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Connected metrizable subtopologies and partitions into copies of the Cantor set

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Abstract. We prove under Martin's Axiom that every separable metrizable space represented as the union of less than 2^ω zero-dimensional compact subsets is zero-dimensional. On the other hand, we show in ZFC that every separable completely metrizable space without isolated points is the union of 2^ω pairwise disjoint copies of the Cantor set.

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1. Introduction

We say that a space X *condenses* onto a space Y if there exists a continuous bijection $f: X \to Y$. A space X has a weaker connected topology (also called *connected subtopology*) if X condenses onto a connected space. In the article [6] on connected subtopologies, the following two results were proved:

(a) Let C_α be the Cantor set, for each $\alpha \in A$, and $X = \bigoplus\{C_\alpha : \alpha \in A\}$ the disjoint topological union of the spaces C_α. The space X has a weaker T_3 connected topology iff it has a weaker Tychonoff connected topology iff $|A| \geq \omega_1$.

(b) If I_α is the unit segment $[0,1]$, for each $\alpha \in A$, and $X = \bigoplus\{I_\alpha : \alpha \in A\}$ is the disjoint topological union of the spaces I_α, then space X has a weaker T_3 connected topology iff $|A| \geq \omega$.

In Section 3 of this article, we study the problem of when the spaces of the form $\bigoplus\{C_\alpha : \alpha \in A\}$ or $\bigoplus\{I_\alpha : \alpha \in A\}$ admit connected metrizable subtopologies. Here we mention three more results on connected subtopologies presented in [4] that will be used in the sequel.

Theorem 1.1. *Let X be a metrizable space of weight $\kappa \leq 2^\omega$. If there exists a closed set $P \subseteq X$ which admits a condensation onto a connected non-compact metrizable space, then X condenses onto a connected separable metrizable space.*

Theorem 1.2. *Every metrizable space of weight 2^ω admits a condensation onto a connected separable metrizable space.*

Theorem 1.3. *Every metrizable space of weight $\kappa \geq 2^\omega$ with achievable extent admits a weaker connected metrizable topology.*

Recall that a metrizable space X has an *achievable extent* if X contains a closed discrete subset of the size equal to the weight of X.

The authors of [6] rose the following problem:

Problem 1.4. *Is it true in ZFC that the topological sum of ω_1 copies of the Cantor set condenses onto a connected compact space?*

Delay and Just in [3] gave the negative answer to the above problem under the assumption that the real line cannot be covered by ω_1 nowhere dense sets.

In Section 2 of the article we show under MA that if separable metrizable space X is a union of less than 2^ω zero-dimensional compact subspaces, then X is also zero-dimensional. In particular, under $MA + \neg CH$, all metrizable subtopologies on the space $\bigoplus\{C_\alpha : \alpha \in \omega_1\}$ are zero-dimensional. This result and Theorem 1.2 together imply that the existence of a connected metrizable subtopology on the topological sum of ω_1 copies of the Cantor set does not depend on ZFC.

We also prove in Section 2 that the topological sum $\bigoplus\{I_\alpha : \alpha \in A\}$, where each I_α is a copy of the unit segment $[0, 1]$, admits a connected metrizable subtopology iff $|A| \geq \omega$.

In connection with Problem 1.4 we show in Section 3 that the topological sum of 2^ω copies of the Cantor set condenses onto the closed unit interval and, even more, it condenses onto every compact metrizable space without isolated points. Finally, we generalize the latter fact and prove that every separable complete metrizable space without isolated points can be represented as a disjoint union of 2^ω copies of the Cantor set.

The reader can consult Kunen's book [5] for details about Martin's Axiom (for short, MA).

2. Connected metrizable subtopologies

Let C_α be the Cantor set for each $\alpha < \omega_1$ and let $X = \bigoplus\{C_\alpha : \alpha < \omega_1\}$. We consider the question whether X admits a connected metrizable subtopology. Notice that under CH, the answer is affirmative, since one can apply Theorem 1.2.

