3-permutable subgroups of finite groups

A. A. Heliel · A. Ballester-Bolinches · R. Esteban-Romero · M. O. Almestady

Abstract Let \( \mathfrak{Z} \) be a complete set of Sylow subgroups of a finite group \( G \), that is, a set composed of a Sylow \( p \)-subgroup of \( G \) for each \( p \) dividing the order of \( G \). A subgroup \( H \) of \( G \) is called \( \mathfrak{Z} \)-permutable if \( H \) permutes with all members of \( \mathfrak{Z} \). The main goal of this paper is to study the embedding of the \( \mathfrak{Z} \)-permutable subgroups and the influence of \( \mathfrak{Z} \)-permutability on the group structure.

Keywords finite group, \( p \)-soluble group, \( p \)-supersoluble group, \( \mathfrak{Z} \)-permutable subgroup, subnormal subgroup

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1 Introduction and statements of results

All groups in this paper will be finite.

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A subgroup $H$ of a group $G$ is said to permute with a subgroup $K$ of $G$ if $HK$ is a subgroup of $G$. $H$ is said to be $S$-permutable in $G$ if $H$ permutes with all Sylow subgroups of $G$. This property extends normality and permutability and was introduced by Kegel in [11]. In this paper, he proved some interesting properties which turn out to be useful in establishing results concerning the group structure. In particular, it is proved there that every $S$-permutable subgroup must be subnormal ([11, Satz 1]).

On the other hand, we say that a set $\mathcal{S}$ of Sylow subgroups of a group $G$ is a complete set of Sylow subgroups of $G$ if $\mathcal{S}$ contains exactly one Sylow subgroup for each prime dividing the order of $G$; $\mathcal{S}$ is called a Sylow basis of $G$ if the Sylow subgroups in $\mathcal{S}$ are pairwise permutable. Sylow bases were introduced and studied by Hall in [7]. The results of this paper show that the existence and conjugacy of Sylow bases is a characteristic property of soluble groups.

In [1], Asaad and Heliel introduced and studied the notion of a $\mathcal{Z}$-permutable subgroup, where $\mathcal{Z}$ is a complete set of Sylow subgroups of a group $G$. A subgroup of $G$ is called $\mathcal{Z}$-permutable if it permutes with every member of a complete set $\mathcal{S}$ of Sylow subgroups of $G$. It is clear that $S$-permutability implies $\mathcal{Z}$-permutability but the converse does not hold in general. In fact, $\mathcal{Z}$-permutable subgroups are not subnormal in general, and subnormal $\mathcal{Z}$-permutable subgroups are not $S$-permutable either as the following example shows:

**Example 1** Let $E = \langle x, y \rangle$ be an extraspecial group of order 27 and exponent 3. Let $a$ be an automorphism of order 2 of $G$ given by $x^a = x^{-1}, y^a = y^{-1}$. Let $G = E \rtimes \langle a \rangle$ be the corresponding semidirect product. Then $\mathcal{S} = \{E, \langle a \rangle\}$ is a complete set of Sylow subgroups of $G$. The subgroup $H = \langle x \rangle$ is $\mathcal{Z}$-permutable, but it does not permute with the Sylow 2-subgroup $\langle ay \rangle$. Therefore, $H$ is not $S$-permutable. However, $H$ is a subnormal subgroup of $G$.

Throughout the first part of our paper, proving important properties of $S$-permutable type of the subnormal $\mathcal{Z}$-permutable subgroups has been our main focus.

The embedding of $S$-permutable subgroups was studied by Kegel [11, Satz 1] and Deskins [4, Theorem 1] (see also [3, Theorem 1.2.14]). They proved that if $A$ is an $S$-permutable subgroup of $G$, then $\langle A^G \rangle / \text{Core}_G(A)$ is nilpotent. Our first main result shows how a subnormal $\mathcal{Z}$-permutable subgroup is embedded in the group.

**Theorem 1** Let $\mathcal{S}$ be a complete set of Sylow subgroups of a group $G$. Let $A$ be a subnormal $\mathcal{Z}$-permutable subgroup of $G$. Then $\langle A^G \rangle / \text{Core}_G(A)$ is soluble. If, in addition, $\mathcal{S}$ is a Sylow basis of $G$, then $\langle A^G \rangle / \text{Core}_G(A)$ is nilpotent.

The alternating group of degree 6 is a non-subnormal $\mathcal{Z}$-permutable subgroup of the alternating group of degree 7 which is not soluble. Moreover, every core-free maximal subgroup of a soluble primitive group is $\mathcal{Z}$-permutable. Therefore subnormality is necessary in the above theorem.

A classical result of Kegel ([11, Satz 2], see also [3, Theorem 1.2.19]) shows that the set of all $S$-permutable subgroups of a group $G$ is a sublattice of the subgroup lattice of $G$. Kegel’s result also holds for subnormal $\mathcal{Z}$-permutable subgroups. It is consequence of the following theorem.
**Theorem 2** Let \( p \) be a prime and \( U \) and \( V \) subgroups of a group \( G \). If \( U \) and \( V \) permute with a Sylow \( p \)-subgroup \( G_p \) of \( G \) and \( U \) is subnormal in \( G \), then \( U \cap V \) permutes with \( G_p \).

The hypothesis of the subnormality of \( U \) is necessary in the above theorem, even for soluble groups, as an example of Doerk [5, Beispiel 1] shows.

**Corollary 1** Let \( \mathfrak{S} \) be a complete set of Sylow subgroups of a group \( G \). Then the set of all subnormal \( \mathfrak{S} \)-permutable subgroups of a group \( G \) is a sublattice of the lattice of all subgroups of \( G \).

