Fréchet space and let $(T(t))_{t\geq 0}$ be a C_0-semigroup on X with generator A. Assume that there exists an open connected subset U and weakly holomorphic functions $f_j : U \to X, j \in J$, such that*

(i) $U \bigcap i\mathbb{R} \neq \emptyset$,
(ii) $f_j(\lambda) \in \ker(\lambda I - A)$ *for every* $\lambda \in U; j \in J$,
(iii) *for any* $x^* \in X^*$, $\langle f_j(\lambda), x^*\rangle = 0$ *for all* $\lambda \in U, j \in J$ *implies* $x^* = 0$,

then the C_0-semigroup $(T(t))_{t\geq 0}$ is Devaney chaotic (and hypercyclic).

Distributional chaos always appears whenever this theorem can be applied, see for instance [8, 13]. Distributionally chaotic C_0-semigroups on Banach spaces has been found in [1, 10].

We extend this definition to the setting of Fréchet spaces. For this purpose, the *upper density* of a set $D \subseteq \mathbb{R}_0^+$ is defined by

$$\overline{\mathrm{Dens}}(D) := \limsup_{t\to+\infty} \frac{\mu(D \cap [0, t])}{t}, \tag{12}$$

where $\mu(\cdot)$ denotes the Lebesgue's measure on $\mathbb{R}_0^+$. A C_0-semigroup $(T(t))_{t\geq 0}$ on X, where X is a infinite-dimensional separable Fréchet space, is said to be *distributionally chaotic* if there exist an uncountable set $S \subseteq X$ and $\sigma > 0$ such that for each $\varepsilon > 0$ and for each pair $x,\ y \in S$ of distinct points we have that

$$\overline{\mathrm{Dens}}(\{s \geq 0 : d(T(s)x, T(s)y) \geq \sigma\}) = 1 \text{ and} \tag{13}$$

$$\overline{\mathrm{Dens}}(\{s \geq 0 : d(T(s)x, T(s)y) < \epsilon\}) = 1. \tag{14}$$

Moreover, if we can choose S to be dense in X, then $(T(t))_{t\geq 0}$ is said to be *densely distributionally chaotic*.

The question whether a C_0-semigroup $(T(t))_{t\geq 0}$ is distributionally chaotic or not is closely connected with the existence of *distributionally irregular vectors* [1, Th. 3.4], i.e., those elements $x \in X$ such that for each $\sigma > 0$ we have that

$$\overline{\mathrm{Dens}}(\{s \geq 0 : d(T(s)x, 0) > \sigma\}) = 1 \text{ and} \tag{15}$$

$$\overline{\mathrm{Dens}}(\{s \geq 0 : d(T(s)x, 0) < \sigma\}) = 1, \tag{16}$$

Remark 2.2. *The definitions of distributional chaos and distributionally irregular vectors can also be considered for strongly continuous families of operators $(T(t))_{t\geq 0} \subset L(X)$, i.e. the ones satisfying $\lim_{t\to 0^+} T_t x = T_0 x$ for every $x \in X$. In case that $(T(t))_{t\geq 0} \subset L(X, Y)$, with Y a Fréchet space, the conditions in* (13)–(16) *should be written in terms of d_Y, the translation invariant distance on Y induced by the fundamental system $(p_n^Y)_{n\in\mathbb{N}}$ of increasing seminorms on Y, as it is indicated in* (1)*.*

3. Distributional chaos for sequences of operators

Throughout this section we will always assume that Y is another separable Fréchet space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. By $L(X, Y)$ we denote the space of all continuous linear operators from X into Y. The main purpose of this section is to inquire into the basic distributionally chaotic properties of a sequence $(T_k)_{k\in\mathbb{N}}$ of linear mappings between the Fréchet spaces X and Y. We start with the following definition:

Definition 3.1. *Let $(T_k)_{k\in\mathbb{N}}$ be a sequence of linear (not necessarily continuous) operators $T_k : D(T_k) \to Y$, $k \in \mathbb{N}$, and let $\widetilde{X}$ be a closed linear subspace of X. We say that the sequence $(T_k)_{k\in\mathbb{N}}$ is*

$\widetilde{X}$-distributionally chaotic *if there exist an uncountable set* $S \subseteq \bigcap_{k=1}^{\infty} D(T_k) \cap \widetilde{X}$ *and* $\sigma > 0$ *such that for each* $\varepsilon > 0$ *and for each pair* $x, y \in S$ *of distinct points we have that*

$$\overline{\text{dens}}\Big(\big\{k \in \mathbb{N} : d_Y\big(T_k x, T_k y\big) \geq \sigma\big\}\Big) = 1 \text{ and} \tag{17}$$

$$\overline{\text{dens}}\Big(\big\{k \in \mathbb{N} : d_Y\big(T_k x, T_k y\big) < \epsilon\big\}\Big) = 1. \tag{18}$$

The set S *is said to be a* $\sigma_{\widetilde{X}}$-scrambled set *(σ-scrambled set in the case that $\widetilde{X} = X$) for the sequence of operators $(T_k)_{k\in\mathbb{N}}$. If S can be chosen to be dense in $\widetilde{X}$, then the sequence $(T_k)_{k\in\mathbb{N}}$ is said to be* densely $\widetilde{X}$-distributionally chaotic. *In the case that $\widetilde{X} = X$, we say that the sequence $(T_k)_{k\in\mathbb{N}}$ is* distributionally chaotic.

Remark 3.2. *In particular, a linear operator $T : D(T) \to Y$ is said to be* (densely) $\widetilde{X}$-distributionally chaotic *if the sequence of its powers $(T^k)_{k\in\mathbb{N}}$ is (densely) $\widetilde{X}$-distributionally chaotic; in the case that $\widetilde{X} = X$ then the operator T is said to be (densely) distributionally chaotic.*

Remark 3.3. *Conditions* (17) *and* (18) *can be rephrased in terms of the family of continuous seminorms $(p_n^Y)_{n\in\mathbb{N}}$. More precisely, condition* (17) *can be replaced by:*

If $0 < \sigma < 1$ and there exists some $n_0 \in \mathbb{N}$ and $\sigma' > \sigma/(1-\sigma)$ such that

$$\overline{\text{dens}}(\{k \in \mathbb{N} : p_{n_0}^Y(T_k x - T_k y) \geq \sigma'\}) = 1. \tag{19}$$

On the other hand, we can exchange Condition (18) *by the following formulation:*

There exists some $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ we have

$$\overline{\text{dens}}(\{k \in \mathbb{N} : p_n^Y(T_k x - T_k y) < \epsilon\}) = 1. \tag{20}$$

Of course, it is interesting to know the minimal linear subspace $\widetilde{X}$ for which the sequence $(T_k)_{k\in\mathbb{N}}$ is $\widetilde{X}$-distributionally chaotic, because it is then $\widetilde{\widetilde{X}}$-distributionally chaotic for any other linear subspace $\widetilde{\widetilde{X}}$ of X such that $\widetilde{X} \subseteq \widetilde{\widetilde{X}}$. Some of the following definitions have been introduced for linear and continuous operators from one space onto itself, see [18, Def. 14].

Definition 3.4. *Let $\widetilde{X}$ be a closed linear subspace of X, $x \in \cap_{k=1}^{\infty} D(T_k)$ and $m \in \mathbb{N}$. Let $(T_k)_{k\in\mathbb{N}}$ be a sequence of linear (not necessarily continuous) operators $T_k : D(T_k) \to Y$, $k \in \mathbb{N}$. We say that:*

1. *the orbit of x under $(T_k)_{k\in\mathbb{N}}$, the set $\{x\} \cup \{T_k x : k \in \mathbb{N}\}$, is* distributionally near to 0 *if there exists $\mathcal{A}_x \subseteq \mathbb{N}$ such that $\overline{\text{dens}}(\mathcal{A}_x) = 1$ and $\lim_{k\in\mathcal{A}_x, k\to\infty} T_k x = 0$;*
2. *the orbit of x under $(T_k)_{k\in\mathbb{N}}$ is* distributionally m-unbounded *if there exists $\mathcal{B} \subseteq \mathbb{N}$ such that $\overline{\text{dens}}(\mathcal{B}) = 1$ and $\lim_{k\in\mathcal{B}, k\to\infty} p_m^Y(T_k x) = \infty$;*
3. *the orbit of x under $(T_k)_{k\in\mathbb{N}}$ is said to be* distributionally unbounded *if there exists $q \in \mathbb{N}$ such that this orbit is distributionally q-unbounded;*
4. *x is a* $\widetilde{X}$-distributionally irregular vector for the sequence $(T_k)_{k\in\mathbb{N}}$ *(distributionally irregular vector for the sequence $(T_k)_{k\in\mathbb{N}}$, in the case that $\widetilde{X} = X$) if $x \in \bigcap_{k=1}^{\infty} D(T_k) \cap \widetilde{X}$ and the orbit of x under $(T_k)_{k\in\mathbb{N}}$ is both distributionally near to 0 and distributionally unbounded.*

Suppose that $X' \subseteq \widetilde{X}$ is a linear manifold. We say that

1. *X' is an* $\widetilde{X}$-distributionally irregular manifold for the sequence $(T_k)_{k\in\mathbb{N}}$ *if any element $x \in (X' \cap \bigcap_{k=1}^{\infty} D(T_k)) \setminus \{0\}$ is $\widetilde{X}$-distributionally irregular vector for the sequence $(T_k)_{k\in\mathbb{N}}$.*
2. *X' is a* uniformly $\widetilde{X}$-distributionally irregular manifold *for the sequence $(T_k)_{k\in\mathbb{N}}$ if there exists $m \in \mathbb{N}$ such that the orbit of each vector $x \in (X' \cap \bigcap_{k=1}^{\infty} D(T_k)) \setminus \{0\}$ under $(T_k)_{k\in\mathbb{N}}$ is both distributionally m-unbounded and distributionally near to 0.*
3. *and if X' is dense in $\widetilde{X}$, then we say that X' is a* dense $\widetilde{X}$-distributionally irregular manifold.

Remark 3.5. *In the same way as in Remark 3.2, it is clear that we have all these notions for a linear operator* $T : D(T) \to Y$ *and a vector* $x \in D_\infty(T) := \cap_{k=1}^{\infty} D(T^k)$ *if we consider the sequence of operators* $(T^k)_{k\in\mathbb{N}}$. *Inspired by Definition 3.1, we can also define a distributionally irregular manifold and an uniformly distributionally irregular manifold in the case that* $\widetilde{X} = X$.

Note that if $x \in \widetilde{X} \cap \bigcap_{k=1}^{\infty} D(T_k)$ *is a* $\widetilde{X}$*-distributionally irregular vector for the sequence* $(T_k)_{k\in\mathbb{N}}$, *then* $X' = \operatorname{span}\{x\}$ *is a uniformly* $\widetilde{X}$*-distributionally irregular manifold for this same sequence* $(T_k)_{k\in\mathbb{N}}$.

In some way, the next lemma enables us to reduce our further study to the case in which $\widetilde{X} = X$. The straightforward proof is left to the reader.