The next theorem, combined with the well-known fact that every metrizable space of weight $\leq 2^\omega$ admits a separable metrizable subtopology (see Lemma 2.5 in [4]), implies that the answer to the question is "no" under $MA + \neg CH$.

In what follows the family of all finite subsets of a set A is denoted by $[A]^{<\omega}$.

Theorem 2.1. *Suppose that MA holds. Let X be a separable metrizable space represented as $X = \bigcup_{\alpha<\kappa} K_\alpha$, where $\kappa < 2^\omega$ and every K_α is a zero-dimensional compact subspace of X. Then X is also zero-dimensional.*

Proof. Let $\mathcal{B}$ be a countable base for X. Pick a point $x_0 \in X$ and take any element $W \in \mathcal{B}$ such that $x_0 \in W$. We have to show that there exists a clopen set U in X such that $x_0 \in U \subseteq W$.

Notice that for every $\alpha < \kappa$, the open subspace $K_\alpha \cap W$ of K_α is σ-compact since K_α is compact and metrizable. Hence, we can represent each K_α in the form $K_\alpha = \bigcup_{n\in\omega} K_{\alpha,n} \cup (K_\alpha \setminus W)$, where the sets $K_{\alpha,n}$ are compact and satisfy $K_{\alpha,n} \subseteq W$. Therefore, we can assume from the very beginning that $K_\alpha \subseteq W$ or $K_\alpha \cap W = \varnothing$ for each $\alpha < \kappa$.

Let us consider the following family:

$$\mathcal{P} = \Big\{ (F,\gamma,\lambda) : \ F \in [\kappa]^{<\omega}, \ \gamma,\lambda \in [\mathcal{B}]^{<\omega}, \ x_0 \in \bigcup \gamma, \\ \overline{\bigcup \gamma} \subseteq W, \ \overline{\bigcup \gamma} \cap \overline{\bigcup \lambda} = \varnothing, \ \bigcup_{\alpha\in F} K_\alpha \subseteq (\bigcup \gamma) \cup (\bigcup \lambda) \Big\}.$$

It is easy to see that the family $\mathcal{P}$ is not empty. Now we introduce a partial order $\le$ in $\mathcal{P}$ by the following rule:

$$(F_1,\gamma_1,\lambda_1) \le (F_2,\gamma_2,\lambda_2) \iff F_2 \subseteq F_1 \ \& \ \gamma_2 \subseteq \gamma_1 \ \& \ \lambda_2 \subseteq \lambda_1.$$

We claim that the poset $(\mathcal{P},\le)$ satisfies the countable chain condition. Indeed, let $Q \subseteq P$ be a subset of cardinality $\aleph_1$. Since $|\mathcal{B}| = \aleph_0$ there exist $Q^* \subseteq Q$, $\gamma^* \in [\mathcal{B}]^{<\omega}$, and $\lambda^* \in [\mathcal{B}]^{<\omega}$ such that $|Q^*| = \aleph_1$ and all elements of Q^* have the form (F,γ^*,λ^*) with $F \in [\kappa]^{<\omega}$. Take two elements $P_1 = (F_1,\gamma^*,\lambda^*)$ and $P_2 = (F_2,\gamma^*,\lambda^*)$ of Q^*. It is easy to see that $P = (F_1 \cup F_2, \gamma^*,\lambda^*)$ is in $\mathcal{P}$, $P \le P_1$ and $P \le P_2$. This proves our claim. In fact, almost the same argument shows that $(\mathcal{P},\le)$ is σ-centered.