If \( \mathfrak{S} \) is a Sylow basis of a group \( G \), the set of all \( \mathfrak{S} \)-permutable subgroups is a sublattice of the subgroup lattice of \( G \) ([6, Chapter I, Theorem 4.29]). However, we do not know whether the set of all \( \mathfrak{S} \)-permutable subgroups (not necessarily subnormal) of a group \( G \) is a sublattice of the lattice of all subgroups of \( G \).

There are several papers in the literature where global information about a group is obtained by assuming that some distinguished subgroups are \( \mathfrak{S} \)-permutable ([1,8,9,12,13,14,15,17]). The second part of the paper concerns situations in this spirit, but we require only that some \( p \)-subgroups, for a given prime \( p \), have the required property.

In order to state our results in this direction, we recall that a group is said to be \( p \)-supersoluble if it is \( p \)-soluble and every \( p \)-chief factor has order \( p \), where \( p \) is a prime that we hold fixed.

In the sequel, \( \mathfrak{S} = \{G_q \mid q \in \pi(G)\} \) will denote a complete set of Sylow subgroups of a group \( G \), where \( G_q \) is a Sylow \( q \)-subgroup of \( G \).

Asaad and Heliel [1, Theorem 3.1] showed that if all maximal subgroups of the Sylow subgroups in \( \mathfrak{S} \) are \( \mathfrak{S} \)-permutable, then \( G \) is supersoluble. A local approach to this theorem is the following.

**Theorem 3** Let \( G \) be a group. Assume that all maximal subgroups of \( G_p \in \mathfrak{S} \) are \( \mathfrak{S} \)-permutable. Then either \( G_p \) is cyclic or \( G \) is \( p \)-supersoluble.

If \( p \) is the smallest prime dividing the order of \( G \), and the Sylow \( p \)-subgroups are cyclic, then \( G \) is \( p \)-nilpotent by [10, IV, Satz 2.8]. Furthermore, if \( G \) is \( p \)-supersoluble, then \( G \) every \( p \)-chief factor is central and so \( G \) is \( p \)-nilpotent. Therefore we have:

**Corollary 2** ([14, Theorem 3.1]) If \( p \) is the smallest prime dividing the order of a group \( G \) and the maximal subgroups of \( G_p \in \mathfrak{S} \) are \( \mathfrak{S} \)-permutable, then \( G \) is \( p \)-nilpotent.

**Corollary 3** ([1, Theorem 3.1]) Assume that \( G \) is a group whose maximal subgroups of the Sylow subgroups in \( \mathfrak{S} \) are \( \mathfrak{S} \)-permutable. Then \( G \) is supersoluble.

The next natural step in our analysis to consider the structural impact of the \( \mathfrak{S} \)-permutability of the second maximal subgroups of the Sylow \( p \)-subgroup in \( \mathfrak{S} \).

Suppose that every \( 2 \)-maximal subgroup of \( G_p \in \mathfrak{S} \) is \( \mathfrak{S} \)-permutable and that \( G_p \) does not have cyclic maximal subgroups. Then every maximal subgroup of \( G_p \) is \( \mathfrak{S} \)-permutable and \( G_p \) is not cyclic. By Theorem 3, \( G \) is \( p \)-supersoluble. Therefore we have:
Corollary 4 Let $G$ be a group. Suppose that all 2-maximal subgroups of $G_p \in \mathfrak{S}$ are $\mathfrak{3}$-permutable. Either $G_p$ has a cyclic maximal subgroup or $G$ is $p$-supersoluble.

In [14, Theorem 3.3], the authors proved the following result:

Theorem 4 ([14, Theorem 3.3]) Assume that $p$ is the smallest prime dividing the order of a group $G$. Suppose that all 2-maximal subgroups of $G_p \in \mathfrak{S}$ are $\mathfrak{3}$-permutable. If $G$ is $A_4$-free, then $G$ is $p$-nilpotent.

Our goal in the sequel is to present an improvement of this theorem. According to Corollary 4, if all 2-maximal subgroups of $G_p \in \mathfrak{S}$ are $\mathfrak{3}$-permutable, then either $G_p$ has a cyclic maximal subgroup or $G$ is $p$-supersoluble. Furthermore, a $p$-supersoluble group $G$ is $p$-nilpotent provided that $p$ is the smallest prime dividing the order of $G$. Hence we only must consider the case when $G_p$ has a cyclic maximal subgroup. We prove:

Theorem 5 Assume that $p$ is the smallest prime dividing the order of a group $G$. Suppose that all 2-maximal subgroups of $G_p \in \mathfrak{S}$ are $\mathfrak{3}$-permutable. If $G_p$ has a cyclic maximal subgroup, then $G$ is $p$-soluble.

A key fact for the proof of Theorem 5 is that $G$ cannot be non-abelian simple. This was established in Step 3 of the proof of [14, Theorem 3.3].

Theorem 6 Assume that $p$ is the smallest prime dividing the order of a group $G$. If every 2-maximal subgroup of $G_p \in \mathfrak{S}$ is $\mathfrak{3}$-permutable, then either $G$ is $p$-nilpotent or $G$ has an epimorphic image isomorphic to $\Sigma_4$.

In [12, Theorem 3.3], the authors proved that if $p$ is the smallest prime dividing the order of a group of $G$ and every cyclic subgroup of $G_p$ with order $p$ or order 4 (if $p = 2$) is $\mathfrak{3}$-permutable in $G$, then $G$ is $p$-nilpotent.

Our last results concern the $\mathfrak{3}$-permutability of the minimal subgroups of the Sylow $p$-subgroup in $\mathfrak{S}$ and include the above result as a particular case.