Lemma 3.6. *Let* $(T_k)_{k\in\mathbb{N}}$ *be a sequence of operators and let* $\widetilde{X}$ *be a closed linear subspace of* X. *If* $D(\mathcal{T}_k) := D(T_k) \cap \widetilde{X}$, *let us define the linear operators* $\mathcal{T}_k : D(\mathcal{T}_k) \to Y$ *by and* $\mathcal{T}_k x := T_k x$, $x \in D(\mathcal{T}_k)$, *for every* $k \in \mathbb{N}$.

1. *The sequence* $(T_k)_{k\in\mathbb{N}}$ *is* $\widetilde{X}$-distributionally chaotic *if, and only if, the sequence* $(\mathcal{T}_k)_{k\in\mathbb{N}}$ *is distributionally chaotic.*
2. *A vector* x *is an* $\widetilde{X}$-distributionally irregular vector *for the sequence* $(T_k)_{k\in\mathbb{N}}$ *if, and only if, the vector* x *is a distributionally irregular vector for the sequence* $(\mathcal{T}_k)_{k\in\mathbb{N}}$.
3. *A linear manifold* X' *is a* (uniformly) $\widetilde{X}$-distributionally irregular manifold for the sequence $(T_k)_{k\in\mathbb{N}}$ *if, and only if, the manifold* X' *is a (uniformly) distributionally irregular manifold for the sequence* $(\mathcal{T}_k)_{k\in\mathbb{N}}$.

Observe that there exist some important cases in which the sequence of linear mappings $(T_{k|\widetilde{X}})_{k\in\mathbb{N}}$, acting between the spaces $\widetilde{X}$ and $Y = \widetilde{X}$, is distributionally chaotic; see e.g. the proof of implication (i) $\Rightarrow$ (ii) of [18, Th. 12]. There, the set $\widetilde{X}$ is taken as the closure of $X_0 := \{x \in X : \lim_{n\in A, n\to\infty} T^n x = 0\}$, with T a linear and continuous operator satisfying the *Distributional Chaos Criterion (DCC)*, e.g. [18, Sec. 2]. The next theorem includes the corresponding version of the (DCC) for sequences of operators between Fréchet spaces.

Theorem 3.7. *Let* $(T_k)_{k\in\mathbb{N}}$ *be a sequence in* $L(X, Y)$ *and let* X_0 *be a dense linear subspace of* X.

1. Distributional Chaos Criterion (DCC): *If the following conditions hold:*
   (a) *for every* $x \in X_0$ *there exists* $\mathcal{A}_x \subseteq \mathbb{N}$ *with* $\overline{\operatorname{dens}}(\mathcal{A}_x) = 1$ *and* $\lim_{k\in\mathcal{A}_x, k\to\infty} T_k x = 0$.
   (b) *there exists a zero sequence* $(y_k)_k \subset X$, $\varepsilon > 0$ *and a strictly increasing sequence* $(N_k)_{k\in\mathbb{N}}$ *in* $\mathbb{N}$ *such that for every* $k \in \mathbb{N}$ *we have*

$$\frac{\operatorname{card}(\{1 \leq j \leq N_k : d_Y\big(T_j y_k, 0\big) > \epsilon\})}{N_k} \geq 1 - \frac{1}{k}, \tag{21}$$

   then there exists a distributionally irregular vector for the sequence $(T_k)_{k\in\mathbb{N}}$.
2. *If the following conditions hold:*
   (a) $\lim_{k\to\infty} T_k x = 0$, *for every* $x \in X_0$,
   (b) *there exists* $x \in X$ *such that its orbit under* $(T_k)_{k\in\mathbb{N}}$ *is distributionally unbounded,*
   then there exists a dense uniformly distributionally irregular manifold for $(T_k)_{k\in\mathbb{N}}$.

In particular, in both cases the sequence $(T_k)_{k\in\mathbb{N}}$ *is distributionally chaotic.*

Proof. We will only outline the most relevant details of the proof. If Y is a Fréchet space, the assertion 3.7.1 can be simply proved by replacing the powers of the operator T by the corresponding operators T_k, $k \in \mathbb{N}$, throughout the proofs of [18, Prop. 7 and 9].

A careful inspection of the proof of [18, Th. 15] shows that the assertion 3.7.2 holds provided that X and Y were Fréchet spaces and $p_m^Y(T_i x) \leq p_{i+m}(x)$, for every $x \in X, i, m \in \mathbb{N}$. We point out that we can always construct a fundamental system $(p'_n(\cdot))_{n\in\mathbb{N}}$ of increasing seminorms on the space X, inducing the same topology, so that this condition holds, i.e. $p^Y_m(T_i x) \leq p'_{i+m}(x)$, for $x \in X$, and $i, m \in \mathbb{N}$. Therefore, we only have to replace the powers of the operator, T^k, appearing in the proof of [18, Th. 15], by the corresponding operators T_k, $k \in \mathbb{N}$. In such a way, we may conclude that the assertion [18, Th. 15] holds provided that X and Y were Fréchet spaces. □

We present three examples of operators that generate C_0-semigroups which have already been considered in the theory of linear dynamics.

Example 3.8. [29, Ex. 4.2] *Let $a, b, c > 0$, $c < \frac{b^2}{2a} < 1$, and*

$$\Lambda := \left\{ \lambda \in \mathbb{C} : \left| \lambda - \left(c - \frac{b^2}{4a} \right) \right| \leq \frac{b^2}{4a},\ \Im\lambda \neq 0 \text{ if } \Re\lambda \leq c - \frac{b^2}{4a} \right\}. \tag{22}$$

Consider the operator $-A : D(A) \to L^2(\mathbb{R}^+_0)$ defined by $-Au := au_{xx} + bu_x + cu$, with $D(-A) := \{f \in W^{2,2}(\mathbb{R}^+_0) : f(0) = 0\}$, being $W^{2,2}(\mathbb{R}^+_0)$ a Sobolev space. Let $P(z) = \sum_{j=0}^n a_j z^j$ be a non-constant complex polynomial such that $a_n > 0$ and

$$P(-\Lambda) \ \cap\ \left\{ z \in \mathbb{C} : |z| = 1 \right\} \ \neq \emptyset. \tag{23}$$

Then $-P(A)$ generates an analytic C_0-semigroup of angle $\pi/2$, $P(-\Lambda) \subseteq \sigma_p(P(A))$ and $P(A)$ is a linear and continuous operator [36] *that is densely distributionally chaotic applying Theorem 3.7.2 and Remark 3.5.*

Example 3.9. [21, 52] *Suppose $c > b/2 > 0$, $\Omega := \{\lambda \in \mathbb{C} : \Re\lambda < c - b/2\}$ and $A_c u := u'' + 2bxu' + cu$ is the bounded perturbation of the one-dimensional Ornstein-Uhlenbeck operator defined from $D(A_c) := \{u \in L^2(\mathbb{R}) \cap W^{2,2}_{loc}(\mathbb{R}) : A_c u \in L^2(\mathbb{R})\}$ on $L^2(\mathbb{R})$. Denote by $\mathcal{F}$ and $\mathcal{F}^{-1}$ the Fourier transform on the real line and its inverse transform, respectively. Then A_c generates a C_0-semigroup with $\Omega \subseteq \sigma_p(A_c)$.*

Moreover, for any open connected subset $\Omega' \subseteq \Omega$ which admits a cluster point in Ω, one has that the set $E = \operatorname{span}\{g_i(\lambda) : \lambda \in \Omega',\ i = 1, 2\}$, where $g_1 : \Omega \to X$ and $g_2 : \Omega \to X$ are defined by $g_1(\lambda) := \mathcal{F}^{-1}(e^{-\frac{\xi^2}{2b}} \xi |\xi|^{-(2+\frac{\lambda - c}{b})})(\cdot)$ and $g_2(\lambda) := \mathcal{F}^{-1}(e^{-\frac{\xi^2}{2b}} |\xi|^{-(1+\frac{\lambda - c}{b})})(\cdot)$ for every $\lambda \in \Omega$, is dense in $L^2(\mathbb{R})$. In the same way as in the previous example, the operator A_c is densely distributionally chaotic. Besides, this property also holds for the multi-dimensional Ornstein-Uhlenbeck operators considered in [21, Sec. 4].

Example 3.10. [32] *Suppose $r > 0, \sigma > 0, \nu = \sigma/\sqrt{2}, \gamma = r/\mu - \mu, s > 1, s\nu > 1$ and $\tau \geq 0$. Set*

$$Y^{s,\tau} := \left\{ u \in C((0,\infty)) : \lim_{x\to 0} \frac{u(x)}{1 + x^{-\tau}} = \lim_{x\to\infty} \frac{u(x)}{1 + x^s} = 0 \right\}. \tag{24}$$

endowed with the norm

$$\|u\|_{s,\tau} := \sup_{x>0} \left| \frac{u(x)}{\left(1 + x^{-\tau}\right)\left(1 + x^s\right)} \right|,\ \textit{for every } u \in Y^{s,\tau}, \tag{25}$$

becomes a separable Banach space. Let $D_\mu := \nu x d/dx$, with maximal domain in $Y^{s,\tau}$, and let the Black-Scholes operator B be defined by $B := D^2_\nu + \gamma D_\mu - r$. The operator B generates a Devaney chaotic C_0-semigroup, see [32, Th. 2.6]. *Moreover, by* [32, Lem. 3.3], *the proof of* [32, Lem. 3.5] *(especially the Figure 1 in the abovementioned paper, in the $Ox'y'$ coordinate system, with $x' = x/\nu$ and $y' = y/\nu$) and the previous consideration, it readily follows that the operator B is densely distributionally chaotic.*

By [27, Th. 2.3] and the proof of [3, Th. 2.1], it is not difficult to see that the operators considered in the previous example are also chaotic (in the sense of [27, Def. 2.1]). Recently, Menet has shown the existence of a Devaney chaotic operator that is not distributionally chaotic [51].

Motivated by the research of Bès et al [19], where the chaotic behaviour of the abstract Laplace operator Δ has been analyzed, we propose the following problem:

Problem 3.11. *Suppose $1 \leq p < \infty$, $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is an open (possibly unbounded) set, and the operator Δ acts on $L^p(\Omega)$ with maximal distributional domain and without any boundary conditions. Is it true that Δ is densely distributionally chaotic?*

Suppose now that $T : D(T) \subseteq X \to X$ is a linear mapping, $C \in L(X)$ is an injective operator with range $R(C)$, satisfying

$$R(C) \subseteq D_\infty(T) \text{ and } T^n C \in L(X) \text{ for all } n \in \mathbb{N}. \tag{26}$$

We also recall that the C-resolvent set of A, denoted by $\rho_C(A)$, is defined by

$$\rho_C(A) := \{\lambda \in \mathbb{C} : \lambda - A \text{ is injective and } (\lambda - A)^{-1}C \in L(X)\}. \tag{27}$$

The conditions in (26) imply that for every $n \in \mathbb{N}$ the mapping $T_n : R(C) \to X$ defined by $T_n(Cx) := T^n Cx$, with $x \in X$, is an element of the space $L([R(C)], X)$. By Theorem 3.7, we immediately obtain the following result.