For every $\alpha \in \kappa$, put $D_\alpha = \{(F,\gamma,\lambda) \in \mathcal{P} : \alpha \in F\}$. Let us verify that D_α is dense in $(\mathcal{P},\le)$. Take an element $P = (F,\gamma,\lambda) \in \mathcal{P}$. It suffices to find an element $P^* \in D_\alpha$ such that $P^* \le P$. If $K_\alpha \subseteq \bigcup\gamma \cup \bigcup\lambda$, then $P^* = (F \cup \{\alpha\},\gamma,\lambda) \in \mathcal{D}_\alpha$ and, clearly, $P^* \le P$. Otherwise, we put $T_1 = K_\alpha \cap \overline{\bigcup\gamma}$ and $T_2 = K_\alpha \bigcap \overline{\bigcup\lambda}$. Then T_1 and T_2 are disjoint, and we may assume that $T_1 \ne \varnothing$. Then, clearly, $K_\alpha \subseteq W$. Since K_α is zero-dimensional, there exist clopen sets A and B in K_α such that $A \cup B = K_\alpha$, $A \cap B = \varnothing$, and $T_1 \subseteq A$, $T_2 \subseteq B$. Evidently, the sets A and B are closed in X. Therefore, the sets $A \cup \overline{\bigcup\gamma}$ and $B \cup \overline{\bigcup\lambda}$ are closed in X and disjoint. We choose disjoint open sets O_1 and O_2 in X such that $A \cup \overline{\bigcup\gamma} \subseteq O_1$ and $B \cup \overline{\bigcup\lambda} \subseteq O_2$. In addition, we can take O_1 with $O_1 \subseteq W$, since $A \cup \overline{\bigcup\gamma} \subseteq W$. Let γ' be an open cover of A by elements of the base $\mathcal{B}$ such that $\overline{\bigcup\gamma'} \subseteq O_1$, and λ' an open cover of B by elements of $\mathcal{B}$ such that $\overline{\bigcup\lambda'} \subseteq O_2$. It is easy to see that $P^* = (F \cup \{\alpha\}, \gamma \cup \gamma', \lambda \cup \lambda')$ is an element of $\mathcal{P}$. In addition, $P^* \in \mathcal{D}_\alpha$ and $P^* \le P$. In the case when $T_1 = \varnothing$ we take $P^* = (F \cup \{\alpha\}, \gamma, \lambda \cup \lambda')$. Then again $P^* \in \mathcal{D}_\alpha$ and $P^* \le P$.

We have thus shown that D_α is dense in $(\mathcal{P}, \leq)$ for each $\alpha < \kappa$. It is also clear that the cardinality of the family $\mathcal{D} = \{D_\alpha : \alpha \in \kappa\}$ is not greater than $\kappa < 2^\omega$. Hence, MA implies that there exists a $\mathcal{D}$-generic filter $\mathcal{G}$ in $(\mathcal{P}, \leq)$, that is, a filter $\mathcal{G} \subseteq \mathcal{P}$ such that $\mathcal{G} \cap D_\alpha \neq \varnothing$ for each $\alpha < \omega_1$. Now we define two open subsets U and V of X by

$$U = \bigcup\{\bigcup\gamma : \exists\ F \in [\kappa]^{<\omega}\ \exists\ \lambda \in [\mathcal{B}]^{<\omega} \text{ such that } (F, \gamma, \lambda) \in \mathcal{G}\},$$

$$V = \bigcup\{\bigcup\lambda : \exists\ F \in [\kappa]^{<\omega}\ \exists\ \gamma \in [\mathcal{B}]^{<\omega} \text{ such that } (F, \gamma, \lambda) \in \mathcal{G}\}.$$

Let us check that the sets U and V satisfy the following conditions:

(1) $x_0 \in U \subseteq W$;
(2) $U \cap V = \varnothing$;
(3) $U \cup V = X$.

