On nearly Hausdorff compactifications

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Abstract. We introduce and study here the notion of nearly Hausdorffness, a separation axiom, stronger than $T_1$ but weaker than $T_2$. For a space $X$, from a subfamily of the family of nearly Hausdorff spaces, we construct a compact nearly Hausdorff space $rX$ containing $X$ as a densely $C^*$-embedded subspace. Finally, we discuss when $rX$ is $\beta X$.

2000 AMS Classification: Primary 54C45, Secondary 54D35.

Keywords: Regular closed set, filter, compactification, Wallman base.

1. Introduction

A closed subset $F$ in a topological space $X$ is called a regular closed set if $F = Cl(Int F)$. We denote the family of all regular closed subsets of $X$ by $R(X)$. Observe that $R(X)$ is closed under finite union. Also, if $F \in R(X)$, then $Cl(X - F) = X - Int F \in R(X)$. In Section 2, we define and study the notion of a nearly Hausdorff space (nh-space). We introduce a topological property $\Pi$ and note that a space with property $\Pi$ is an nh-space if and only if it is Urysohn. A flow diagram showing various implications about separation axioms supported by necessary counter examples is included in this section.

A map $f: X \to Y$ is called a density preserving map (dp-map) if for $A \subseteq X$, $Int(Cl f(A)) \neq \phi$ whenever $Int A \neq \phi$ [2]. We provide here an example showing that the nh-property is not preserved even under continuous dp-maps. Note that if $X$ is an nh-space then $R(X)$ forms a base for closed sets in $X$.

In Section 3, we obtain a $\beta X$ like' compactification of an nh-space $X$ with property $\Pi$. Since $R(X)$ need not be closed under finite intersections, we form a new collection $Rf(X)$, of all possible finite intersections of members of $R(X)$. We observe that for an nh-space $X$ with the property $\Pi$, the set $rX = \{\alpha \subseteq Rf(X) \mid \alpha$ is an $r$-ultrafilter} with the natural topology, is a nearly Hausdorff compact space which contains $X$ as a dense $C^*$-embedded subspace. The natural question when $rX = \beta X$ is discussed in Section 4. We observe that
an nh-space $X$ for which $Rf(X)$ is a Wallman base, is a completely regular Hausdorff space and hence for such a space $X$, $rX = \beta X$, the Stone-Čech compactification of $X$. In particular, if $X$ is normal or zero-dimensional then $rX = \beta X$. The problem whether $rX = \beta X$ for any Tychonoff space $X$ is still open.

2. Nearly Hausdorff spaces

**Definition 2.1.** Distinct points $x$ and $y$ in a topological space $X$ are said to be separated by subsets $A$ and $B$ of $X$ if $x \in A - B$ and $y \in B - A$.

**Definition 2.2.** A topological space $X$ is called nearly Hausdorff (nh-) if for every pair of distinct points of $X$ there exists a pair of regular closed sets separating them.

**Definition 2.3.** A space $X$ is said to have property II if for every $F \in R(X)$ and $x \notin F$ there exists an $H \in R(X)$ such that $x \in \text{Int}H$ and $H \cap F = \emptyset$. The symbol $X(\Pi)$ denotes a space $X$ having property II.

**Remark 2.4.** Henceforth all our regular spaces are Hausdorff. Recall that a space $X$ is Urysohn [5] if for each pair of distinct points of $X$, we can find disjoint regular closed sets of $X$ containing the points in their respective interiors. We have following implications:

\[
\text{Regular } \Rightarrow \text{Urysohn } \iff \text{Nearly Hausdorff } \Rightarrow \text{Urysohn} \quad \downarrow \\
\text{T}_1 \iff \text{Nearly Hausdorff } \iff \text{Hausdorff}
\]

Examples given below (refer [4, 5]) justify that unidirectional implications in the above flow diagram need not be reversible. In addition, Example 2.5(b) shows that nearly Hausdorffness is not a closed hereditary property.

**Example 2.5.**

(a) An infinite cofinite space is a $T_1$ space but not an nh-space. The one-point compactification of the space $X$ in our Note 2 is a non-Hausdorff compact nh-space.