Theorem 7 Let $G$ be a $p$-soluble group such that every cyclic subgroup of $G_p$ with order $p$ or order 4 (if $p = 2$) is $\mathfrak{3}$-permutable in $G$. Then $G$ is $p$-supersoluble.

Theorem 8 Let $G$ be a group such that every cyclic subgroup of $G_p$ with order $p$ or order 4 (if $p = 2$) is $\mathfrak{3}$-permutable in $G$. Either $G_p$ has order $p$ or $G$ is $p$-soluble.

Corollary 5 Let $G$ be a group such that every cyclic subgroup of $G_p$ with order $p$ or order 4 (if $p = 2$) is $\mathfrak{3}$-permutable in $G$. Either $G_p$ has order $p$ or $G$ is $p$-soluble.

Corollary 6 ([12, Theorem 3.3]) If $p$ is the smallest prime dividing the order of $G$ and every cyclic subgroup of $G_p$ with order $p$ or order 4 (if $p = 2$) is $\mathfrak{3}$-permutable in $G$, then $G$ is $p$-nilpotent.
2 Preliminaries

Suppose that $G$ is a group and $N$ a normal subgroup of $G$. Following [1], we write

$\mathfrak{Z}N = \{G_qN : G_q \in \mathfrak{Z}\}$, $\mathfrak{Z}N/N = \{G_qN/N : G_q \in \mathfrak{Z}\}$, and $\mathfrak{Z}\cap X = \{G_q \cap X : G_q \in \mathfrak{Z}\}$

for all subgroups $X$ of $G$.

**Lemma 1** ([1, Lemma 2.1]) Let $G$ be a group and $N$ a normal subgroup of $G$.

1. $\mathfrak{Z}\cap N$ and $\mathfrak{Z}N/N$ are complete sets of Sylow subgroups of $N$ and $G/N$, respectively.
2. If $U$ is a $\mathfrak{Z}$-permutable subgroup of $G$, then $UN/N$ is $\mathfrak{Z}N/N$-permutable. If $U$ is contained in $N$, then $U$ is $\mathfrak{Z}\cap N$-permutable.

The following well-known fact, which follows from the repeated application of [10, Kapitel I, Hilfssatz 7.7a]), will be used in this paper without further notice.

**Lemma 2** Let $S$ be a subnormal subgroup of a group $G$ and let $Q$ be a Sylow $q$-subgroup of $G$, where $q$ is a prime. Then $Q\cap S$ is a Sylow $q$-subgroup of $S$.

The following result, due to Vdovin, turns out to be crucial in the proofs of some of our results.

**Theorem 9** If, for every prime $q \neq p$, $G$ possesses a Hall $\{p,q\}$-subgroup, then $G$ is $p$-soluble.

The above theorem is a consequence of the following lemma, whose proof requires a bit of notation.

Let $q$ be a natural number, and $r$ an odd prime such that $\gcd(q,r) = 1$. Let $e(q,r)$ denote the multiplicative order of $q$ modulo $r$, that is, the least natural number $t$ with $q^t \equiv 1 \pmod{r}$. For an odd $q$, we set $e(q,2) = 1$ if $q \equiv 1 \pmod{4}$ and $e(q,2) = 2$ otherwise.

**Lemma 3** Let $r$ be a prime. Then, for every simple group $S$ with $r \in \pi(S)$, there exists $s \in \pi(S)$ such that $S$ does not possess a Hall $\{r,s\}$-subgroup.

**Proof** Suppose, by contradiction, that there exists a finite simple group $S$ and a prime $r \in \pi(S)$ such that, for every $s \in \pi(S)$, $S$ possesses a Hall $\{r,s\}$-subgroup $H$. Burnside’s $p^aq^b$-theorem implies that $H$ is soluble. We proceed case by case.

Assume first that $S$ is an alternating group $A_n$ of degree $n \geq 5$. If $r \neq 2$, then [16, Table 2] implies that for every odd $s \in \pi(G) \setminus \{p\}$, we have that $S$ does not possess a Hall $\{r,s\}$-subgroup. If $r = 2$, then [16, Table 2] implies that $S$ does not have a Hall $\{2,5\}$-subgroup.

Now assume that $S$ is sporadic. Then the claim follows from [16, Tables 3 and 4].

Finally, assume that $S$ is a finite simple group of Lie type over a field of characteristic $p$ and order $q$. If $r = p$, then [16, Theorem 8.3] implies that every Hall $\pi$-subgroup of $S$ with $r \in \pi$ is contained in a Borel subgroup $B$ or is parabolic. Since $B$ is a proper subgroup of $S$, there exists $s \in \pi(|S : B|)$, and so $B$ cannot contain a Hall $\{r,s\}$-subgroup of $S$. Therefore a Hall $\{r,s\}$-subgroup $H$ of $S$ is parabolic. Theorems 8.5, 8.6, and 8.7 and Table 6 from [16] imply that in this case $\{r,s\} = \{2,3\}$.
and $S \in \{\text{SL}_3(2), \text{SL}_3(3), \text{SL}_4(2), \text{SL}_5(2)\}$. In all these cases, there is no Hall $(r,s)$-subgroup for $s \in \pi((q^6 - 1)/(q - 1))$, which is always contained in $\pi \cap \pi(G)$ if $G$ has a proper Hall $\pi$-subgroup with $|\pi| \geq 2$.