Condition (1) follows from the definition of $\mathcal{P}$. To show that (2) holds we note that if $P_1 = (F_1, \gamma_1, \lambda_1) \in \mathcal{G}$ and $P_2 = (F_2, \gamma_2, \lambda_2) \in \mathcal{G}$, then there exists $P = (F, \gamma, \lambda) \in \mathcal{G}$ such that $P \leq P_1$ and $P \leq P_2$. Therefore, $\gamma_1 \cup \gamma_2 \subseteq \gamma$ and $\lambda_1 \cup \lambda_2 \subseteq \lambda$. Since $\overline{\bigcup\gamma} \cap \overline{\bigcup\lambda} = \varnothing$, we have that $\overline{\bigcup\gamma_1} \cap \overline{\bigcup\lambda_2} = \varnothing$ and $\overline{\bigcup\gamma_2} \cap \overline{\bigcup\lambda_1} = \varnothing$. Hence, $\bigcup\gamma \cap \bigcup\lambda = \varnothing$ for arbitrary γ and λ that are used to form U and V. To check (3) we note that for each $\alpha < \kappa$, there exists $P = (F^*, \gamma^*, \lambda^*) \in \mathcal{G} \cap D_\alpha$. Hence, $\alpha \in F^*$ and $K_\alpha \subseteq (\bigcup\gamma^*) \cup (\bigcup\lambda^*)$. Since $\bigcup\gamma^* \subseteq U$ and $\bigcup\lambda^* \subseteq V$, we have $K_\alpha \subseteq U \cup V$. It follows from (1) and (3) that $V \neq \varnothing$ and, hence, U and V are clopen sets in X. Thus, the clopen set $U \subseteq X$ satisfies $x_0 \in U \subseteq W$. □

Corollary 2.2. *The existence of a connected metrizable subtopology on the space* $X = \bigoplus\{C_\alpha : \alpha \in \omega_1\}$*, where each* C_α *is a copy of the Cantor set, does not depend on* ZFC*.*

In the second part of this section we study the problem whether there exists a connected metrizable subtopology on the space $X = \bigoplus\{I_\alpha : \alpha \in A\}$, where each I_α is the unit segment.

Example 2.3. *Let* I_n *be the unit segment* $[0, 1]$ *for each* $n \in \omega$ *and* $X = \bigoplus\{I_n : n \in \omega\}$*. Then the space* X *condenses onto a connected non-compact subspace of the plane* $\mathbb{R}^2$*.*

Proof. Let $\{r_n : n \in \mathbb{N}\}$ be the set of rational numbers in $[0, 1]$. For each $n \in \mathbb{N}$, put

$$J_n = \{(x, r_n) \in \mathbb{R}^2 : 1/n \leq x \leq 1\}$$

and $J_0 = \{0\} \times [0, 1]$. Let us consider $Y = \bigcup\{J_n : n \in \omega\}$. We claim that Y is a connected subspace of $\mathbb{R}^2$. Indeed, let U be a clopen set of Y such that $J_0 \subseteq U$. Since J_0 is compact, there exists $\epsilon > 0$ such that if $\widetilde{U} = \{(x, y) \in \mathbb{R}^2 : 0 \leq x < \epsilon,\ 0 \leq y \leq 1\}$, then $J_0 \subseteq \widetilde{U} \cap Y \subseteq U$. Take $n_0 \in \mathbb{N}$ such that $1/n_0 < \epsilon$. Then for every $n \geq n_0$, $c_n = (1/n, r_n) \in \widetilde{U} \cap Y \subseteq U$. Since $c_n \in J_n$, J_n is connected, and U is a clopen set in Y, we conclude that $J_n \subseteq U$ for every $n \geq n_0$.

Take $m < n_0$ and let V be an open neighborhood of the point $c_m = (1/m, r_m)$ in Y. Evidently,

$$|V \cap \{(1/m, r_n) : n \in \mathbb{N}\}| = \aleph_0.$$

Therefore, there exists $k > n_0$ such that $(1/m, r_k) \in J_k \cap V \subseteq U \cap V \neq \varnothing$. It follows that $c_m = (1/m, r_m) \in cl_Y(U)$. Since $c_m \in J_m$, we conclude that $J_m \subseteq U$ for every $m < n_0$. Thus, $U = Y$ and this proves that Y is connected.