Corollary 3.12. *Let T and C be linear operators satisfying condition (26) and let X_0 be a dense linear subspace of X.*

1. *If the following conditions hold:*
   (a) *for every $x \in X_0$ there exists $\mathcal{A}_x \subseteq \mathbb{N}$ with $\overline{\text{dens}}(\mathcal{A}_x) = 1$ and $\lim_{k \in \mathcal{A}_x, k \to \infty} T^k C(x) = 0$.*
   (b) *there exists a zero sequence $(z_k)_k \subset X$, $\varepsilon > 0$ and a strictly increasing sequence $(N_k)_{k \in \mathbb{N}}$ in $\mathbb{N}$ such that for every $k \in \mathbb{N}$ we have*

$$\frac{\text{card}(\{1 \leq j \leq N_k : d_Y\big(T^j C z_k, 0\big) > \epsilon\})}{N_k} \geq 1 - \frac{1}{k}, \tag{28}$$

   then there exists a distributionally irregular vector $x \in R(C)$ for the operator T. In particular, T is distributionally chaotic and a σ-scrambled set S of T can be chosen to be a linear submanifold of $R(C)$.
2. *If the following conditions hold:*
   (a) *$\lim_{k \to \infty} T^k Cx = 0$, for every $x \in X_0$,*
   (b) *there exist $x \in X$, $m \in \mathbb{N}$ and a set $B \subseteq \mathbb{N}$ such that*

$$\overline{\text{dens}}(\mathcal{B}) = 1, \qquad \text{and} \qquad \lim_{k \in \mathcal{B}, k \to \infty} p_m(T^k Cx) = \infty, \tag{29}$$

   then there exists a uniformly distributionally irregular manifold W for the operator T. In particular T is distributionally chaotic. Furthermore, if $R(C)$ is dense in X, then W can be chosen to be dense in X and T is densely distributionally chaotic.

This previous result can be used in order to get that some differential operators that are known to generate certain C_0-semigroups are in fact distributionally chaotic. We state the following result in this line.

Theorem 3.13. *Suppose that $\mathbb{K} = \mathbb{C}$, $C \in L(X)$, and $(A, D(A))$ is a closed linear operator such that and $D(A)$ is dense in X. Let $z_0 \in \mathbb{C} \setminus \{0\}$, $\beta \geq -1$, $m \in (0, 1)$, $0 < \varepsilon < d \leq 1$, $\gamma > -1$, and consider the following regions in the complex plane:*

$$P_{\beta,\varepsilon,m} := \{\xi + i\eta : \xi \geq \varepsilon,\ \eta \in \mathbb{R}, |\eta| \leq m(1 + \xi)^{-\beta}\} \tag{30}$$

and also

$$P_{z_0,\beta,\varepsilon,m} := e^{i\arg(z_0)}\big(|z_0| + (P_{\beta,\varepsilon,m} \cup B_d)\big), \tag{31}$$

where B_d is the open disk of center 0 and radius d, such that

(i) $P_{z_0,\beta,\varepsilon,m} \subseteq \rho_C(A)$.
(ii) *The family $\{(1+|\lambda|)^{-\gamma}(\lambda - A)^{-1}C : \lambda \in P_{z_0,\beta,\varepsilon,m}\}$ is equicontinuous, and*
(iii) *for every fixed $x \in E$ the mapping $\lambda \mapsto (\lambda - A)^{-1}Cx$, with $\lambda \in P_{z_0,\beta,\varepsilon,m}$, is continuous.*

In addition, if there exist a dense subset X_0 of X such that $\lim_{k\to\infty} A^k x = 0$ for all $x \in X_0$, and $\lambda \in \sigma_p(A)$ with $|\lambda| > 1$, then the operator μA^n is densely distributionally chaotic on A for all $n \in \mathbb{N}$ and $|\mu| = 1$.

Proof. Let us denote the set $\partial P_{z_0,\beta,\varepsilon,m}$ by $\Gamma(z_0,\beta,\varepsilon,m)$, or simply Γ if there is no confusion. This is a continuous piecewise smooth curve that can be defined as $\Gamma = \cup_{i=1}^3 \Gamma_i$, where:

- $\Gamma_1 = \{e^{i\arg(z_0)}(|z_0| + \xi + i\eta) : \xi \geq \varepsilon, \eta = -m(1+\xi)^{-\beta}\}$,
- $\Gamma_2 = \{e^{i\arg(z_0)}(|z_0| + \xi + i\eta) : \xi^2 + \eta^2 = d^2, \xi \leq \varepsilon\}$, and
- $\Gamma_3 = \{e^{i\arg(z_0)}(|z_0| + \xi + i\eta) : \xi \geq \varepsilon, \eta = m(1+\xi)^{-\beta}\}$.

are curves in the complex plane. We can give an orientation to Γ such that the imaginary part of λ decreases along $e^{-i\arg(z_0)}\Gamma_1$. Figure 1 illustrates an example of such a region.

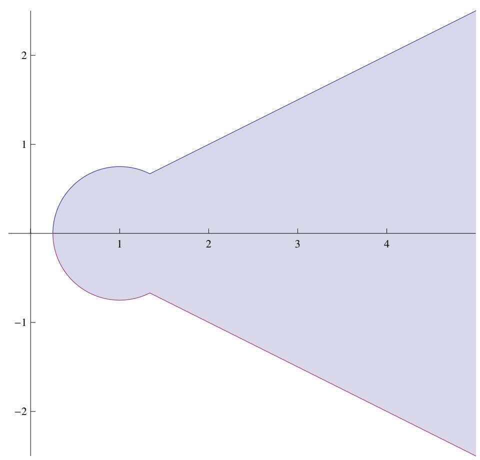

FIGURE 1. The set $P_{1,-1,1/4,1/2}$.

Let us define $\Sigma_\delta := \{\lambda \in \mathbb{C} : \lambda \neq 0, |\arg(\lambda)| < \delta\}$ for $\delta \in (0,\pi]$. Let $b \in (0,1/2)$ be fixed and we set $\delta_b := \arctan(\cos \pi b)$ and $A_0 := e^{-i\arg(z_0)}A - |z_0|$. Now we define

$$C_b(z)x := \frac{1}{2\pi i}\int_\Gamma e^{-z(-\lambda)^b}(\lambda - A_0)^{-1}Cx\, d\lambda,\ x \in X. \tag{32}$$

for every $z \in \Sigma_{\delta_b}$. The operator $C_b(z) \in L(X)$ is injective and has dense range in X for any $z \in \Sigma_{\delta_b}$ (see e.g. the proofs of [46, Th. 3.15 and 3.16]). Furthermore, $C_b(z)A \subseteq AC_b(z)$, $z \in \Sigma_{\delta_b}$ and the condition (26) holds with T and C replaced respectively by A and $C_b(z)$, for all $z \in \Sigma_{\delta_b}$.

Densely distributionally chaotic operators are rotationally invariant, therefore it suffices to show that the operator A^n is densely distributionally chaotic. On the one hand, since $\lambda^n \in \sigma_p(A^n)$, then there exists $x \in X \setminus \{0\}$ such that $A^n x = \lambda^n x$, $n \in \mathbb{N}$. On the other hand, $C_b(z)$ is injective, therefore $C_b(z)x \neq 0$ and there exists $m \in \mathbb{N}$ such that $p_m(C_b(z)x) \neq 0$.

Since $C_b(z)$ commutes with A, we have

$$\lim_{k\to\infty} p_m(A^{nk}C_b(z)x) = \lim_{k\to\infty} \lambda^{nk} p_m(C_b(z)x) = +\infty.$$

One can similarly prove that the condition 3.12.2(a) is satisfied and then the hypothesis of Corollary 3.12.2 hold, so that A is distributionally chaotic. In particular, there exists a uniformly distributionally irregular manifold W for the operator A. □

4. Distributional chaos for C_0-semigroups

The dynamical notions presented in Definitions 3.1 and 3.4 can also be stated to C_0-semigroups with slightly modifications. We omit the details here. In the following statements we deepen into (densely) distributionally chaotic properties of C_0-semigroups. The following continuous version of Corollary 3.12.2 will be essentially used in our further analysis of distributionally chaotic semigroups and α-times integrated semigroups.

Theorem 4.1. *Suppose that X_0 is a dense linear subspace of X, $(T(t))_{t\geq 0} \subseteq L(X,Y)$ is a strongly continuous operator family, satisfying:*

(a) $\lim_{t\to\infty} T(t)x = 0$, *for every $x \in X_0$,*

(b) *there exist $x \in X$, $m \in \mathbb{N}$ and a set $\mathcal{B}_x \subseteq \mathbb{R}^+$ such that $\overline{\text{Dens}}(\mathcal{B}_x) = 1$ and*

$$\lim_{t\in\mathcal{B}_x, t\to\infty} p_m(T(t)x) = \infty, \tag{33}$$

then $(T(t))_{t\geq 0}$ is a distributionally chaotic strongly continuous operator family.

Proof. The proof is very similar to those of [18, Th. 15] and Corollary 3.12.2. The family $(T(t))_{t\geq 0} \subseteq L(X,Y)$ is locally equicontinuous because it is strongly continuous and X is barrelled ([43, Prop. 1.1]). Hence, for every $l, n \in \mathbb{N}$, there exist $c_{l,n} > 0$ and $a_{l,n} \in \mathbb{N}$ such that $p_l^Y(T(t)x) \leq c_{l,n} p_{a_{l,n}}(x)$, for every $x \in X$ and $t \in [0,n]$. Let us suppose that

$$p_k^Y(T(t)x) \leq p_{k+\lceil t\rceil}(x), \quad x \in X,\ t \geq 0,\ k \in \mathbb{N}. \tag{34}$$

This can be assumed if we introduce the following fundamental system of increasing seminorms $p_n'(\cdot)$, $n \in \mathbb{N}$, on X:

$$\begin{aligned} p_1'(x) &\equiv p_1(x), \quad x \in X, \\ p_2'(x) &\equiv p_1'(x) + c_{1,1}p_{a_{1,1}}(x) + p_2(x), \quad x \in X, \\ &\dots \\ p_{n+1}'(x) &\equiv p_n'(x) + c_{1,n}p_{a_{1,n}}(x) + \cdots + c_{n,1}p_{a_{n,1}}(x) + p_{n+1}(x), \quad x \in X, \\ &\dots \end{aligned} \tag{35}$$

Without loss of generality, we may assume $m = 1$. Then one can find a sequence $(x_k)_{k\in\mathbb{N}}$ in X_0 such that $p_k(x_k) \leq 1$, $k \in \mathbb{N}$, and a strictly increasing sequence of positive real numbers $(t_k)_{k\in\mathbb{N}}$ tending to infinity such that:

$$\frac{\overline{\text{Dens}}(\{1 \leq s \leq t_k : p_1(T(s)x_k) > k2^k\})}{t_k} \geq 1 - \frac{1}{k^2} \tag{36}$$

and

$$\frac{\overline{\text{Dens}}(\{1 \leq s \leq t_k : p_k(T(s)x_l) < k^{-1}\})}{t_k} \geq 1 - \frac{1}{k^2}, l = 1, \dots, k-1. \tag{37}$$

Let $(r_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{N}$ satisfying:

$$r_{j+1} \geq 1 + r_j + \lceil t_{r_j+1} \rceil, \quad \text{for every } j \in \mathbb{N}. \tag{38}$$

Repeating literally the argumentation used in the remaining part of the proof of [18, Th. 15], we obtain the existence of a dense subspace S of X such that, for every $x \in S$, there exist two sets $\mathcal{A}_x,\ \mathcal{B}_x \subseteq [0, \infty)$ such that $\overline{\text{Dens}}(\mathcal{A}_x) = \overline{\text{Dens}}(\mathcal{B}_x) = 1$, and

$$\lim_{t\to\infty, t\in\mathcal{A}_x} T(t)x = 0 \quad \text{and} \quad \lim_{t\to\infty, t\in\mathcal{B}_x} p_1(T(t)x) = \infty, \tag{39}$$

Then the conclusion yields as in the discrete case.