Assume that $r \neq p$ is that $r$ is odd. If $S$ is an exceptional group of Lie type and $S$ is neither a Suzuki nor a Ree group, then [16, Table 7] implies that $S \in E_{(x)}$ if and only if $e(q,r) = e(q,s)$. Now by the decomposition of $|S|$ as a product of polynomials in $q$, there exists an odd $s \in \pi(S)$ with $e(q,r) \neq e(q,s)$, and so $S$ does not have a Hall $(r,s)$-subgroup. If $S$ is either a Suzuki or a Ree group, then the claim follows immediately from [16, Table 8]. If $S$ is a classical group of Lie type, then [16, Table 7] implies that $S$ has a Hall $(r,s)$-subgroup only if either $e(q,r) = e(q,s) = b(r)$, where $b(r) \in \{1, r\}$ if $G = \text{PSL}_q(q)$, $b(r) \in \{2, 2r\}$ if $G = \text{PSU}_q(q)$, $b(r) = 2e(q,r)$ if $e(q,r)$ is odd and $G = \text{PSU}_q(q)$, $b(r) = e(q,r)/2$ if $e(q,r)$ is even, $q$ does not divide $e(q,r)$ and $G = \text{PSU}_q(q)$. In particular, if $e(q,s) \neq e(q,r)$, then $e(q,s)$ can take at most two values. If the rank of $S$ is at least 2, then $|\{e(q,s) \mid s \in \pi(S) \setminus \{p\}\}| \geq 3$, and so there exists $s$ such that $e(q,r) \notin \{e(q,r), b(r)\}$ and therefore $S \not\in E_{(x)}$. If the rank of $S$ is less than 2, then $S \cong \text{PSL}_2(q)$. If $S = \text{PSL}_2(q)$ and $e(q,r) = 1$, then there exists $s \in \pi(S)$ with $e(q,s) = 2$ and [16, Tables 7 and 10] imply that $S \not\in E_{(x)}$. If $S \cong \text{PSL}_2(q)$ and $e(q,r) = 2$, then $S \not\in E_{(x)}$.

Finally, assume that $r = 2$ and $r \neq p$. If $S \neq \text{SL}_3(3)$, then $S \not\in E_{(2,p)}$ by the arguments presented for the case $r = p$. If $S = \text{SL}_3(3)$, then $S \not\in E_{(2,13)}$.

**Corollary 7** Let $G$ be a group and $p \in \pi(G)$. Assume that

1. all maximal subgroups of $G_p \in \pi$ are $3$-permutable and $G_p$ is not cyclic, or
2. all $2$-maximal subgroups of $G_p \in \pi$ are $3$-permutable and $G_p$ has no cyclic maximal subgroups.

Then $G$ is $p$-soluble.

**Proof** Assume that 1 holds. Then $G_p$ possesses two maximal subgroups $M_1$ and $M_2$, both $3$-permutable. Then $M_1M_2 = G_p$ is $3$-permutable. This implies that $G_pG_p$ is a Hall $(p,q)$-subgroup of $G$ for each $q \neq p$. By Theorem 9, $G$ is $p$-soluble.

Assume that 2 holds. Let $M_1$ be a maximal subgroup of $G_p$. Since $M_1$ is not cyclic, $M_1$ possesses two maximal subgroups $M_{11}$ and $M_{12}$. Since both of them are $3$-permutable, $M_1 = M_{11}M_{12}$ is also $3$-permutable. Hence 1 holds and $G$ is $p$-soluble.

### 3 Proofs of the main results

**Proof (of Theorem 1)** We prove that $A/\text{Core}_G(A)$ is soluble by induction on the order of $G$. Since $A/\text{Core}_G(A)$ is $(3\text{Core}_G(A))/\text{Core}_G(A)$-permutable in $G/\text{Core}_G(A)$ by Lemma 1, we can assume that $\text{Core}_G(A) = 1$. Let $r$ be a prime dividing $|G|$ and let $R$ be the Sylow $r$-subgroup of $G$ in $\pi$. Consider $X = AR$. Let $q$ be a prime different from $r$ and let $G_q$ be the Sylow $q$-subgroup of $G$ in $\pi$. Since $A$ is subnormal in $G$, $G_q \cap A$ is a Sylow $q$-subgroup of $A$. Moreover, $A$ is $(3\cap X)$-permutable, because $3 \cap X = \{R\} \cup \{G_q \cap A \mid q \neq r\}$. Moreover $A$ is subnormal in $X$. Assume that $X$ is a proper subgroup of $G$. By induction, the soluble residual $A^S$ of $A$ is contained in $\text{Core}_X(A) = \text{Core}_R(A)$. Consequently, $A^S = (A^S)^S \leq \text{Core}_R(A)^S \leq A^S$. It follows
that $A^g = \text{Core}_G(A)^g$ is a normal subgroup of $X$. In particular, $R \leq N_G(A^g)$. Suppose that for every Sylow subgroup $R$ of $G$ in $\mathfrak{3}$, $AR$ is a proper subgroup of $G$. It follows that $R \leq N_G(A^g)$ for each $R \in \mathfrak{3}$. Hence $A^g$ is a normal subgroup of $G$. Thus $A^g \leq \text{Core}_G(A) = 1$. Consequently $A$ is soluble, as wanted.

Therefore there exists a prime $r$ and a Sylow $r$-subgroup $R$ of $G$ in $\mathfrak{3}$ such that $G = AR$. Let $q$ be a prime different from $r$ and let $Q$ be a Sylow $q$-subgroup of $G$. The subnormality of $A$ implies that $Q \cap A$ is a Sylow $q$-subgroup of $A$. By order considerations, $Q \cap A = Q$ and so $Q$ is a Sylow $q$-subgroup of $A$. It follows that $O^r(G) \leq \text{Core}_G(A) = 1$. In particular, $G$ is a $r$-group and so $G$ is soluble. Hence, $A$ is soluble.

We conclude that $(A^g)/\text{Core}_G(A)$ is soluble.