For each $n \in \omega$, take a continuous bijection $f_n \colon I_n \to J_n$. Then the sum of the functions f_n, say,

$$f = \nabla_{n\in\omega} f_n \colon X \to Y$$

is a continuous bijection (see Proposition 2.1.11 in [2]). Hence, f is a condensation of X onto the connected separable metrizable space $Y \subseteq \mathbb{R}^2$. Finally, Y is not closed in $\mathbb{R}^2$ and is not compact. □

Corollary 2.4. *Let* $X = \bigoplus\{I_\alpha : \alpha \in A\}$, *where each* I_α *is the unit segment* $[0, 1]$. *Then the space* X *admits a connected metrizable subtopology if and only if* $|A| \leq 1$ *or* $|A| \geq \omega$.

Proof. If $1 < |A| < \omega$, then X is a disconnected compact space and, hence, it does not admit a connected Hausdorff subtopology.

If $|A| = \omega$, it follows from Example 2.3 that X has a connected separable metrizable subtopology.

If $\omega < |A| \leq 2^\omega$, then $X \supseteq P = \bigoplus\{I_n : n \in \omega\}$. The set P is closed in X and condenses onto a connected non-compact metrizable space. Apply Theorem 1.1 to conclude that X admits a connected separable metrizable subtopology.

Let $|A| > 2^\omega$. In each I_α, take a point x_α. Then $P = \{x_\alpha : \alpha \in A\}$ is a closed discrete set in X whose size equal to the weight of X, that is, X has achievable extent. It follows from Theorem 1.3 that X admits a connected metrizable subtopology. □

3. Condensations of the disjoint topological union of 2^ω copies of the Cantor set

In Theorem 3.3 we prove that the topological sum of 2^ω copies of the Cantor set condenses onto the unit interval $I = [0, 1]$. Then we extend this fact to compact metrizable spaces without isolated points (Theorem 3.8). We finish with Theorem 3.10 that shows that the conclusion is valid for every separable complete metrizable space without isolated points.

We will use the following notation. The Cantor set is $C = \{0, 1\}^\omega$. If $A \subseteq \omega$, then $\pi_A \colon C \to \{0, 1\}^A$ is the projection and, for $n \in \omega$, π_n is the projection of C onto the nth factor. If $A \subseteq B \subseteq \omega$, then π^B_A is the projection of $\{0, 1\}^B$ onto $\{0, 1\}^A$. For an open canonical set $U \subseteq \{0, 1\}^A$, we put

$$coord(U) = \{i \in A : |\pi_i(U)| = 1\}.$$

If $A \subseteq B \subseteq \omega$ and $f \in \{0, 1\}^A$, then the set $O(f) = (\pi^B_A)^{-1}(f)$ is called the *cylinder over* f in $\{0, 1\}^B$.

Let X be a space. We say that $S \subseteq X$ is a first category set in X if $S = \bigcup_{n=1}^{\infty} S_n$, where each S_n is a nowhere dense subset of X.

We start with a lemma.

Lemma 3.1. *Let S be a subset of the Cantor set C. If there exists a family $\{A_n : n \in \omega\}$ of pairwise disjoint infinite subsets of ω such that $\omega = \bigcup\{A_n : n \in \omega\}$ and $|\{0,1\}^{A_n} \setminus \pi_{A_n}(S)| = \mathfrak{c}$ for each $n \in \omega$, then C is the union of $\mathfrak{c} = 2^\omega$ pairwise disjoint copies of the Cantor set such that S intersects each of these copies in at most one point.*

Proof. Let $C_n = \{0,1\}^{A_n}$ be a copy of the Cantor set for each $n \in \omega$. Clearly, $C \cong \prod_{n=0}^{\infty} C_n$. For every $n \in \omega$, put $S_n = \pi_{A_n}(S)$. Since $|C_n \setminus S_n| = \mathfrak{c}$, it is possible to represent each C_n as the union $C_n = \bigcup\{P_{n,\alpha} : \alpha \in \Gamma_n\}$, where each $P_{n,\alpha}$ is a doubleton, $P_{n,\alpha} \cap P_{n,\beta} = \varnothing$ if $\alpha \neq \beta$, $|S_n \cap P_{n,\alpha}| \leq 1$, and Γ_n is an index set of cardinality $\mathfrak{c}$.