(b) Consider $\mathbb{N}$, the set of natural numbers with cofinite topology and $\mathbb{I} = [0, 1]$ with the usual topology. Let $X = \mathbb{N} \times \mathbb{I}$ and define a topology on $X$ as follows: neighborhoods of $(n, y)$, $y \neq 0$ will be usual neighborhoods $\{(n, z) \mid y - \epsilon < z < y + \epsilon\}$ in $\mathbb{I}$, for small positive $\epsilon$; neighborhoods of $(n, 0)$ will have the form $\{(m, z) \mid m \in U, 0 \leq z < \epsilon_m\}$, where $U$ is a neighborhood of $n$ in $\mathbb{N}$ and $\epsilon_m$ is a small positive number for each $m \in U$. The resulting space $X$ is a non-Hausdorff, nh-space without property II. It is easy to observe that the subspace $\mathbb{N}$ of $X$ is closed but is a non-nh, $T_1$ space.

(c) Let $A$ be the linearly ordered set $\{1, 2, 3, \ldots, \omega, \ldots, -3, -2, -1\}$ with the interval topology and let $\mathbb{N}$ be the set of natural numbers with the discrete topology. Define $X$ to be $A \times \mathbb{N}$ together with two distinct
points say $a$ and $-a$ which are not in $A \times \mathbb{N}$. The topology $\mathcal{S}$ on $X$ is determined by the product topology on $A \times \mathbb{N}$ together with basic neighborhoods $M_n(a) = \{a\} \cup \{(i, j) \mid i < \omega, j > n\}$ and $M_n(-a) = \{-a\} \cup \{(i, j) \mid i > \omega, j > n\}$ about $a$ and $-a$ respectively. Resulting space $X$ is a non-Urysohn Hausdorff space without property II. In fact, there does not exist any regular closed set containing $a$ and disjoint from $M_n(-a)$. This example also justifies that a Hausdorff space need not have property II.

(d) Let $S$ be the set of rational lattice points in the interior of the unit square except those whose $x$-coordinate is $\frac{1}{2}$. Define $X$ to be $S \cup \{(0, 0)\} \cup \{(1, 0)\} \cup \{(\frac{1}{2}, r \sqrt{2}) \mid r \in \mathbb{Q}, 0 < r \sqrt{2} < 1\}$. Topologize $X$ as follows: local basis for points in $X$ from the interior of unit square are same as those inherited from the Euclidean topology and for other points following local bases are taken:

$$U_n(0, 0) = \{(x, y) \in S \mid 0 < x < \frac{1}{4}, 0 < y < \frac{1}{n}\} \cup \{(0, 0)\}, \quad U_n(1, 0) = \{(x, y) \in S \mid \frac{3}{4} < x < 1, 0 < y < \frac{1}{n}\} \cup \{(1, 0)\}, \quad U_n(\frac{1}{2}, r \sqrt{2}) = \{(x, y) \in S \mid \frac{1}{4} < x < \frac{3}{4}, |y-r \sqrt{2}| < \frac{1}{n}\}.$$

The resulting space $X$ is a Urysohn space without property II.

(e) Let $X$ be the set of real numbers with neighborhoods of non-zero points as in the usual topology, while neighborhoods of 0 will have the form $U - A$, where $U$ is a neighborhood of 0 in the usual topology and $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Note that $X$ is a non regular Urysohn space with property II.

**Theorem 2.6.** A nonempty product of an nh-space is an nh-space if and only if each factor is an nh-space.

Proof. Let $\{X_\gamma\}_{\gamma \in \Lambda}$ be a family of nh-spaces, $\lambda \neq \phi$ and let $x, y \in X = \prod_{\gamma \in \Lambda} X_\gamma, x \neq y$. Then $x_\gamma \neq y_\gamma$, for some $\gamma \in \Lambda$. Since each $X_\gamma$ is an nh-space, there exist regular closed sets $F_x$ and $F_y$ separating $x_\gamma$ and $y_\gamma$. Define $U = \prod_{\beta \in \Lambda} U_\beta$ and $V = \prod_{\beta \in \Lambda} V_\beta$, where $V_\beta = U_\beta = X_\beta$, for $\beta \neq \gamma$ and $U_\gamma = \text{Int} F_x$, $V_\gamma = \text{Int} F_y$. The regular closed sets $\text{Cl} U$ and $\text{Cl} V$ in $X$ separate $x$ and $y$.

