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On nearly Hausdorff compactifications

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Abstract. We introduce and study here the notion of nearly Hausdorffness, a separation axiom, stronger than T_1 but weaker than T_2. For a space X, from a subfamily of the family of nearly Hausdorff spaces, we construct a compact nearly Hausdorff space rX containing X as a densely C*-embedded subspace. Finally, we discuss when rX is βX.

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1. Introduction

A closed subset F in a topological space X is called a *regular closed set* if $F = Cl(IntF)$. We denote the family of all regular closed subsets of X by $R(X)$. Observe that $R(X)$ is closed under finite union. Also, if $F \in R(X)$, then $Cl(X - F) = X - IntF \in R(X)$. In Section 2, we define and study the notion of a nearly Hausdorff space (nh-space). We introduce a topological property Π and note that a space with property Π is an nh-space if and only if it is Urysohn. A flow diagram showing various implications about separation axioms supported by necessary counter examples is included in this section. A map $f{:}X{\to}Y$ is called a *density preserving map* (dp-map) if for $A \subseteq X$, $Int(Clf(A)) \neq \phi$ whenever $IntA \neq \phi$ [2]. We provide here an example showing that the nh-property is not preserved even under continuous dp-maps. Note that if X is an nh-space then $R(X)$ forms a base for closed sets in X.

In Section 3, we obtain a 'βX like' compactification of an nh-space X with property Π. Since $R(X)$ need not be closed under finite intersections, we form a new collection $Rf(X)$, of all possible finite intersections of members of $R(X)$. We observe that for an nh-space X with the property Π, the set $rX = \{\alpha \subseteq Rf(X) \mid \alpha$ is an r-ultrafilter$\}$ with the natural topology, is a nearly Hausdorff compact space which contains X as a dense C*-embedded subspace. The natural question when $rX = \beta X$ is discussed in Section 4. We observe that an nh-space X for which $Rf(X)$ is a Wallman base, is a completely regular Hausdorff space and hence for such a space X, $rX = \beta X$, the Stone-Čech compactification of X. In particular, if X is normal or zero$-$dimensional then $rX = \beta X$. The problem whether $rX = \beta X$ for any Tychonoff space X is still open.

2. Nearly Hausdorff spaces

Definition 2.1. *Distinct points x and y in a topological space X are said to be* ***separated*** *by subsets A and B of X if $x \in A - B$ and $y \in B - A$.*

Definition 2.2. *A topological space X is called* ***nearly Hausdorff*** *(nh-) if for every pair of distinct points of X there exists a pair of regular closed sets separating them.*

Definition 2.3. *A space X is said to have* ***property*** Π *if for every $F \in R(X)$ and $x \notin F$ there exists an $H \in R(X)$ such that $x \in IntH$ and $H \cap F = \phi$. The symbol $X(\Pi)$ denotes a space X having property* Π.

Remark 2.4. Henceforth all our regular spaces are Hausdorff. Recall that a space X is *Urysohn* [5] if for each pair of distinct points of X, we can find disjoint regular closed sets of X containing the points in their respective interiors. We have following implications:

$$\begin{array}{ccccccc}
\text{Regular} & \Rightarrow & \text{Urysohn } (\Pi) & \Leftrightarrow & \text{Nearly Hausdorff } (\Pi) & \Rightarrow & \text{Urysohn} \\
 & & & & & & \Downarrow \\
 & & T_1 & \Leftarrow & \text{Nearly Hausdorff} & \Leftarrow & \text{Hausdorff}
\end{array}$$

Examples given below (refer [4, 5]) justify that unidirectional implications in the above flow diagram need not be revertible. In addition, Example 2.5(b) shows that nearly Hausdorffness is not a closed hereditary property.

Example 2.5.

(a) An infinite cofinite space is a T_1 space but not an nh-space. The one-point compactification of the space X in our Note 2 is a non-Hausdorff compact nh-space.