For an element $\gamma = (\gamma_n)_{n\in\omega}$ of $\Gamma = \prod_{n=0}^{\infty} \Gamma_n$, put $C_\gamma = \prod_{n=0}^{\infty} P_{n,\gamma_n}$. It is routine to verify that:

(i) $C_\gamma \subseteq C$ is a copy of the Cantor set for every $\gamma \in \Gamma$;
(ii) $C = \bigcup\{C_\gamma : \gamma \in \Gamma\}$ and $|\Gamma| = 2^\omega$;
(iii) $C_\gamma \cap C_\lambda = \varnothing$ if $\gamma \neq \lambda$;
(iv) $|S \cap C_\gamma| \leq 1$ for every $\gamma \in \Gamma$.

Our lemma is proved. □

Corollary 3.2. *Let S be a countable subset of the Cantor set C. Then C is the union of 2^ω pairwise disjoint copies of the Cantor set such that S intersects each of these copies in at most one point.*

Proof. Clearly, there exists a family $\{A_n : n \in \omega\}$ of disjoint infinite sets such that $\omega = \bigcup_{n=0}^{\infty} A_n$. We have that $|\pi_{A_n}(S)| = \aleph_0$ and, therefore, the complement $\{0,1\}^{A_n} \setminus \pi_{A_n}(S)$ has cardinality $\mathfrak{c}$, for each $n \in \omega$. Now apply Lemma 3.1. □

Theorem 3.3. *The unit segment $I = [0,1]$ is the union of 2^ω pairwise disjoint copies of the Cantor set.*

Proof. We take the Cantor set $C = \{0,1\}^\omega$ and consider the mapping $f\colon C \to I$ defined by the formula

$$f(x) = \sum_{i=0}^{\infty} \frac{x_i}{2^{i+1}},$$

where x_i is ith coordinate of the point $x \in C$.

It is known that f is a continuous mapping onto I (see 3.2.b of [2]). Although f is not one-to-one, it is easy to see that there is a countable infinite subset $S \subset C$ such that $|f^{-1}(f(x))| = 2$ for every $x \in S$ and $|f^{-1}(f(x))| = 1$ for $x \in C \setminus S$. It follows from Corollary 3.2 that there exists a representation $C = \bigcup\{C_\gamma : \gamma \in \Gamma\}$ such that the family $\{C_\gamma : \gamma \in \Gamma\}$ and the set S satisfy conditions (i)–(iv) in the proof of Lemma 3.1. Put $K_\gamma = f(C_\gamma)$. Since $|C_\gamma \cap S| \leq 1$, it is evident that $f_\gamma = f\restriction_{C_\gamma}\colon C_\gamma \to K_\gamma$ is a homeomorphism; therefore,

K_γ is a copy of the Cantor set for every $\gamma \in \Gamma$. Notice that $I = \bigcup\{K_\gamma : \gamma \in \Gamma\}$. It is easy to see that if $C_\gamma \cap S = \varnothing$, then K_γ does not meet any other K_λ, and if $C_\gamma \cap S = \{x\}$, then there is a unique $\lambda \in \Gamma \setminus \{\gamma\}$ such that $K_\gamma \cap K_\lambda \neq \varnothing$. In this case, $K_\gamma \cap K_\lambda = \{y\}$, where $y = f(x)$.

Hence, for every $y \in f(S)$ there exists a unique pair $(\alpha_y, \beta_y) \in [\Gamma]^2$ such that $K_{\alpha_y} \cap K_{\beta_y} = \{y\}$. Put $L_y = K_{\alpha_y} \cup K_{\beta_y}$. Clearly, each L_y is a copy of the Cantor set and $L_y \cap L_z = \varnothing$ if $y \neq z$. We obtain that

$$I = \bigcup\{K_\gamma : \gamma \in \Gamma,\ C_\gamma \cap S = \varnothing\} \cup \bigcup\{L_y : y \in f(S)\},$$

that is, I is the union of 2^ω disjoint copies of the Cantor set. □

In the sequel we will show that Theorem 3.3 remains valid if the unit interval I is replaced by a compact metrizable space without isolated points. First we need some lemmas. The following result is trivial.