□

The next result is a characterization of distributionally chaotic C_0-semigroups on Fréchet spaces from several results already known to hold in the Banach case.

Theorem 4.2. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on X. The following are equivalent:*

1. *$(T(t))_{t\geq 0}$ is distributionally chaotic.*
2. *$T(t_0)$ is distributionally chaotic for every $t_0 > 0$.*
3. *$T(t_0)$ is distributionally chaotic for some $t_0 > 0$.*
4. *$T(t_0)$ satisfies the (DCC) for single operators for every $t_0 > 0$, i.e., there exist two sequences $(x_k)_{k\in\mathbb{N}}, (y_k)_{k\in\mathbb{N}} \subseteq X$, $\varepsilon > 0$, a strictly increasing sequence $(N_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$, and a subset $\mathcal{A} \subseteq \mathbb{N}$ with $\overline{\text{dens}}(\mathcal{A}) = 1$ such that:*
   (a) $\lim_{n\in\mathcal{A}, n\to\infty} T(t_0)^n x_k = 0$, *for every* $k \in \mathbb{N}$.
   (b) $(y_k)_{k\in\mathbb{N}} \subseteq \overline{\text{span}\{x_l : l \in \mathbb{N}\}}$, $\lim_{k\to\infty} y_k = 0$ *and*
$$\frac{\text{card}(\{1 \leq j \leq N_k : d(T^j y_k, 0) > \epsilon\})}{N_k} \geq 1 - \frac{1}{k}, \quad \textit{for every } k \in \mathbb{N}. \tag{40}$$
5. *$T(t_0)$ satisfies the (DCC) for single operators for some $t_0 > 0$.*
6. *$(T(t))_{t\geq 0}$ satisfies the (DCC) for semigroups, i.e., there exist two sequences $(x_k)_{k\in\mathbb{N}}, (y_k)_{k\in\mathbb{N}} \subseteq X$, $\varepsilon > 0$, a strictly increasing sequence $(\rho_k) \subseteq \mathbb{R}^+$, and a set $\mathcal{D} \subseteq \mathbb{R}^+$ with $\overline{\text{Dens}}(\mathcal{D}) = 1$ such that:*
   (a) $\lim_{s\in\mathcal{D}, s\to\infty} T(s)x_k = 0$, $k \in \mathbb{N}$.
   (b) $y_k \in \overline{\text{span}\{x_l : l \in \mathbb{N}\}}$, $\lim_{k\to\infty} y_k = 0$ *and*
$$\frac{m(\{s \in [0, \rho_k] : d(T(s)y_k, 0) > \epsilon\})}{\rho_k} \geq 1 - \frac{1}{k}, \quad \textit{for every } k \in \mathbb{N}. \tag{41}$$
7. *$T(t_0)$ has a distributionally irregular vector for every $t_0 > 0$.*
8. *$T(t_0)$ has a distributionally irregular vector for some $t_0 > 0$.*
9. *$(T(t))_{t\geq 0}$ has a distributionally irregular vector.*

Proof. Using the fact that $(T(t))_{t\geq 0}$ is locally equicontinuous, the proofs of Lemma 2.4, Theorem 3.1, Proposition 1, Proposition 3, Remark 2 and Theorem 3.4 in [1] can be slightly modified replacing norms by seminorms, and obtaining the corresponding results for C_0-semigroups defined on Fréchet spaces.

In ([18, Th. 12]); it is proved that statements $2, 4$, and 7 are equivalent, and also $3, 5$, and 8. Extending the results proved in [1] to Fréchet spaces as mentioned above, we have that $1, 2, 5, 6$ and 9 are equivalent, too. More specifically, the equivalence between $1, 6$, and 9 is proved in Theorem 3.4 of [1] and the other equivalences are contained in Proposition 1 of [1].

□

We wish to observe that Theorem 4.2, in combination with [18, Th. 16] and the proof of [22, Th. 5], implies that the assertion of [8, Th. 3.7] continues to hold in the setting of separable Fréchet spaces.

Theorem 4.3. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup in $L(X)$ with infinitesimal generator $(A, D(A))$. If the following conditions hold:*

1. *there is a dense subset $X_0 \subseteq X$ with $\lim_{t\to\infty} T(t)x = 0$, for each $x \in X_0$, and*
2. *there is some $\lambda \in \sigma_p(A)$ with $\Re(\lambda) > 0$,*

then $(T(t))_{t\geq 0}$ has a dense distributionally irregular manifold. In particular, $(T(t))_{t\geq 0}$ is distributionally chaotic.

In the case that $\mathbb{K} = \mathbb{R}$, condition (2) should be restated by assuming that the corresponding eigenvalue $\lambda \in \sigma_p(A)$ satisfies $\lambda > 0$.

Example 4.4. *Let us consider the translation semigroup $(T(t))_{t\geq 0}$ on the space $C^m([0,\infty),\mathbb{K})$, where $m \in \mathbb{N}_0 \cup \{\infty\}$. Since the point spectrum of the generator d/dx of this semigroup equals to $\mathbb{K}$, then $(T(t))_{t\geq 0}$ is densely distributionally chaotic.*

Throughout the rest of this section we provide a spectral criterion that implies the existence of rescaled distributionally chaotic C_0-semigroups. In particular, given a C_0-semigroup with infinitesimal generator A acting on a complex Fréchet space X, we provide a spectral criterion that implies that for each operator B in the commutant of A there exists $m \in \mathbb{C}$ such that the rescaled semigroup $e^{mt}e^{Bt}$ is distributionally chaotic. Chaotic properties of some rescaled C_0-semigroups have been studied before in [21]. Operators with hypercyclic commutant have been widely studied. Since the seminal paper [35], different questions related to this problem have been considered, see for example [15, 40, 53]. The next result is inspired in a similar one stated for single operators [37, Th. 2.1]. Although both proofs run parallel, we include the proof of the next result for the sake of completeness.

Theorem 4.5. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a complex space X with infinitesimal generator $A \in L(X)$. Suppose that the spectrum of A contains a non-empty connected open bounded set $U \subset \mathbb{C}$ such that the following conditions hold:*

(a) *Every $\lambda \in U$ is a simple eigenvalue of A.*
(b) $\operatorname{span}\{\ker(A-\lambda I) : \lambda \in U\}$ *is dense in X.*
(c) *There exists a holomorphic function $\hat{x} : U \to X$ defined as $\hat{x}(\lambda) = x_\lambda$ for every $\lambda \in U$, such that $0 \neq x_\lambda \in \ker(A - \lambda I)$.*

Then for all B in the commutant of A there exists $m \in \mathbb{C}$ such that the rescaled C_0-semigroup $(e^{(mI+B)t})_{t\geq 0}$ is distributionally chaotic.

Proof. First let us see that given a non-empty open subset $V \subset U$, the set $\operatorname{span}\{\ker(A-\lambda I) : \lambda \in V\}$ remains dense in X. Otherwise, there exists $x^* \neq 0$ in X^* such that $x^*(\hat{x}(\lambda)) = 0$ for all $\lambda \in V$. By the analyticity condition (c) we have that $x^*(\hat{x}(\lambda)) = 0$ for all $\lambda \in U$, and by (a) and (b) we have $x^* = 0$, a contradiction. Let $B \in L(X)$ be in the commutant of A with $B \neq \mu I$. For each $\lambda \in U$ we have $Ax_\lambda = \lambda x_\lambda$, hence $ABx_\lambda = \lambda Bx_\lambda$. By condition (a) we have $Bx_\lambda = b(\lambda)x_\lambda$ for some complex number $b(\lambda)$. In this way we can define $b : U \to \mathbb{C}$ such that $\lambda \to b(\lambda)$. This function b is non constant since $B \neq \mu I$.

Let us see that $b : U \to \mathbb{C}$ is holomorphic. For each $0 \neq y^* \in X^*$ our function b satisfies

$$b(\lambda) = \frac{y^*(B(x_\lambda))}{y^*(x_\lambda)}. \tag{42}$$

Thus it is holomorphic on $U \setminus Z$, where $Z = \{\lambda : y^*(x_\lambda) = 0\}$ is a discrete subset of U. But for $\lambda_0 \in Z$ we can take $u^* \in X^*$ such that $u^*(x_{\lambda_0}) \neq 0$. This implies that $b(\lambda)$ is holomorphic on λ_0, hence $b(\lambda)$ is holomorphic on U. Since $b(\lambda)$ is non-constant, $b(U)$ is an open subset of the complex plane. Then for some $m \in \mathbb{C}$, the set $m + b(U)$ intersects the imaginary axis $i\mathbb{R}$. We will use Theorem 2.1 in order to prove that $e^{(mI+B)t}$ is distributionally chaotic. It is sufficient to check that the subsets:

$$\operatorname{span}\{\ker(mI + B - \lambda I) : \Re(\lambda) < 0\} \quad \text{and} \quad \operatorname{span}\{\ker(mI + B - \lambda I) : \Re(\lambda) > 0\} \tag{43}$$

are dense in X. It is clear that

$$\operatorname{span}\{\ker(A - \lambda I) : \lambda \in (b^{-1}\{\mu : \Re(\mu) < 0\} - m)\} \subset \operatorname{span}\{\ker(mI + B) - \lambda I) : \Re(\lambda) < 0\}. \tag{44}$$

Indeed, let $x_\lambda \in \ker(A-\lambda I)$, with $\lambda \in (b^{-1}\{\mu : \Re(\mu) < 0\}-m)$, then $(mI+B)x_\lambda = (m+b(\lambda))x_\lambda$, so we have that $x_\lambda \in \{\ker(mI + B - \lambda I), \Re(\lambda) < 0\}$. By the considerations at the beginning of the proof we know that $\operatorname{span}\{\ker(A - \lambda I) : \lambda \in (b^{-1}\{\mu : \Re(\mu) < 0\} - m)\}$ is dense in X and a similar argument is valid for the other set. We get that the semigroup $(e^{(mI+B)t})_{t\geq 0}$ is distributionally chaotic. □

Remark 4.6. *In the case that the simplicity of the eigenvalues in U fails, the same conclusions hold for operators in the commutant of A such that $\{\ker(A - \lambda I) : \lambda \in U\} \subset \{\ker(B - \mu I) : \mu \in \mathbb{C}\}$.*

Now we show other three examples that illustrate Theorem 4.5.