Proof of the converse is similar. $\square$

**Lemma 2.7.** Let $X$ be an nh-space and let $f : X \to Y$ be a dp-epimorphism. Then for a regular closed subset $H$ of $Y$ we have $\text{Cl} f(\text{Cl} f^{-1}(\text{Int} H)) = H$ and hence $R(Y) = \{\text{Cl} f(F) \mid F \in R(X)\}$.

Proof. Clearly for $H \in R(Y)$, $\text{Cl} f(\text{Cl} f^{-1}(\text{Int} H)) \subseteq H$. For the reverse containment, if $y \in H - \text{Cl} f(\text{Cl} f^{-1}(\text{Int} H))$ then there exists an open set $U$ containing $y$ satisfying $f^{-1}(U \cap \text{Int} H) = \phi$ which contradicts $y \in H = \text{Cl} \text{Int} H$. $\square$

**Note 1.** Lemma 2.7 is stated in note 2.2 of [2] for a regular space. Further, observe that the first projection of the space $N \times I$ in example 2.5 (b) shows that continuous image of an nh-space need not be an nh-space. On the other hand, if we consider second projection of $N \times I$ on $[0, 1]$ with cofinite topology then we get that even a continuous density preserving image of an nh-space need not be an nh-space.
3. The Space \( rX \)

For an nh-space \( X \), a filter \( \alpha \subseteq R f(X) - \{ \varnothing \} \) is called an \( r \)-filter. A maximal \( r \)-filter is called an \( r \)-ultrafilter. The family of all \( r \)-ultrafilters in \( X \) is denoted by \( rX \). Observe that for \( x \in X \), there exists a unique \( r \)-ultrafilter \( \alpha_x \) in \( rX \) such that \( \cap \alpha_x = \{ x \} \). Further, if \( X \) is compact then each \( r \)-ultrafilter in \( X \) is fixed. The converse is also true: If \( C \) is an open cover of \( X \) then \( B = \{ F \in R(X) \mid X - U \subset F, \text{ for some } U \in C \} \) does not have finite intersection property for otherwise \( B \) will generate a fixed \( r \)-ultrafilter which will contradict that \( C \) is a cover of \( X \). Hence \( C \) has a finite subcover. Topologize the set \( rX \) by taking \( B = \{ F \in R(X) \mid F \in \alpha \} \) as a base for closed sets in \( rX \), where \( F = \{ \alpha \in rX \mid F \in \alpha \} \) and \( F \in R(X) \). The map \( rX \rightarrow rX \) defined by \( r(x) = \alpha_x \), where \( \alpha_x = \{ F \in Rf(X) \mid x \in F \} \) is an embedding.

**Lemma 3.1.** Let \( X \) be an nh-space with property \( \Pi \). Then the space \( rX \) of all \( r \)-ultrafilters in \( X \) is a compact nh-space which contains \( X \) as a dense subspace.

**Proof.** Clearly \( \alpha_x = \{ F \in Rf(X) \mid x \in F \} \) is an \( r \)-filter. For maximality of \( \alpha_x \), suppose \( A = \cap_{i=1}^{n} A_i \) in \( Rf(X) \) be such that \( A \cap F \neq \varnothing \), for each \( F \) in \( \alpha_x \). If possible suppose for some \( i \), \( A_i \notin \alpha_x \). Then \( x \notin A_i \). By the property \( \Pi \), there exists an \( H \in R(X) \) such that \( x \in IntH \) and \( H \cap A_i = \varnothing \). Therefore \( H \in \alpha_x \) and hence \( H \cap A \neq\varnothing \). But this implies \( \varnothing \neq H \cap A \subset H \cap A_i \neq \varnothing \), a contradiction. Further \( Cl_{rX}(F) = \overline{F} \) for all \( F \in R(X) \) implies \( r \) is a dense embedding. \( \square \)