Suppose that $\mathfrak{3}$ is a Sylow basis of $G$. We shall show that $A^g/\text{Core}_G(A)$ is nilpotent. Without loss of generality we may assume that $\text{Core}_G(A) = 1$. Let $B = \bigcap\limits_{p} O^q(A)$ be the nilpotent residual of $A$. Let $r$ be a prime dividing $|G|$ and $g \in G$. Then $g = xy$, where $x$ is an element of $G_r$ and $y$ is a $r'$-element of $Z = \prod_{q \neq r} G_q$. It follows that $B_r = B \cap G_r = O^r(A) \cap G_r$ is a Sylow $r$-subgroup of $B$. Applying [3, Lemma 1.1.11], we have that $O^r(A) = O^r(AG_r)$, which is a normal subgroup of $AG_r$, and that $O^r(A) = O^r(AZ)$, which is a normal subgroup of $AZ$. In particular, $G_r$ normalises $O^r(A)$ and $Z$ normalises $O^r(A)$. Moreover, $B_r$ is contained in $O^r(A)^g = O^r(A)$. Since $G_r$ normalises $B_r$, it follows that $B_r^g = B_r^g$ is contained in $O^r(A)$ and so it is a subgroup of $A$. Consequently, the normal closure $(B_r^g)$ is contained in $A$ and then $(B_r^g) \leq \text{Core}_G(A) = 1$. Hence $B = 1$ and $A$ is nilpotent.

Therefore, $(A^g)/\text{Core}_G(A)$ is nilpotent.

\textbf{Proof (of Theorem 2)} We argue by induction on $|G|$. Assume that $VG_p$ is a proper subgroup of $G$. Then $U \cap VG_p$ is a subnormal subgroup of $VG_p$. Since $UG_p \cap VG_p = (U \cap VG_p)G_p$ is a subgroup of $G$, $U \cap VG_p$ permutes with the Sylow $p$-subgroup $G_p$ of $VG_p$. The induction hypothesis implies that $G_p$ permutes with $(U \cap VG_p) \cap V = U \cap V$. Therefore we may assume that $G = VG_p$. An analogous argument with the subnormal subgroup $U$ of $UG_p$ and $V \cap UG_p$ shows that $G = UG_p$. Let $q \neq p$ be a prime dividing $|G|$ and let $G_q$ be a Sylow $q$-subgroup of $G$ contained in $V$. Then $G_q \cap U$ is a Sylow $q$-subgroup of $U$ since $U$ is subnormal in $G$. Hence $G_q$ is contained in $U$ by order considerations. This means that $U \cap V$ contains a Sylow $q$-subgroup of $G$ for all primes $q \neq p$. Therefore $|G : U \cap V|$ is a power of $p$. Applying [6, Chapter A, Lemma 1.6 (b)], we conclude that $G = G_p(U \cap V)$ and $G_p$ permutes with $U \cap V$, as required.

\textbf{Proof (of Theorem 3)} Suppose that every maximal subgroup of $G_p \in \mathfrak{3}$ is $3$-permutable and that $G_p$ is not cyclic. By Corollary 7, $G$ is $p$-soluble. Assume that $G$ is not $p$-supersoluble and consider $G$ of least possible order. Let $N$ be a minimal normal subgroup of $G$. Let $M/N$ be a maximal subgroup of $PN/N$. Then $M/N = M_1/N$ for some maximal subgroup $M_1$ of $P$. Since $M_1$ is $3$-permutable, it follows that $M/N$ is $3$-permutable by Lemma 1. Then $G_pN/N$ has all maximal subgroups $3$-permutable. Assume that $N$ is a $p'$-group. Since $G_pN/N \cong G_p$, we have that $G_pN/N$ is not cyclic and so $G/N$ is $p'$-supersoluble by the choice of $G$. This implies that $G$ itself is $p'$-supersoluble, against the hypothesis. Hence $N$ is a $p$-group. Suppose that $N$
is contained in $\Phi(G_p)$, the Frattini subgroup of $G_p$. Then $G_p/N$ is not cyclic. Hence $G/N$ is $p$-supersoluble by the choice of $G$. Moreover, $N$ is also contained in $\Phi(G)$. Since the class of all $p$-supersoluble groups is a saturated formation, it follows that $G$ is $p$-supersoluble, contrary to assumption.

Consequently, $N$ is not contained in $\Phi(G_p)$. Let $M_1$ be a maximal subgroup of $G_p$ such that $NM_1 = G_p$. Let $q \in \pi(G) \setminus \{ p \}$ and let $G_q$ be the Sylow $q$-subgroup of $G$ in $\mathfrak{Z}$. Thus $1 = G_q \cap G_p = G_q \cap NM_1 = (G_q \cap N)(G_q \cap M_1)$. By [6, Chapter A, Lemma 1.2], $(N \cap M_1)G_q = NG_q \cap M_1 G_q$ is a subgroup of $G$. Furthermore, $(N \cap M_1)G_q \cap N = (N \cap M_1)(G_q \cap N) = N \cap M_1$ is a normal subgroup of $(N \cap M_1)G_q$. Hence $G_q$ normalises $N \cap M_1$. On the other hand, $N \cap M_1$ is a normal subgroup of $M_1$ and, since $N$ is abelian, is centralised by $N$. Therefore $N \cap M_1$ is normalised by $NM_1 = G_p$. Consequently, $N \cap M_1$ is a normal subgroup of $G$ properly contained in $N$. Hence $N \cap M_1 \neq 1$. But $|G_p : M_1| = |NM_1 : M_1| = |N : N \cap M_1| = p$, hence $N$ has order $p$. If $M_1$ were not cyclic, then $G/N$ would be $p$-soluble by the minimal choice of $G$. Thus, $G$ would be $p$-soluble, against supposition. Therefore, $M_1$ is cyclic and $G/N$ has cyclic Sylow $p$-subgroups. This implies that every $p$-chief factor of $G/N$ is cyclic and $G/N$ is $p$-soluble. Thus, $G$ is $p$-soluble. This final contradiction completes the proof.