(b) Consider N, the set of natural numbers with cofinite topology and $\mathbf{I} = [0, 1]$ with the usual topology. Let $X = N \times \mathbf{I}$ and define a topology on X as follows: neighborhoods of (n, y), $y \neq 0$ will be usual neighborhoods $\{(n, z) \mid y - \epsilon < z < y + \epsilon\}$ in $\mathbf{I}_n = \{n\} \times \mathbf{I}$ for small positive ϵ; neighborhoods of $(n, 0)$ will have the form $\{(m, z) \mid m \in U, 0 \leq z < \epsilon_m\}$, where U is a neighborhood of n in N and ϵ_m is a small positive number for each $m \in U$. The resulting space X is a non Hausdorff, nh-space without property Π. It is easy to observe that the subspace N of X is closed but is a non-nh, T_1 space.

(c) Let A be the linearly ordered set $\{1, 2, 3, ..., \omega, ..., -3, -2, -1\}$ with the interval topology and let $\mathbf{N}$ be the set of natural numbers with the discrete topology. Define X to be $A \times \mathbf{N}$ together with two distinct points say a and $-a$ which are not in $A\times\mathbf{N}$. The topology $\Im$ on X is determined by the product topology on $A\times\mathbf{N}$ together with basic neighborhoods $M_n(a) = \{a\} \cup \{(i,j) \mid i < \omega, j > n\}$ and $M_n(-a) = \{-a\} \cup \{(i,j) \mid i > \omega, j > n\}$ about a and $-a$ respectively. Resulting space X is a non-Urysohn Hausdorff space without property Π. In fact, there does not exist any regular closed set containing a and disjoint from $\overline{M_n(-a)}$. This example also justifies that a Hausdorff space need not have property Π.

(d) Let S be the set of rational lattice points in the interior of the unit square except those whose x-coordinate is $\frac{1}{2}$. Define X to be $S \cup \{(0,0)\} \cup \{(1,0)\} \cup \{(\frac{1}{2}, r\sqrt{2}) \mid r \in Q,\ 0 < r\sqrt{2} < 1\}$. Topologize X as follows: local basis for points in X from the interior of unit square are same as those inherited from the Euclidean topology and for other points following local bases are taken:
$U_n(0,0) = \{(x, y) \in S \mid 0 < x < \frac{1}{4},\ 0 < y < \frac{1}{n}\} \cup \{(0, 0)\}$, $U_n(1, 0) = \{(x, y) \in S \mid \frac{3}{4} < x < 1,\ 0 < y < \frac{1}{n}\} \cup \{(1,0)\}$, $U_n(\frac{1}{2}, r\sqrt{2}) = \{(x, y) \in S \mid \frac{1}{4} < x < \frac{3}{4},\ |y - r\sqrt{2}| < \frac{1}{n}\}$.
The resulting space X is a Urysohn space without property Π.

(e) Let X be the set of real numbers with neighborhoods of non-zero points as in the usual topology, while neighborhoods of 0 will have the form $U - A$, where U is a neighborhood of 0 in the usual topology and $A = \{\frac{1}{n} \mid n \in \mathbf{N}\}$. Note that X is a non regular Urysohn space with property Π.

Theorem 2.6. *A nonempty product of an nh-space is an nh-space if and only if each factor is an nh-space*

Proof. Let $\{X_\gamma\}_{\gamma\in\lambda}$ be a family of nh-spaces, $\lambda \neq \phi$ and let $x, y \in X = \prod_{\gamma\in\lambda} X_\gamma$, $x \neq y$. Then $x_\gamma \neq y_\gamma$ for some $\gamma \in \lambda$. Since each X_γ is an nh-space, there exist regular closed sets F_x and F_y separating x_γ and y_γ. Define $U = \prod_{\beta\in\lambda} U_\beta$ and $V = \prod_{\beta\in\lambda} V_\beta$, where $V_\beta = U_\beta = X_\beta$, for $\beta \neq \gamma$ and $U_\gamma = IntF_x$, $V_\gamma = IntF_y$. The regular closed sets ClU and ClV in X separate x and y. Proof of the converse is similar. □