Example 4.7. *Let B_w be a bilateral weighted forward shift defined by $B_w(e_n) = w_n e_{n+1}$ where $\{e_n : n \in \mathbb{Z}\}$ is the natural orthonormal basis of $\ell_p(\mathbb{Z})$, $1 \leq p < \infty$ and (w_n) is a bounded sequence of strictly positive numbers. In* [37], *it is proved that if $r_3^+(B_w) < r_2^+(B_w)$, where*

$$r_3^+ := \limsup_{n\to\infty} (w_0 \dots w_{n-1})^{\frac{1}{n}} \tag{45}$$

$$r_2^+ := \liminf_{n\to\infty} (w_{-1} \dots w_{-n})^{\frac{1}{n}}, \tag{46}$$

then B_w satisfies the hypothesis of Theorem 4.5. Therefore, for every operator A in the commutant of B_w there exists m, such that the rescaled C_0-semigroup $(e^{(mI+A)t})_{t\geq 0}$ is Devaney and distributionally chaotic.

Example 4.8. *Let B_w be a unilateral weighted backward shift on the complex Hilbert space of sequences ℓ_2. Take its natural orthonormal basis $\{e_n : n \in \mathbb{N}\}$ and a bounded sequence of strictly positive numbers $(w_n)_n$, and define the weigthed backward shift operator as $B(e_n) = w_{n-1}e_{n-1}$, if $n \geq 2$, and $Be_1 = 0$.*

In [37], *it is proved that if $r_2(B_w) > 0$, where*

$$r_2(B_w) := \liminf_{n\to\infty} (w_1 \dots w_n)^{\frac{1}{n}}, \tag{47}$$

then B_w satisfies the hypothesis of Theorem 4.5. Then for every operator A in the commutant of B_w there exists m, such that the rescaled C_0-semigroup $(e^{(mI+A)t})_{t\geq 0}$ is Devaney and distributionally chaotic.

Example 4.9. *Let us consider the following infinite system of ODE's associated with a linear kinetic model, see for instance* [4, 5, 24]

$$\begin{cases} \frac{\partial f_n}{\partial t} = -\alpha_n f_n + \beta_n f_{n+1}, & n \geq 1, \\ f_n(0) = a_n, & n \geq 1 \end{cases} \tag{48}$$

where $(\alpha_n)_n$ and $(\beta_n)_n$ are bounded positive sequences and $(a_n)_n \in \ell_1$ is a real sequence. Consider the operator $A \in L(\ell_1)$ defined as

$$Af = (-\alpha_n f_n + \beta_n f_{n+1})_n \text{ for every } f = (f_n)_n \in \ell_1. \tag{49}$$

The operator A generates a C_0- semigroup $(T(t))_{t\geq 0}$ which is solution of 48. It is shown in [39], *that if*

$$\alpha := \sup_{n\geq 1} \alpha_n < \liminf_{n\to\infty} \beta_n := \beta, \tag{50}$$

then the generator A satisfies (b) and (c) of theorem 4.5. Let us take $\alpha/2 < r < \beta/2$. In order to prove (b) and (c) let us consider the open disk of radius r centered at $-\alpha/2$, namely $U \subset \mathbb{C}$, that intersects the imaginary axis. Given $f = (f_n)_n \in \ell_1$ such that $Af = \lambda f, \lambda \in U$, we get that

$$f_n = \gamma f_1 \text{ with } \gamma_n = \prod_{k=1}^{n-1} \frac{\lambda + \alpha_k}{\beta_k}, \text{ for every } n \geq 2, \text{ and } \gamma_1 = 1. \tag{51}$$

It is clear that if there exists $g = (g_n)_n \in \ell_1$ such that $Ag = \lambda g$, then there exists $\mu \in \mathbb{C}$ such that $f = \mu g$, and then all the eigenvalues in U are simple. As a result, for every operator B in the commutant of A there exists m, such that the rescaled C_0-semigroup $(e^{(mI+B)t})_{t\geq 0}$ is Devaney and distributionally chaotic.

Let us calculate explicitly some element B in the commutant of A. Let B be of the form

$$B = \begin{pmatrix} a_1 & b_1 & 0 & & \\ c_2 & a_2 & b_2 & 0 & \\ 0 & c_3 & a_3 & b_3 & \ddots \\ & 0 & c_4 & a_4 & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{52}$$

As $AB = BA$, solving the system of equations we get that:

$$\begin{aligned} &b_i := b_1 \tfrac{\beta_i}{\beta_1}, \text{ for every } i \geq 1 \\ &a_{i+1} = a_i - \tfrac{b_1}{\beta_1}(\alpha_{i+1} - \alpha_i), \text{ for every } i \geq 2 \\ &c_i := 0, \text{ for every } i \geq 2. \end{aligned} \tag{53}$$

As a result, we obtain that for all semigroups whose generator B verifies (53)*, there exists a rescaled semigroup of it which is Devaney and distributionally chaotic.*

5. Distributional chaos for α-times integrated C-semigroups

First of all, we will introduce the definition of α-times integrated C-semigroups, see for instance [48] and [56, Def. 3.1].

Definition 5.1. *Let $\alpha \geq 0$, and let A be a closed linear operator on X. If there exists a strongly continuous operator family $(S_\alpha(t))_{t\geq 0} \subseteq L(X)$ such that:*

1. *$S_\alpha(t)A \subseteq AS_\alpha(t)$, for every $t \geq 0$,*
2. *$S_\alpha(t)C = CS_\alpha(t)$, for every $t \geq 0$,*
3. *for all $x \in X$ and $t \geq 0$ we have*

$$\int_0^t S_\alpha(s)x\, ds \in D(A) \tag{54}$$

and

$$A\int_0^t S_\alpha(s)x\, ds = S_\alpha(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}Cx, \tag{55}$$

then it is said that $(S_\alpha(t))_{t\geq 0}$ *is an* α-times integrated C-semigroup with subgenerator A.

Furthermore, it is said that $(S_\alpha(t))_{t\geq 0}$ *is an* exponentially equicontinuous α-times integrated C-semigroup with subgenerator A *if, in addition, there exists* $\omega \in \mathbb{R}$ *such that*

1' *the family* $\{e^{-\omega t}S_\alpha(t) : t \geq 0\}$ *is equicontinuous,*
2' $(\omega, \infty) \subset \rho_C(A)$ *and*
3'

$$(\lambda - A)^{-1}Cx = \lambda^\alpha \int_0^\infty e^{-\lambda t}S_\alpha(t)x dt, \qquad \textit{for every } x \in X. \tag{56}$$

If $\alpha = 0$, then $(S_0(t))_{t\geq 0}$ is said to be a C-regularized semigroup with subgenerator A. Moreover, if $\alpha \geq 0$ and $(S_\alpha(t))_{t\geq 0}$ is an α-times integrated C-semigroup with subgenerator A, then $(S_\alpha(t))_{t\geq 0}$ is a locally equicontinuous family, i.e. for every $\kappa > 0$ and $n \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and $c > 0$ such that $p_n(S_\alpha(t)x) \leq cp_m(x)$, for every $t \in [0, \kappa]$, and $x \in X$.

The *integral generator* $\widehat{A}$ of $(S_\alpha(t))_{t\geq 0}$ is a closed linear operator that is an extension of any subgenerator of $(S_\alpha(t))_{t\geq 0}$. The graph of an *integral generator* of $(S_\alpha(t))_{t\geq 0}$ is defined as

$$\operatorname{graph}(\widehat{A}) := \left\{(x, y) \in X \times X : S_\alpha(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}Cx = \int_0^t S_\alpha(s)y\, ds, \text{ for every } t \geq 0\right\}. \tag{57}$$

Arguing as in the proofs of [48, Prop. 2.1.6, Prop. 2.1.19], we have that the integral generator $\widehat{A}$ of $(S_\alpha(t))_{t\geq 0}$ is a maximal element for the inclusion relation on the set of admissible subgenerators of $(S_\alpha(t))_{t\geq 0}$. Furthermore, the following equality holds $\widehat{A} = C^{-1}AC$ on X.

Denote by $Z_1(A)$ the space that consists of those elements $x \in X$ for which there exists a unique continuous mapping $u : \mathbb{R}_0^+ \times X \to X$ satisfying $\int_0^t u(s, x)\, ds \in D(A)$ and $A\int_0^t u(s, x)\, ds = u(t, x) - x$, for every $t \geq 0$, i.e., the unique mild solution of the corresponding Abstract Cauchy Problem (ACP_1) :

$$(ACP_1) \begin{cases} u'(t) = Au(t), \ t \geq 0, \\ u(0) = x, \ x \in X. \end{cases} \tag{58}$$

Suppose that A is a subgenerator (the integral generator) of an α-times integrated C-semigroup $(S_\alpha(t))_{t\geq 0}$. There is only one (trivial) mild solution of (58) with $x = 0$, so that $Z_1(A)$ is a linear subspace of X. Moreover, for every $\beta > \alpha$, the operator A is a subgenerator (the integral generator) of a β-times integrated C-semigroup $(S_\beta(t))_{t>0}$, where $S_\beta(t)x := \frac{1}{\Gamma(\beta-\alpha)}\int_0^t (t-s)^{\beta-\alpha-1}S_\alpha(s)xds$ for every $t > 0, x \in X$. The space $Z_1(A)$ consists exactly of those elements $x \in X$ for which the mapping from $\mathbb{R}_0^+$ on X defined as $t \mapsto C^{-1}S_{\lceil\alpha\rceil}(t)x$, is well defined and $\lceil\alpha\rceil$-times continuously differentiable on $\mathbb{R}_0^+$; see e.g [48]. Set

$$T(t)x := \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}}C^{-1}S_{\lceil\alpha\rceil}(t)x, \text{ for every } t \geq 0, x \in Z_1(A). \tag{59}$$

We have $T(t)(Z_1(A)) \subseteq Z_1(A)$; $T(t)C \subseteq CT(t)$ for every $t \geq 0$ and

$$T(t)T(s)x = T(t+s)x, \qquad \text{for every } t, s \geq 0, x \in Z_1(A). \tag{60}$$

We point out that the Fréchet topology on $Z_1(A)$ induces the following family of seminorms $(p_{l,m})_{l,m\in\mathbb{N}}$, where $p_{l,m}(x) := \sup_{t\in[0,m]} p_l(T(t))x)$, $x \in X$. We also indicate that the solution space $Z_1(A)$ is independent of the choice of $(S_\alpha(t))_{t\geq 0}$ in the following sense: If $C_1 \in L(X)$ is another injective operator with $C_1A \subseteq AC_1$, for every $\gamma \geq 0, x \in X$ and A is a subgenerator (the integral generator) of a γ-times integrated C_1-semigroup $(S_\gamma(t))_{t\geq 0}$, then the mapping $t \mapsto C^{-1}S_{\lceil\alpha\rceil}(t)x$, $t \geq 0$ is well defined and $\lceil\alpha\rceil$-times continuously differentiable on $\mathbb{R}_0^+$ if, and only if, the mapping $t \mapsto C_1^{-1} S^{\lceil\gamma\rceil}(t)x$, $t \geq 0$ is well defined and $\lceil\gamma\rceil$-times continuously differentiable on $\mathbb{R}_0^+$.