\textbf{Proof (of Theorem 5)} We prove that $G$ is $p$-soluble by induction on the order of $G$. Applying Step 3 of the proof of [14, Theorem 3.3], $G$ cannot be non-abelian simple. Let $M$ be a maximal normal subgroup of $G$. Assume that $G_p$ is contained in $M$. By Lemma 1, $\mathfrak{Z} \cap M$ is a complete set of Sylow subgroups of $M$ and every 2-maximal subgroup of $G_p \cap M \in \mathfrak{Z} \cap M$ is $\mathfrak{Z}$-permutable. By induction, $M$ is $p$-soluble.

Futhermore, $G/M$ is a $p'$-group. Thus $G$ is $p$-soluble. Therefore we may assume that $p$ divides $|G/M|$. Then $M_p = M \cap G_p$ is a proper subgroup of $G_p$. Let $S$ be a maximal subgroup of $G_p$ containing $M_p$. Suppose that $S$ is cyclic. Then $G_p/M$ has a cyclic maximal subgroup. By Lemma 1, $\mathfrak{Z} M/M$ is a complete set of Sylow subgroups of $G/M$ and every 2-maximal subgroup of $G/M$ is $\mathfrak{Z} M/M$-permutable. Therefore, by induction, $G/M$ is $p$-soluble. Futhermore, since $M_p$ is cyclic, we have that $M$ is $p$-nilpotent by [10, Kapitel IV, Satz 2.8]. Therefore, $M$ is $p$-soluble and so is $G$. Hence we may assume that $S$ is not cyclic. Then $S$ has two different maximal subgroups which are $\mathfrak{Z}$-permutable. Thus $S$ is $\mathfrak{Z}$-permutable. Let $q \in \pi(G) \setminus \{ p \}$ and let $G_q$ be the Sylow $q$-subgroup of $G$ in $\mathfrak{Z}$. It follows that $M_q = M \cap G_q$ is a Sylow $q$-subgroup of $M$. Now, $G_q$ permutes with $S$ and $M$. Applying Theorem 2, $G_q$ permutes with $M \cap S = M \cap G_q = M_p$. Hence, $M_q M_q = M_q (M \cap G_q) = M \cap M_q G_q$ is a Hall $\{ p, q \}$-subgroup of $G$. By Theorem 9, $M$ is $p$-soluble. Consequently, $G$ is $p$-soluble, as wanted.

\textbf{Proof (of Theorem 6)} Suppose that $G$ is soluble and every 2-maximal subgroup of $G_p \in \mathfrak{Z}$ is $3$-permutable. Assume, arguing by contradiction, that neither $G$ is a $p$-nilpotent group nor $G$ has an epimorphic image isomorphic to $\Sigma_4$. By Corollary 4 and Theorem 5, $G$ is $p$-soluble.

Let $N$ be a minimal normal subgroup of $G$. The quotient group $G/N$ inherits the hypothesis of the theorem. Therefore $G/N$ is $p$-nilpotent. Since the class of all $p$-nilpotent groups is a saturated formation, it follows that $N = \text{Soc}(G)$ is a minimal normal subgroup of $G$ which is complemented in $G$ by a core-free maximal
p-nilpotent subgroup of \( G, M \) say. Moreover, \( C_G(N) = N \) and \( N \) is a \( p \)-group. Hence \( N \leq G_p \), the Sylow \( p \)-subgroup in \( G \). Then \( G_p = N(G_p \cap M) \) and there exists a maximal subgroup \( M_1 \) of \( G_p \) containing \( M_p = G_p \cap M \) such that \( NM_1 = G_p \). Assume that \( M_p \) is a maximal subgroup of \( G_p \). Then \( |N : [G_p : M_p]| = p \) and \( G \) is \( p \)-supersoluble. This implies that \( G \) is \( p \)-supersoluble, which contradicts our assumption. Therefore \( M_p \) is not a maximal subgroup of \( G \) and so \( M_p \) is contained in a 2-maximal subgroup \( S \) of \( G_p = NS \). Let \( p \neq q \in \pi(G) \). Thus \( 1 = G_q \cap G_p = G_q \cap NS = (G_q \cap N)(G_q \cap S) \). By [6, Chapter A, Lemma 1.2], \( (N \cap S)G_q = NG_q \cap SG_q \) is a subgroup of \( G \). In particular, \( (N \cap S)G_q \cap N = (N \cap S)(G_q \cap N) = N \cap S \) is a normal subgroup of \( (N \cap S)G_q \) and \( G_q \) normalises \( N \cap S \). On the other hand, \( N \cap S \) is a normal subgroup of \( G_p \). Consequently, \( N \cap S \) is a normal subgroup of \( G \) properly contained in \( N \). Hence \( N \cap S = 1 \) and so \( |N| = p^2 \). Since \( C_G(N) = N \), it follows that \( q \) divides \( p + 1 \). Hence \( p = 2 \) and \( G/N \) is isomorphic to \( \Sigma_3 \). Consequently, \( G \) is isomorphic to \( \Sigma_4 \). This contradiction proves the theorem.

Our hypothesis in the next two theorems is that subgroups of \( G_p \) with order \( p \) or 4 (if \( p = 2 \)) are \( \Sigma \)-permutable. Let us collect together the arguments common to these two results.