Lemma 2.7. *Let X be an nh-space and let $f{:}X\to Y$ be a dp-epimorphism. Then for a regular closed subset H of Y we have $Clf(Clf^{-1}(IntH)) = H$ and hence $R(Y) = \{Clf(F) \mid F \in R(X)\}$.*

Proof. Clearly for $H \in R(Y)$, $Clf(Clf^{-1}(IntH)) \subseteq H$. For the reverse containment, if $y \in H - Clf(Clf^{-1}(IntH))$ then there exists an open set U containing y satisfying $f^{-1}(U \cap IntH) = \phi$ which contradicts $y \in H = ClIntH$. □

Note 1. *Lemma 2.7 is stated in note 2.2 of* [2] *for a regular space. Further, observe that the first projection of the space $N\times\mathbf{I}$ in example 2.5* (b) *shows that continuous image of an nh-space need not be an nh-space. On the other hand, if we consider second projection of $N\times\mathbf{I}$ on* $[0, 1]$ *with cofinite topology then we get that even a continuous density preserving image of an nh-space need not be an nh-space.*

3. The space rX

For an nh-space X, a filter $\alpha \subseteq Rf(X)-\{\phi\}$ is called an *r-filter*. A maximal r-filter is called an *r-ultrafilter*. The family of all r-ultrafilters in X is denoted by rX. Observe that for $x \in X$, there exists a unique r-ultrafilter α_x in rX such that $\cap\alpha_x = \{x\}$. Further, if X is compact then each r-ultrafilter in X is fixed. The converse is also true: If C is an open cover of X then $B = \{F \in R(X) | X - U \subset F$, for some $U \in C\}$ does not have finite intersection property for otherwise B will generate a fixed r-ultrafilter which will contradict that C is a cover of X. Hence C has a finite subcover. Topologize the set rX by taking $B = \{\overline{F} \mid F \in R(X)\}$ as a base for closed sets in rX, where $\overline{F} = \{\alpha \in rX \mid F \in \alpha\}$ and $F \in R(X)$. The map $r{:}X \to rX$ defined by $r(x) = \alpha_x$, where $\alpha_x = \{F \in Rf(X) \mid x \in F\}$ is an embedding.

Lemma 3.1. *Let X be an nh-space with property Π. Then the space rX of all r-ultrafilters in X is a compact nh-space which contains X as a dense subspace.*

Proof. Clearly $\alpha_x = \{F \in Rf(X) | x \in F\}$ is an r-filter. For maximality of α_x, suppose $A = \cap_{i=1}^{n} A_i$ in $Rf(X)$ be such that $A \cap F \neq \phi$, for each F in α_x. If possible suppose for some i, $A_i \notin \alpha_x$. Then $x \notin A_i$. By the property Π, there exists an H in $R(X)$ such that $x \in IntH$ and $H \cap A_i = \phi$. Therefore $H \in \alpha_x$ and hence $H \cap A \neq \phi$. But this implies $\phi \neq H \cap A \subset H \cap A_i = \phi$, a contradiction. Further $Cl_{rX} r(F) = \overline{F}$ for all $F \in R(X)$ implies r is a dense embedding. □

Note 2. *A compactification of a non-Urysohn space without property Π may also be an nh-space. For example, consider the subspace*

$$Y = \{(\frac{1}{n}, \frac{1}{m}) \mid n \in \mathbf{N}, |m| \in \mathbf{N}\} \cup \{(\frac{1}{n}, 0) \mid n \in \mathbf{N}\}$$

of the usual space $\boldsymbol{R}^2$. Take $X = Y \cup \{p^+, p^-\}$; $p^+, p^- \notin Y$ and topologize it by taking sets open in Y as open in X and a set U containing p^+ (respectively p^-) to be open in X if for some $r \in \mathbf{N}$, $\{(\frac{1}{n}, \frac{1}{m}) \mid n \geq r, m \in \mathbf{N}\} \subseteq U$ (respectively $\{(\frac{1}{n}, \frac{1}{m}) \mid n \geq r, -m \in \mathbf{N}\} \subseteq U$). The resulting space X is a non-Urysohn Hausdorff space without property Π and its one point compactification is an nh-space.