**Note 2.** A compactification of a non-Urysohn space without property \( \Pi \) may also be an nh-space. For example, consider the subspace

\[ Y = \{ \left( \frac{1}{n}, \frac{1}{m} \right) \mid n \in \mathbb{N}, m \in \mathbb{N} \} \cup \{ \left( \frac{1}{n}, 0 \right) \mid n \in \mathbb{N} \} \]

of the usual space \( \mathbb{R}^2 \). Take \( X = Y \cup \{ p^+, p^- \}; p^+, p^- \notin Y \) and topologize it by taking sets open in \( Y \) as open in \( X \) and a set \( U \) containing \( p^+ \) (respectively \( p^- \)) to be open in \( X \) if for some \( r \in \mathbb{N} \), \( \{(\frac{1}{n}, \frac{1}{m}) \mid n \geq r, m \in \mathbb{N} \} \subseteq U \) (respectively \( \{(\frac{1}{n}, \frac{1}{m}) \mid n \geq r, -m \in \mathbb{N} \} \subseteq U \). The resulting space \( X \) is a non-Urysohn Hausdorff space without property \( \Pi \) and its one point compactification is an nh-space.

**Proposition 3.2.** Let the space \( X \) and \( rX \) be as in Lemma 3.1. Then \( X \) is \( C^* \)-embedded in \( rX \).

**Proof.** Let \( f \in C^*(X) \). Suppose range of \( f \subseteq [0, 1] = I \). For \( \alpha \in rX \), define \( f^\sharp(\alpha) = \{ H_1 \cup H_2 \in R(I) \mid Cl_X f^{-1}(Int H_1 \cup Int H_2) \in \alpha \} \). Note that if \( H_1 \cup H_2 \in f^\sharp(\alpha) \) then either \( H_1 \in f^\sharp(\alpha) \) or \( H_2 \in f^\sharp(\alpha) \). Also \( f^\sharp(\alpha) \) satisfies finite intersection property. Thus \( \cap f^\sharp(\alpha) \neq \varnothing \). We assert that \( \cap f^\sharp(\alpha) = \{ t \} \), for some \( t \in I \).

Assuming the assertion in hand, we define \( rf: rX \rightarrow I \) by \( rf(\alpha) = \cap f^\sharp(\alpha) \). Clearly \( rf \) restricted to \( X \) is \( f \). We now establish continuity of \( rf \). Let \( \alpha \in rX \). Then choose an open set \( G \) of \( I \) such that \( t \in G \), where \( rf(\alpha) = t \). Using
regularity of $I$ successively we obtain open sets $G_1$, $G_2$ such that $t \in G_1 \subseteq ClG_1 \subseteq G_2 \subseteq ClG_2 \subseteq G$. Set $F_t = ClG_2$ and $H_t = Cl(I - ClG_1)$. Since $IntF_t \cup IntH_t = I$. We have $F_t \cup H_t \in f^t(\alpha)$ and as $t \notin H_t$, $F_t \in f^t(\alpha)$ and $H_t \notin f^t(\alpha)$. If $L_t = ClXf^{-1}(IntH_t)$, then $\alpha \notin L_t$ and the open set $rX - L_t$ contains $\alpha$. Finally the containment $rf(rX - L_t) \subseteq G$ establishes the continuity. For the assertion, one may use the above technique to note that \{ $F \in R(I) \mid t \in IntF$ \} \subseteq f^t(\alpha), for each $t \in f^t(\alpha)$. \hfill $\square$

**Theorem 3.3.** Let $X$ be an nh-space with property $\Pi$. Then there exists a compact nh-space $rX$ in which $X$ is densely $C^*$-embedded.

**Proof.** Follows from Lemma 3.1 and Proposition 3.2. \hfill $\square$

**Corollary 3.4.** If $X$ is a regular space, then it is densely $C^*$-embedded in $rX$.

4. When $rX = \beta X$?

Let $X$ be an nh-space such that $Rf(X)$ is a Wallman base. Then by 19L(7) in [5], $X$ is a completely regular space. Therefore by Corollary 3.4, $X$ is $C^*$-embedded in $rX$. Further if $X$ is an nh-space such that $Rf(X)$ forms a Wallman base then by 19L(5) in [5], $rX$ is Hausdorff. Hence we have the following result:

**Theorem 4.1.** Let $X$ be an nh-space such that $Rf(X)$ is a Wallman base. Then $rX = \beta X$.

**Corollary 4.2.** If $X$ is normal or zero-dimensional then $rX = \beta X$.

**Question:** Is $rX = \beta X$ when $X$ is a Tychonoff space?

**Acknowledgements.** We thank the referee for his/her valuable suggestions.

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