Every subgroup of \( G_p O_p'(G)/O_p(G) \) of order \( p \) or 4 (if \( p = 2 \)) is of the form \( T O_p'(G)/O_p'(G) \) for some subgroup \( T \) of \( G_p \) with order \( p \) or 4 (if \( p = 2 \)). Then, by Lemma 1, every subgroup of \( G_p O_p'(G)/O_p'(G) \) is a \( \Sigma \)-permutable. Hence, arguing by induction or minimal counterexample, we assume that \( O_p'(G) = 1 \). Hence \( F(G) \), the Fitting subgroup of \( G \), is a \( p \)-group.

Assume that \( 1 \neq F(G) \) and let \( z \) be an element of \( Z(F(G)) \) of order \( p \) and let \( y \) be an element of order \( p \) of \( F(G) \). Then \( \langle z, y \rangle \) is an elementary abelian subgroup of \( G_p \) and \( G_q \) normalises \( \langle w \rangle \) for each \( w \in \langle z, y \rangle \) and each \( q \neq p \) because \( \langle w \rangle = \langle w \rangle G_q \cap F(G) \) is a normal subgroup of \( \langle w \rangle G_q \). Hence \( p'- \) elements of \( G \) induce power automorphisms in the abelian socle \( S \) of \( G \). Applying [3, Lemma 2.1.3], all the \( G \)-chief factors of \( G \) below \( S \) are cyclic and \( G \)-isomorphic.

If \( N \) is a central minimal normal subgroup of \( G \), then \( \Omega_1(O_p'(G)) \) is centralised by all \( p'- \) elements of \( G \). Furthermore, if \( p = 2 \), every subgroup \( Z \) of order 4 of \( F(G) \) is normalised by every \( 2'- \) element of \( G \). Since the automorphism group of \( Z \) is of order 2, it follows that \( O^2(G) \) centralises every subgroup with order 2 and order 4 of \( F(G) \). In this case, we can apply [10, IV, Satz 5.12], to conclude that \( O^p(G) \) centralises \( F(G) \).

If \( G \) is \( p \)-soluble, then \( C_G(F(G)) \leq F(G) \) by [10, VI, Hilfsatz 6.5]. Consequently, \( G \) is a \( p \)-group.

**Proof (of Theorem 7)** Assume that all subgroups of \( G_p \in \Sigma \) with order \( p \) and order 4 (if \( p = 2 \)), with \( G \) a \( p \)-soluble, non-\( p \)-supersoluble group of the smallest possible order, are \( \Sigma \)-permutable.

By the above arguments, \( p \) is odd, \( O_p'(G) = 1 \). Since \( G \) is \( p \)-soluble, it follows that \( S \), the abelian socle of \( G \), is just \( \text{Soc}(G) \) and every minimal normal subgroup of \( G \) is not central in \( G \) and has order \( p \). Let \( N \) be one of them. Then \( C_G(N) \) is a proper normal subgroup of \( G \). Let \( M \) be a maximal normal subgroup of \( G \) containing \( C_G(N) \). Since \( N \) has order \( p \), \( G/C_G(N) \) is a cyclic group of order dividing \( p - 1 \). In particular, \( [G : M] \) is a \( p' \)-group. Since \( O_p'(M) \leq O_p'(G) = 1 \), it follows that \( O_{p', p}(M) = O_p'(M) \). The
minimal choice of \(G\) implies that \(M\) is a \(p\)-supersoluble group. Hence \(M/O_p(M)\) is an abelian group of exponent dividing \(p - 1\). Therefore \(O_p(M)\) is a Sylow \(p\)-subgroup of \(G\). In particular, \(O_p(M) = G_p\) is a normal subgroup of \(G\).

Since \(G\) is not \(p\)-supersoluble, then \(G\) contains a minimal non-\(p\)-supersoluble subgroup \(H\). Hence \(H\) is one of the groups of \([2, \text{Theorem } 9]\). We will follow the notation of this paper.

Assume that \(|H|\) is divisible only by two primes, \(p\) and \(q\). Then the Sylow \(p\)-subgroup \(H_p\) of \(H\) is contained in \(G_p\). The Sylow \(q\)-subgroup \(H_q\) of \(H\) is contained in a conjugate \(G^x_q\) of \(G_q\), with \(x \in G\). By taking \(H^{-1}\) if necessary, we can assume that \(H_q\) is contained in \(G_q\). Let \(x\) be an element of order \(p\) of \(H\). Then \(\langle x \rangle G_q\) is a subgroup of \(G\). Now \(\langle x \rangle G_q \cap G_p = \langle x \rangle (G_q \cap G_p) = \langle x \rangle\) is a normal subgroup of \(\langle x \rangle G_q\). In particular, \(H_q\) normalises \(\langle x \rangle\). This rules out the groups of types 2, 4, 6, 8, and 10. Moreover, every element of order a power of \(q\) acts in the same way on all elements of order \(p\). This rules out the groups of type 3, 5, 7, and 9, since there are elements \(x\) of \(H_q\) such that \(H_q\) does not normalise \(\langle x \rangle\). Suppose that \(H\) is a group of type 1. If \(s = 1\), we consider the generator \(c\) of \(C\), of order \(p\), which is not normalised by the Sylow \(q\)-subgroup \(H_q\). If \(s > 1\), then if \(c\) is a generator of \(C\), \(c^{(s)^{-1}}\) has order \(p\) and is centralised by \(H_q\), but \(H_q\) does not centralise the elements of order \(p\) of \(M\). This contradicts the fact that \(H_q\) induces the same automorphism on all cyclic subgroups of order \(p\).