To sum up, we have that $u(t;x) := \frac{d^{\lceil\gamma\rceil}}{dt^{\lceil\gamma\rceil}} C_1^{-1} S^{\lceil\gamma\rceil}(t)x$, $t \geq 0$ is a unique mild solution of the corresponding Abstract Cauchy Problem (58). Furthermore, $(S_\alpha(t))_{t\geq 0}$ and $(S_\gamma(t))_{t\geq 0}$ share the same (subspace) distributionally chaotic properties. We state here the corresponding definitions of distributional chaos for α-times integrated C-semigroups.

Definition 5.2. *Let $\widetilde{X}$ be a closed linear subspace of X, and let $\alpha \geq 0$. An α-times integrated C-semigroup $(S_\alpha(t))_{t\geq 0}$ with a subgenerator A is said to be $\widetilde{X}$-distributionally chaotic if there are an uncountable set $\mathcal{S} \subseteq Z_1(A) \cap \widetilde{X}$ and a number $\sigma > 0$ such that for every $\epsilon > 0$ and for every pair $x, y \in \mathcal{S}$ of distinct points we have*

$$\overline{\mathrm{Dens}}(\{s \geq 0 : d(T(s)x, T(s)y) \geq \sigma\}) = 1, \quad \text{and} \quad \overline{\mathrm{Dens}}(\{s \geq 0 : d(T(s)x, T(s)y) < \epsilon\}) = 1, \tag{61}$$

where $(T(t))_{t>0}$ is defined in (59). If, moreover, $\mathcal{S}$ can be chosen to be dense in $\widetilde{X}$, then $(S_\alpha(t))_{t\geq 0}$ is said to be densely $\widetilde{X}$-distributionally chaotic. In the case that $\widetilde{X} = X$, then it is also said that $(S_\alpha(t))_{t\geq 0}$ is (densely) distributionally chaotic.

First, we state some statements concerning distributionally chaotic α-times integrated C-semigroups by passing to the theory of C_0-semigroups.

Suppose that A is the integral generator of an α-times integrated C-semigroup $(S_\alpha(t))_{t\geq 0}$ on X, and $\widetilde{X}$ is a closed linear subspace of $Z_1(A)$, satisfying $T(t)(X') \subseteq \widetilde{X}$, where $(T(t))_{t>0}$ is defined in (59). Denote by $\widetilde{d}(\cdot,\cdot)$ the metric on $\widetilde{X}$ inherited by the invariant translation metric $d(\cdot,\cdot)$ on X.

Arguing as in [45, Sec. 3], we may conclude that $(T(t))_{|\widetilde{X}})_{t\geq 0}$ is a C_0-semigroup on $\widetilde{X}$, with infinitesimal generator $A_{|\widetilde{X}}$. Therefore, as an immediate consequence of Theorem 4.2, we can state the following version of the (DCC):

Theorem 5.3. *Suppose that A is the integral generator of an α-times integrated C-semigroup $(S_\alpha(t))_{t\geq 0}$ on X, and $\widetilde{X}$ is a closed linear subspace of $Z_1(A)$ with the property that $T(t)(\widetilde{X}) \subseteq \widetilde{X}$, for every $t \geq 0$. Then the following are equivalent:*

1. *$(T(t))_{|\widetilde{X}})_{t\geq 0}$ is a distributionally chaotic C_0-semigroup on $\widetilde{X}$.*
2. *There exist two sequences $(x_k)_k$ and $(y_k)_k$ in $\widetilde{X}$, $\epsilon > 0$, a strictly increasing sequence $(\rho_k)_k$ in $\mathbb{R}_+$, and a set $\mathcal{D} \subseteq \mathbb{R}_0^+$ with $\overline{\mathrm{Dens}}(\mathcal{D}) = 1$ such that:*
   a) *$\lim_{s\to\infty, s\in\mathcal{D}} T(s)x_k = 0$ in $\widetilde{X}$ for every $k \in \mathbb{N}$.*
   b) *$y_k \in \overline{\mathrm{span}\{x_l : l \in \mathbb{N}\}}$, $\lim_{k\to\infty} y_k = 0$ in $\widetilde{X}$, and*

$$\frac{\mu(\{s \in [0,\rho_k] : \widetilde{d}(T(s)y_k, 0) > \epsilon\})}{\rho_k} \geq 1 - \frac{1}{k}, \quad k \in \mathbb{N}. \tag{62}$$

Now we are able to extend the assertion of [8, Th. 3.7] to α-times integrated C-semigroups.

Theorem 5.4. *Suppose that $\alpha \geq 0$, $t_0 > 0$ and A is the infinitesimal subgenerator of an α-times integrated C-semigroup $(S_\alpha(t))_{t\geq 0}$ on X. Let $n \geq \lceil\alpha\rceil$, $[C(D(A^n))]$ be separable, and the following conditions hold:*

1. *There exists a dense subset $\widetilde{X_0}$ of $[C(D(A^n))]$ such that $\lim_{t\to\infty} T(t)x = 0$, for every $x \in \widetilde{X_0}$.*
2. *There exist $x \in C(D(A^n))$ and $m \in \mathbb{N}$ such that $\lim_{t\to\infty} p_m(T(t))x) = \infty$.*

Then $(S_\alpha(t))_{t\geq 0}$ and the operator $T(t_0)$ are distributionally chaotic. If, moreover, $R(C)$ and $D(A^n)$ are dense in X, then $(S_\alpha(t))_{t\geq 0}$ and $T(t_0)$ are densely distributionally chaotic.

Proof. First of all, it should be noted that $C(D(A^n)) \subseteq Z_1(A)$; if $x = Cy \in C(D(A^n))$, then for every $t \geq 0$,

$$T(t)x = \frac{d^n}{dt^n}C^{-1}S_n(t)x = \frac{d^n}{dt^n}S_n(t)y = S_n(t)A^n y + \sum_{i=0}^{n-1} g_{n-i}(t)CA^{n-1-i}y;$$

for some continuous functions g_{n-1}, $0 \leq i \leq n-1$, see [44, Prop. 2.3.3]. Therefore, for every $t \geq 0$, we have that the mapping $T(t) : [C(D(A^n))] \to X$ is linear and continuous. Moreover, the family $(T(t))_{t \geq 0} \subseteq L([C(D(A^n))], X)$ is strongly continuous. Define $T_k := T(kt_0) : [C(D(A^n))] \to X$ for every $k \in \mathbb{N}$. Then $(T_k)_{k \in \mathbb{N}} \subseteq L([C(D(A^n))], X)$ and $T_k x = T(t_0)^k x$, for every $x \in C(D(A^n))$, see formula (60). As an application of Theorem 3.7.2 it yields that the operator $T(t_0)$ is distributionally chaotic on $C(D(A^n)$. If $\mathcal{S}_{t_0}$ is a corresponding σ_{t_0}-scrambled set, $R(C)$ and $D(A^n)$ are dense in X, then $\mathcal{S}_{t_0}$ is dense in $[C(D(A^n))]$ and X. So that $T(t_0)$ is densely distributionally chaotic. Using Theorem 4.1, we can similarly prove the corresponding statements for the α-times integrated C-semigroup $(S_\alpha(t))_{t \geq 0}$. □

Remark 5.5. i) *If $R(C)$ and $D(A)$ are dense in X and $\rho_C(A) \neq \emptyset$, then for every $\lambda \in \rho_C(A)$ the set $((\lambda - A)^{-1}C)^k C(D(A^n))$ is dense in X for every $k \in \mathbb{N}_0$.*

ii) *If $\lambda \in \sigma_p(A)$ and $Ax = \lambda x$ for some $x \in X \setminus \{0\}$, then $x \in Z_1(A)$, see for instance* [47, Rem. 26.i)]*). The corresponding mild solution to* (58) *is given by $T(t)x = e^{\lambda t}x$, for every $t \geq 0$. In particular, if $\lambda \in \mathbb{K}_-$, then $\lim_{t\to\infty} T(t)x = 0$ and if $\lambda \in \mathbb{K}_+$, then there exists $m \in \mathbb{N}$ such that $\lim_{t\to\infty} p_m(T(t)x) = \infty$.*

iii) *We recall that a distributionally chaotic C_0-semigroup $(T(t))_{t\geq 0}$ on a Fréchet space shares the corresponding σ-scrambled set with each operator $T(t_0)$; see* [1, Rem. 1]*. It is not clear whether the above continues to hold for α-times integrated C-semigroups considered in Theorem 5.4.*

Now we introduce the following conditions:

H_0: There exist a number $\lambda_0 \in \mathbb{K}_+ \cap \sigma_p(A)$ and a nonempty open connected subset Ω of $\mathbb{K}_-$ such that $\Omega \subseteq \sigma_p(A)$. Let $f : \Omega \to X$ satisfy $f(\lambda) \in \mathrm{Ker}(A - \lambda) \setminus \{0\}$, for $\lambda \in \Omega$; and set $X_\Omega := \mathrm{span}\{f(\lambda) : \lambda \in \Omega\}$.

H_1: H_0 holds with $f(\lambda_0) \in \overline{X_\Omega}$, where we denote by $f(\lambda_0)$ the eigenfunction corresponding to the eigenvalue λ_0.

H_2: H_0 holds with $\overline{X_\Omega} = X$.

Using the ideas in the proof of Desch-Schappacher-Webb criterion for C_0- semigroups, we prove the following lemma. We leave the details of the proof to an interested reader. Unfortunately, it is still not clear how can we prove that $C(X_0)$ is dense in $[C(D(A^n))]$ provided that $n \geq 1$.