Proposition 3.2. *Let the space X and rX be as in Lemma 3.1. Then X is C^*-embedded in rX.*

Proof. Let $f \in C^*(X)$. Suppose range of $f \subseteq [0, 1] = \mathbf{I}$. For α in rX, define $f^\sharp(\alpha) = \{H_1 \cup H_2 \in R(\mathbf{I}) \mid Cl_X f^{-1}(IntH_1 \cup IntH_2) \in \alpha\}$. Note that if $H_1 \cup H_2 \in f^\sharp(\alpha)$ then either $H_1 \in f^\sharp(\alpha)$ or $H_2 \in f^\sharp(\alpha)$. Also $f^\sharp(\alpha)$ satisfies finite intersection property. Thus $\cap f^\sharp(\alpha) \neq \phi$. We assert that $\cap f^\sharp(\alpha) = \{t\}$, for some $t \in \mathbf{I}$.

Assuming the assertion in hand, we define rf: $rX \to \mathbf{I}$ by $rf(\alpha) = \cap f^\sharp(\alpha)$. Clearly rf restricted to X is f. We now establish continuity of rf. Let $\alpha \in rX$. Then choose an open set G of $\mathbf{I}$ such that $t \in G$, where $rf(\alpha)=t$. Using regularity of $\mathbf{I}$ successively we obtain open sets G_1, G_2 such that $t \in G_1 \subseteq ClG_1 \subseteq G_2 \subseteq ClG_2 \subseteq G$. Set $F_t = ClG_2$ and $H_t = Cl(\mathbf{I} - ClG_1)$. Since $IntF_t \cup IntH_t = \mathbf{I}$. We have $F_t \cup H_t \in f^\sharp(\alpha)$ and as $t \notin H_t$, $F_t \in f^\sharp(\alpha)$ and $H_t \notin f^\sharp(\alpha)$. If $L_t = Cl_X f^{-1}(IntH_t)$, then $\alpha \notin \overline{L}_t$ and the open set $rX - \overline{L}_t$ contains α. Finally the containment $rf(rX - \overline{L}_t) \subseteq G$ establishes the continuity. For the assertion, one may use the above technique to note that $\{F \in R(\mathbf{I}) \mid t \in IntF\} \subseteq f^\sharp(\alpha)$, for each $t \in f^\sharp(\alpha)$. □

Theorem 3.3. *Let X be an nh-space with property Π. Then there exists a compact nh-space rX in which X is densely C^*-embedded.*

Proof. Follows from Lemma 3.1 and Proposition 3.2. □

Corollary 3.4. *If X is a regular space, then it is densely C^*-embedded in rX.*

4. When $rX = \beta X$?

Let X be an nh-space such that $Rf(X)$ is a Wallman base. Then by 19L(7) in [5], X is a completely regular space. Therefore by Corollary 3.4, X is C*-embedded in rX. Further if X is an nh-space such that $Rf(X)$ forms a Wallman base then by 19L(5) in [5], rX is Hausdorff. Hence we have the following result:

Theorem 4.1. *Let X be an nh-space such that $Rf(X)$ is a Wallman base. Then $rX = \beta X$.*

Corollary 4.2. *If X is normal or zero-dimensional then $rX = \beta X$.*

Question: *Is $rX = \beta X$ when X is a Tychonoff space?*

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