Assume now that \(H\) has order divisible by three primes, \(p\), \(q\), and \(r\). Then \(H\) is one of the groups of types 11 or 12. As above, we can assume that the Sylow \(q\)-subgroup \(H_q\) is contained in \(G_q\). As before, \(G_q\) normalises \(\langle x \rangle\) for each \(x \in G_p\), in particular, \(H_q\) normalises \(\langle x \rangle\) for each \(x \in H_p\). If \(G\) is a group of type 12, then \(H_p = P\) is an extraspecial group of order \(p^3\) and exponent \(p\) and the elements of \(H_q = M\) act on the cyclic subgroups of \(P\) in the same way. This is impossible since \(M\) does not centralise \(P\). Assume that \(G\) is a group of type 11. There exists \(z \in G\) such that the Sylow \(r\)-subgroup \(H_r\) of \(H\) is contained in \(G_z\). Let \(c\) be the generator of \(C\). Given \(y \in G_r\), there exists an integer \(t(y)\) such that if \(x\) is an element of order \(p\) of \(G_p\), \(x^t = x^t(c)\) for each \(y \in G_r\). In particular, given an element \(x\) of \(H_p^{(t)}\), \(x^c = x^t(c)\). Since \(c\) acts in the same way on all elements of \(G_p\), for every element \(x\) of \(H_p\), \(x^c = x^t(c)\). But this implies that \(MC\) acts as a group of power automorphisms on \(P\), in particular, \(MC\) acts as an abelian group on \(P\). This implies that \(H\) is \(p\)-supersoluble, a contradiction.

Proof (of Theorem 8) Let \(G\) be a group in which every cyclic subgroup with order \(p\) or order 4 (if \(p = 2\)) of \(G_p\) is 3-permutable. Assume that the order of \(G_p\) is greater than \(p\). We prove that \(G\) is \(p\)-soluble by induction on the order of \(G\). Applying the above arguments, we may assume that \(O_p(G) = 1\) and every abelian minimal normal subgroup of \(G\) is of order \(p\).

Let \(M\) be a maximal normal subgroup of \(G\). Then, by Lemma 1, \(M\) satisfies the hypotheses of the theorem. Therefore either \(M_p\) is of order \(p\) or \(M\) is \(p\)-soluble. If \(M\) is \(p\)-soluble, then \(M\) is \(p\)-supersoluble by Theorem 7. Since \(O_p(M) \leq O_p(G) = 1\), it follows that \(M_p = G_p \cap M\) is a normal Sylow \(p\)-subgroup of \(M\) by \([3, \text{Lemma } 2.1.6]\).

Let \(A\) be a maximal normal subgroup of \(G\) such that \(A \neq M\). Then \(G = AM\). Applying \([3, \text{Theorem } 1.1.19]\), there exist Sylow \(p\)-subgroups \(A_p\) and \(M_p\) of \(A\) and \(M\), respectively, such that \(A_pM_p\) is a Sylow \(p\)-subgroup of \(G\). If \(|A_p| = |M_p| = p\), then
\[ |G_p| = p^2 \] and, by Theorem 3, \( G \) is \( p \)-supersoluble. Suppose that the order of \( M_p \) is greater than \( p \). Then \( M_p \) is normal in \( G \). If \( A \) were \( p \)-supersoluble, then \( A_p \) would be also normal in \( G \) and so would be \( G_p \). Then \( G \) is \( p \)-soluble. Assume that \( |A_p| = p \). Then \( M_p \) is not contained in \( A \) and so \( G = AM_p \). This means that \( G/A \) is of order \( p \) and \( |G_p| = p^2 \). By Theorem 3, \( G \) is \( p \)-supersoluble. Therefore \( G \) is \( p \)-soluble.

Therefore we may assume that \( M \) is the unique maximal normal subgroup of \( G \). Assume that \( MG_p \) is a proper subgroup of \( G \). Then \( \mathfrak{Z} \cap MG_p \) is a complete set of Sylow subgroups of \( MG_p \) and every subgroup of \( G_p \) with order \( p \) or order 4 (if \( p = 2 \)) is \( (\mathfrak{Z} \cap MG_p) \)-permutable. By induction, \( MG_p \) is \( p \)-soluble and \( M_p \) is a normal subgroup of \( G \). Since \( O_p(G) = 1 \), it follows that \( M_p = 1 \) and \( F(G) = F(M) \) is a non-trivial \( p \)-group. Suppose that \( p = 2 \). Since \( M \) is \( 2 \)-soluble, we conclude that \( M \) is a \( 2 \)-group and so \( G \) is \( 2 \)-soluble. Assume that \( p = 3 \) and every abelian minimal normal subgroup of \( G \) is non-central. Let \( N \) be one of them. Then \( C_G(N) \) is contained in \( M \) and so \( G/M \) is a \( p' \)-group. Consequently, \( M_p \) is a normal Sylow \( p \)-subgroup of \( G \) and \( G \) is \( p \)-soluble.

We may therefore assume that \( G = MG_p \) and \( |G : M| = p \). If \( M \) were not \( p \)-soluble, then \( M_p \) must be of order \( p \) and so the order of the Sylow \( p \)-subgroups of \( G \) would be \( p^3 \). By Theorem 3, \( G \) is \( p \)-supersoluble.

Consequently, in all cases, \( G \) is \( p \)-soluble and the induction argument is complete.

**Proof (of Corollary 5)** Assume that \( G \) is a group in which every cyclic subgroup with order \( p \) or order 4 (if \( p = 2 \)) of \( G_p \) is \( \mathfrak{Z} \)-permutable. If the order of \( G_p \) is greater than \( p \), then \( G \) is \( p \)-soluble by Theorem 8. If the order of \( G_p \) is \( p \), then \( G \) has Hall \( \{p,q\} \)-subgroups for all \( q \in \pi(G) \). By Theorem 9, \( G \) is \( p \)-soluble. In both cases, we have that \( G \) is \( p \)-soluble. Applying Theorem 7, \( G \) is \( p \)-supersoluble.

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