Lemma 5.6. *Let $\mathbb{K} = \mathbb{C}$, suppose that the condition H_0 holds with $f : \Omega \to X$ being an analytic function.*

1. *If the assumption $(x^* \circ f)(\lambda) = 0$ for all $\lambda \in \Omega$ and for some $x^* \in X^*$, implies $x^* = 0$, then for each subset Ω' of Ω which admits a cluster point in Ω, we have that* $\mathrm{span}\{f(\lambda) : \lambda \in \Omega'\}$ *is dense in X. In particular, X_Ω is dense in X, H_2 holds and $C(X_\Omega)$ is dense in $[R(C)]$.*
2. *In the case that Ω is an open connected subset of $\mathbb{C}$ which intersects the imaginary axis then the equality $\overline{X_\Omega} = \overline{X_{\Omega'}}$ holds for each subset Ω' of Ω which admits a cluster point in Ω. Hence, condition H_2 holds.*

We have already considered distributionally chaotic properties of α-times integrated C-semigroups. If A subgenerates such a semigroup $(S_\alpha(t))_{t\geq 0}$, and $\lambda \in \rho_C(A)$, then the operator A subgenerates a $((\lambda - A)^{-1}C)^{\lceil \alpha \rceil}$-regularized semigroup, see [48, Prop. 2.3.13] and [48, Cor. 2.1.20]. Therefore, it is natural to formulate the following consequences of Theorem 5.4 for C-regularized semigroups; observe only that the space $[R(C)]$ is always separable provided that X is.

Corollary 5.7. *Let A be the subgenerator of a C-regularized semigroup $(S(t))_{t\geq 0}$ and let $t_0 > 0$.*

1. *Suppose that the following conditions hold:*
   (a) *there exists a dense subset X_0 of X such that $\lim_{t\to\infty} S(t)x = 0$, $x \in X_0$, and*
   (b) *there exist $x \in X$ and $m \in \mathbb{N}$ such that $\lim_{t\to\infty} p_m(S(t)x) = \infty$ ($\lim_{t\to\infty} \|S(t)x\| = \infty$ in the case that X is a Banach space).*

   Then $(S(t))_{t\geq 0}$ and the operator $C^{-1}S(t_0)$ are distributionally chaotic. If $R(C)$ is dense in X, then $(S(t))_{t\geq 0}$ and $C^{-1}S(t_0)$ are densely distributionally chaotic.
2. *If condition (H$_2$) is satisfied, then the conclusions of part 1. still holds.*
3. *Suppose that condition (H$_1$) holds, $\widetilde{X} := \overline{X_\Omega}$, and $C(\widetilde{X}) \subseteq \widetilde{X}$. Then the operator $A_{|\widetilde{X}}$ subgenerates a $C_{|\widetilde{X}}$-regularized semigroup $(S(t)_{|\widetilde{X}})_{t\geq 0}$ on the space $\widetilde{X}$. Furthermore, $(S(t)_{|\widetilde{X}})_{t\geq 0}$ and the operator $C_{|\widetilde{X}}^{-1}S(t_0)_{|\widetilde{X}}$ are distributionally chaotic in the space $\widetilde{X}$. If $R(C_{|\widetilde{X}})$ is dense in $\widetilde{X}$, then $(S(t)_{|\widetilde{X}})_{t\geq 0}$ and $C_{|\widetilde{X}}^{-1}S(t_0)_{|\widetilde{X}}$ are densely distributionally chaotic in the space $\widetilde{X}$.*

Proof. The assertion 1. is an immediate consequence of Theorem 5.4, while the assertion 2. follows from Theorem 5.4 and the equality $S(t)f(\lambda) = e^{\lambda t}Cf(\lambda)$, for every $t \geq 0$ and $\lambda \in \sigma_p(A)$, see Remark 5.5.2. If the assumptions of part 3. hold, then the previous equality in combination with the inclusion $C(\widetilde{X}) \subseteq \widetilde{X}$ implies that $S(t)(\widetilde{X}) \subseteq \widetilde{X}$, for $t \geq 0$. Then it follows that the operator $A_{|\widetilde{X}}$ subgenerates a $C_{|\widetilde{X}}$-regularized semigroup $(S(t)_{|\widetilde{X}})_{t\geq 0}$ on $\widetilde{X}$, which is separable. Furthermore, the condition (H$_2$) holds with the operator $A_{|\widetilde{X}}$. Now the assertion 3. simply follows from 2. □

Before going on, we need the following definition which may be found in [46].

Definition 5.8. *An* entire C-regularized group *is an operator family $(S(z))_{z\in\mathbb{C}}$ on $L(X)$ that satisfies:*

1. *$S(0) = C$*
2. *$S(z+\omega)C = S(z)S(\omega)$ for every $z, \omega \in \mathbb{C}$, and*
3. *the mapping $z \mapsto S(z)x$, with $z \in \mathbb{C}$, is entire for every $x \in X$.*

The integral generator (subgenerator) of $(S(z))_{z\in\mathbb{C}}$ is said to be the integral generator (subgenerator) *of $(S(t))_{t\geq 0}$.*

The next theorem can be applied to the polynomials of generators of strongly continuous semigroups considered in [29], [41], and [48, Ex. 3.3.12]. Its formulation is based in the Desch-Schappacher-Webb criterion, see 2.1. For the sake of clearness we have stated it in full detail, instead of referring to conditions H$_0$–H$_2$.

Theorem 5.9. *Let $\mathbb{K} = \mathbb{C}$, $t_0 > 0$, $\theta \in (0, \frac{\pi}{2})$, $n \in \mathbb{N}$ such that $n(\frac{\pi}{2} - \theta) < \frac{\pi}{2}$, $a_n > 0$, and $a_i \in \mathbb{C}$ for $0 \leq i \leq n-1$. Let us define $p(A) := \sum_{i=0}^{n} a_i A^i$, and suppose that $-A$ generate an exponentially equicontinuous, analytic C_0-semigroup of angle θ.*

1. *Assume that there exists a nonempty open connected subset Ω of $\mathbb{C}$ that satisfies $\sigma_p(-A) \supseteq \Omega$, $p(-\Omega) \subseteq \mathbb{C}_-$, and a complex number $\lambda_0 \in \sigma_p(A)$ such that $\Re(p(\lambda_0)) > 0$. Let $f : \Omega \to X$ be an analytic mapping satisfying $f(\lambda) \in Ker(-A - \lambda) \setminus \{0\}$, for $\lambda \in \Omega$. Assume, also, that if $(x^* \circ f)(\lambda) = 0$, for $\lambda \in \Omega$ and some $x^* \in X^*$, then $x^* = 0$. Then there exists an injective operator $C \in L(X)$, with $R(C)$ being dense in X, such that $p(A)$ generates an entire C-regularized group $(S(z))_{z\in\mathbb{C}}$. Furthermore, $(S(t))_{t\geq 0}$ and the operator $C^{-1}S(t_0)$ are densely distributionally chaotic on X.*
2. *Assume that there exists a nonempty open connected subset Ω of $\mathbb{C}$ that satisfies $\sigma_p(-A) \supseteq \Omega$, $p(-\Omega) \subseteq \mathbb{C}_-$, a complex number $\lambda_0 \in \sigma_p(A)$ such that $\Re(p(\lambda_0)) > 0$, and an element $x_0 \in X \setminus \{0\}$ such that $Ax_0 = \lambda_0 x_0$. Let $f : \Omega \to X$ be an analytic mapping satisfying $f(\lambda) \in \ker(-A - \lambda) \setminus \{0\}$ for every $\lambda \in \Omega$. Let X_Ω be defined as before, $\widetilde{X} := \overline{X_\Omega}$ and let $x_0 \in \widetilde{X}$. Then there exists an injective operator $C \in L(X)$, with range dense in X, such that*

$p(A)$ generates an entire C-regularized group $(S(t))_{t\in\mathbb{C}}$. Furthermore, C can be chosen so that $C(\widetilde{X}) \subseteq \tilde{X}$, $R(C_{|\widetilde{X}})$ is dense in the space $\tilde{X}$, and the operator $p(A)_{|\tilde{X}}$ generates an entire $C_{|\widetilde{X}}$-regularized group $(S(z)_{|\widetilde{X}})_{z\in\mathbb{C}}$ in the space $\widetilde{X}$. The semigroup $(S(t)_{|\widetilde{X}})_{t\geq 0}$ and the operator $C^{-1}_{|\widetilde{X}} S(t_0)_{|\widetilde{X}}$ are densely distributionally chaotic.

Proof. Using [46, Th. 3.13], the argumentation given in [26, Sec. XXIV], as well as the fact that the generators of C_0-semigroups on sequentially complete locally convex are always densely defined (see for example [43, 46] for more details), it readily follows that the operator A^k is densely defined for all $k \in \mathbb{N}$, and the operator $-p(A)$ generates an exponentially equicontinuous, analytic strongly continuous semigroup of angle $\frac{\pi}{2} - n(\frac{\pi}{2} - \theta)$.

By [46, Th. 3.15], we get that there exists an injective operator $C \in L(X)$ such that $p(A)$ generates an entire C-regularized group $(S(z))_{z\in\mathbb{C}}$ with $R(C)$ dense in X. By the proof of [27, Lem. 5.6], we have that $\sigma_p(-p(A)) = -p(-\sigma_p(-A))$ and $f(\lambda) \in \mathrm{Ker}(-p(A) + p(-\lambda))$, for $\lambda \in \Omega$. Without loss of generality, we may assume that $p'(z) \neq 0$, $z \in -\Omega$; otherwise, we can replace Ω by $\Omega \setminus \{\gamma_1, \ldots, \gamma_{n-1}\}$, where $\gamma_1, \ldots, \gamma_{n-1}$ are not necessarily distinct zeros of the polynomial $p'(z)$. Hence, the mapping $\lambda \mapsto p(-\lambda)$ for every $\lambda \in \Omega$ and its inverse mapping $z \mapsto -p^{-1}(z)$ $z \in p(-\Omega)$, are analytic and open.

Consequently, the set $-p(-\Omega)$ is open and connected. Moreover, the mapping $z \mapsto f(-p^{-1}(-z))$ defined for every $z \in -p(-\Omega)$ is analytic, $f(-p^{-1}(-z)) \in \ker(-p(A) - z)$ for every $z \in -p(-\Omega)$, and if for some $x^* \in X^*$ we have $x^*(f(-p^{-1}(-z))) = 0$ for every $z \in -p(-\Omega)$, then we have $x^* = 0$.

Therefore, it is sufficient to prove the assertions 1. and 2. in the case $p(z) = z$. By the foregoing, we have that $f(-\lambda) \in Z(A)$ for every $\lambda \in -\Omega$, $f(-\lambda) \in \mathrm{Ker}(A - \lambda)$ for every $\lambda \in -\Omega$, and

$$S(t)f(\lambda) = e^{-\lambda t} C f(\lambda), \text{ for evert } t \geq 0, \lambda \in \Omega. \tag{63}$$

The assertion 1. follows now from a simple application of Corollary 5.7.2. In order to prove the assertion 2. by [47, Rem. 14.ii)] jointly with the proof of [46, Th. 3.15] imply that $C(\tilde{X}) \subseteq \widetilde{X}$ and $R(C_{|\tilde{X}})$ is dense in $\widetilde{X}$. By (63), we get $S(z)f(\lambda) = e^{-\lambda z} C f(\lambda)$ for every $z \in \mathbb{C}, \lambda \in \Omega$. This implies that $S(z)(\tilde{X}) \subseteq \widetilde{X}$ for every $z \in \mathbb{C}$ and that the operator $p(A)_{|\tilde{X}}$ generates an entire $C_{|\widetilde{X}}$-regularized group $(S(z)_{|\tilde{X}})_{z\in\mathbb{C}}$ on the space $\widetilde{X}$. The remaining part of the proof is a consequence of Corollary 5.7.3. □

Remark 5.10. If $p(-\Omega) \cap i\mathbb{R} \neq \emptyset$, then we have that $x_0 \in \tilde{X}$ in Theorem 5.9.2, (see also Lemma 5.6.2.)

Remark 5.11. *The notions of distributional chaos and distributionally irregular vectors for α-times integrated C-semigroups have been defined in terms of the unbounded operators $G(\delta_t)$, where G is an algebra homomorphisms and δ_t is the Dirac measure concentrated at the point t.*

Let $\mathcal{D}_{\mathbb{K}}$ the space of C^∞ functions on $\mathbb{R}$ with values on $\mathbb{K}$ and compact support. This space is supplied with the usual inductive limit topology. For obtaining a description of Devaney chaos that involves the operators $\mathcal{G}(\varphi)$, with $\varphi \in \mathcal{D}_{\mathbb{K}}$, one can proceed as in [27, Th. 4.6] *and* [48, Th. 3.1.32] *and try to exploit the identity $G(\delta_s)x = \lim_{k\to\infty} C^{-1}\mathcal{G}(\varphi_{k,s})$, $s \geq 1$, $x \in Z_1(A)$, where $\varphi_{k,s}(\cdot) := k\varphi(s + k(\cdot - s))$ for some fixed $\varphi \in \mathcal{D}_{\mathbb{K}}$ with $\mathrm{supp}(\varphi) \subseteq [s-1, s+1]$ and $\int \varphi d\mu = 1$. For the sake of brevity, we will not consider this question in more detail here and this will be the topic of a future research.*

Nevertheless, we leave as an interesting problem to the reader to transfer the assertions of Corollary 3.12.1 and [18, Th. 19], [48, Th. 3.1.34] to fractionally integrated C-semigroups.

It is quite natural to ask whether some integrated C-semigroups that are already known to be hypercyclic and/or Devaney chaotic, possess a certain distributionally chaotic behaviour.

Example 5.12. [44, Ex. 3.1.35(i)] *and* [47, Ex. 38]. *Let us consider $n \in \mathbb{N}$, $\mathbb{K} = \mathbb{C}$ (the established results continue to hold in the case that $\mathbb{K} = \mathbb{R}$), a weight function $\rho(t) := \frac{1}{t^{2n}+1}$ defined for every $t \in \mathbb{R}$ and an integral generator $Af := f'$ with domain $D(A) := \{f \in C_{0,\rho}(\mathbb{R}) : f' \in C_{0,\rho}(\mathbb{R})\}$.*

Let us also consider $X_n := (C_{0,\rho}(\mathbb{R}))^{n+1}$ *and the* $n+1$*-linear operator defined as*

(64)
$$A_n(f_1,\cdots,f_{n+1}) := (Af_1+Af_2, Af_2+Af_3,\cdots, Af_n+Af_{n+1}, Af_{n+1}),\ \text{for every } (f_1,\cdots,f_{n+1}) \in D(A_n),$$

with $D(A_n) := D(A)^{n+1}$.

Then $\pm A_n$ *generate a polynomially bounded* n*-times integrated semigroups* $(S_{n,\pm}(t))_{t\geq 0}$*, and neither* A_n *nor* $-A_n$ *generates a local* $(n-1)$*-times integrated semigroup.*

The above implies that $\pm A_n$ *generate* $(1\mp A)^{-n}$*-regularized semigroups* $(T_{n,\pm}(t))_{t\geq 0}$*, and neither* A_n *nor* $-A_n$ *generates a local* $(1\mp A)^{-(n-1)}$*-regularized semigroup. Then it can be easily proved that, for every* $t \geq 0$ *and* $\varphi_1,\cdot\cdot\cdot,\varphi_{n+1} \in \mathcal{D}_{\mathbb{C}}$,

$$(1\mp A)^n T_{n,\pm}(t)\big(\varphi_1,\cdot\cdot\cdot,\varphi_{n+1}\big)^T = \big(\psi_1,\cdot\cdot\cdot,\psi_{n+1}\big)^T, \tag{65}$$

where

$$\psi_i(\cdot) = \sum_{j=0}^{n+1-i} \frac{(\pm t)^j}{j!}\varphi_{i+j}^{(j)}(\cdot \pm t),\ \text{for every } 1 \leq i \leq n+1. \tag{66}$$

If $0 \leq i \leq n$, $\varphi \in \mathcal{D}_{\mathbb{C}}$ *and* $supp(\varphi) \subseteq [a,b]$, *then*

$$\begin{aligned} t^i \sup_{x\in\mathbb{R}} |\varphi(x\pm t)|\rho(x) &\leq t^i \sup_{x\in[a\mp t, b\mp t]} \frac{1}{x^{2n}+1} \\ &\leq t^i\left(\frac{1}{(a-t)^{2n}+1} + \frac{1}{(a+t)^{2n}+1} + \frac{1}{(b-t)^{2n}+1} + \frac{1}{(b+t)^{2n}+1}\right) \end{aligned}$$

which tends to zero as $|t| \to \infty$.

Set $X_0 := (\mathcal{D}_{\mathbb{C}})^n$. *By* (65)*, we have that* $\lim_{t\to\infty} T_{n,\pm}(t)\vec{\varphi} = 0$, *for* $\vec{\varphi} \in X_0$. *Since* $\rho(\cdot)$ *is an admissible weight function in the sense of* [29]*, and* (65) *holds, it readily follows that the operator* A *generates the translation* C_0*-group* $(T(t))_{t\in\mathbb{R}}$ *on* $C_{0,\rho}(\mathbb{R})$*, as well as that the tuple* $(\varphi,0,\cdot\cdot\cdot,0)^T$ *belongs to the space* $Z_1(\pm A_n)$ *for any* $\varphi \in C_{0,\rho}(\mathbb{R})$*, with*

$$T(\pm t)\big(\varphi\big) := \big(\varphi(\cdot \pm t), 0, \cdot\cdot\cdot, 0\big)^T,\ \text{for every } t \geq 0. \tag{67}$$

Now we can apply [13, Cor. 30] *(with* $X_0 = \mathcal{D}_{\mathbb{C}}$ *and* $B = \mathbb{N}$*), and* [1, Prop. 2]*, so as to conclude that for every* $t_0 > 0$ *the operators* $T(\pm t_0)$ *have dense distributionally irregular manifolds and that the* C_0*-semigroups* $(T(\pm t))_{t\geq 0}$ *are densely distributionally chaotic. For every* $l \in \mathbb{N}$ *define* $T^l_{\pm}(t) := T(t)x$*, for every* $t \geq 0, x \in D(A^l)$*. Then it is well known that* $(T^l_{\pm}(t))_{t\geq 0}$ *are* C_0*-semigroups on the space* $[D(A^l)]$ *and that* $(1-A)^{-l} : C_{0,\rho}(\mathbb{R}) \to D(A^l)$ *is a continuous linear homeomorphism satisfying that* $T^l_{\pm}(t)\circ(1-A)^{-l} = (1-A)^{-l}\circ T_{\pm}(t)$*, for every* $t \geq 0$ *(see e.g. the proof of* [42, Lem. 3.2]*). Using that distributional chaos is preserved under conjugaqy and Theorem 4.2, it readily follows that for each* $l \in \mathbb{N}$ *there exists* $\varphi \in D(A^l)$ *such that*

$$\lim_{t\to\infty} \|T_{\pm}(t)\varphi\|_{C_{0,\rho}(\mathbb{R})} = \infty. \tag{68}$$

Applying Theorem 5.4, we get that $(S_{n,\pm}(t))_{t\geq 0}$ *and* $(T_{n,\pm}(t))_{t\geq 0}$ *are densely distributionally chaotic, as well as that for each* $t_0 > 0$ *the operator* $(1\mp A)^n T_{n,\pm}(t_0)$ *is densely distributionally chaotic.*

Example 5.13. *Devaney chaos was already studied for* C*-regularized semigroups generated by differential operators with constant coefficients* [27] *We return to this setting for studying the distributional chaotic properties of those generators.*

Let us consider the weighted space $L^p_\rho(\mathbb{R})$ *with* $1 \leq p < \infty$ *and* ρ *an admissible weight function in the sense of* [29]*. Let us define*

- $\omega_1 := \sup\{\mu \in \mathbb{R} : \int_0^\infty e^{\mu p s}\rho(s)ds < \infty\}$.
- $\omega_2 := \inf\{\mu \in \mathbb{R} : \int_{-\infty}^0 e^{\mu p s}\rho(s)ds < \infty\}$.

- $V_{\omega_2,\omega_1} := \{z \in \mathbb{C} : \omega_1 \leq \Re(z) \leq \omega_2\}$.
- $Q(z) := \sum_{k=0}^{N} a_k z^k$, *with* $a_k \in \mathbb{C}$ *for every* $0 \leq k \leq N$, *and* $a_N \neq 0$.
- $Q(B) := \sum_{k=0}^{N} a_k B^k$, *where* B *is the derivative operator.*
- *For an arbitrary* $\mu \in \mathbb{C}$, *we define* $h_\mu(s) := e^{\mu s}$, *for every* $s \in \mathbb{R}$.

Finally let us take $t_0 > 0$ *and :*

$$Q(\mathrm{int}(V_{\omega_2,\omega_1})) \cap i\mathbb{R} \neq \emptyset. \tag{69}$$

For every $\mu \in \mathrm{int}(V_{\omega_2,\omega_1})$ *we have* $\pm Q(B)h_\mu = \pm Q(\mu)h_\mu$. *If* $C := (e^{-(-z^2)^N})(B)$, *then we have that the operator* $Q(B)$ *is the integral generator of the* C*-regularized semigroup* $(W_Q(t))_{t\geq 0}$ *on* $L^p_\rho(\mathbb{R})$, *with* $W_Q(t)(z) := e^{tQ(z)}e^{-(-z^2)^N})(B)$ *for every* $t \geq 0, z \in \mathbb{C}$.

Furthermore, the sets $R(C)$, $X_0 := \mathrm{span}\{h_\mu : \Re(Q(\mu)) < 0\}$ *and* $X_1 := \mathrm{span}\{h_\mu : \Re(Q(\mu)) > 0\}$ *are dense in* $L^p_\rho(\mathbb{R}$. *By Corollary 5.7.2, we obtain that the* C*-regularized semigroup* $(W_Q(t))_{t\geq 0}$ *and the operator* $e^{t_0 Q(B)}$ *are densely distributionally chaotic. If we replace the condition* (69) *by*

$$Q(\mathrm{int}(V_{\omega_2,\omega_1})) \cap \{z \in \mathbb{C} : |z| = 1\} \neq \emptyset, \tag{70}$$

then it can be proved, with the help of Corollary 3.12.2, that the unbounded operator $Q(B)$ *is densely distributionally chaotic.*

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