Lemma 3.4. *If $F \subseteq \{0,1\}^\omega$ is a nowhere dense set and $A \subseteq \omega$ is finite, then $\pi_{\omega\setminus A}(F)$ is a nowhere dense subset of $\{0,1\}^{\omega\setminus A}$, where $\pi_{\omega\setminus A}$ is the projection of $\{0,1\}^\omega$ onto $\{0,1\}^{\omega\setminus A}$.*

Lemma 3.5. *Let $S \subseteq C = \{0,1\}^\omega$ be a first category set. Then there are infinite disjoint sets A and B such that $\omega = A \cup B$ and the sets $\pi_A(S)$ and $\pi_B(S)$ are of the first category in $\{0,1\}^A$ and in $\{0,1\}^B$, respectively.*

Proof. Let $S = \bigcup_{m=0}^\infty F_m$, where every F_m is a nowhere dense subset of C. We will construct by induction two families $\{A_n : n \in \omega\}$ and $\{B_n : n \in \omega\}$ which satisfy following conditions for each $n \in \omega$:

(i) $A_n, B_n \in [\omega]^{<\omega}$;
(ii) $A_n \cap B_n = \varnothing$;
(iii) $A_n \subseteq A_{n+1}$, $A_{n+1} \setminus A_n \neq \varnothing$;
(iv) $B_n \subseteq B_{n+1}$, $B_{n+1} \setminus B_n \neq \varnothing$;
(v) For every $f \in \{0,1\}^{A_n}$, there exists a family $\{U_{0f}, U_{1f}, \dots, U_{nf}\}$ of open canonical subsets of $\{0,1\}^{\omega\setminus B_n}$ such that $\pi^{\omega\setminus B_n}_{A_n}(U_{if}) = \{f\}$, $U_{if} \cap \pi_{\omega\setminus B_n}(F_i) = \varnothing$ and $A_n \subseteq \mathit{coord}(U_{if}) \subseteq A_{n+1}$, for each $i \leq n$;
(vi) For every $g \in \{0,1\}^{B_n}$, there exists a family $\{U_{0g}, U_{1g}, \dots, U_{ng}\}$ of open canonical subsets of $\{0,1\}^{\omega\setminus A_{n+1}}$ such that $\pi^{\omega\setminus A_{n+1}}_{B_n}(U_{ig}) = \{g\}$, $U_{ig} \cap \pi_{\omega\setminus A_{n+1}}(F_i) = \varnothing$ and $B_n \subseteq \mathit{coord}(U_{ig}) \subseteq B_{n+1}$, for each $i \leq n$;
(vii) $n \in A_n \cup B_n$.

We start with $A_0 = \{0\}$ and $B_0 = \{1\}$. Now suppose that for some $n \in \omega$ we have constructed $\{A_k : k \leq n\}$ and $\{B_k : k \leq n\}$ which satisfy (i)–(vii).

Let us construct A_{n+1}. Take $f \in \{0,1\}^{A_n}$ and let $O(f)$ be the cylinder over f in $\{0,1\}^{\omega\setminus B_n}$. For each $i = 0, 1, \dots, n$, the set $\pi_{\omega\setminus B_n}(F_i)$ is nowhere dense in $\{0,1\}^{\omega\setminus B_n}$ (see Lemma 3.4). Hence, for each $i \leq n$ there exists an open canonical set $U_{if} \subseteq \{0,1\}^{\omega\setminus B_n}$ such that $U_{if} \subseteq O(f)$ and $U_{if} \cap \pi_{\omega\setminus B_n}(F_i) = \varnothing$.

Clearly, $\pi^{\omega\setminus B_n}_{A_n}(U_{if}) = \{f\}$, $A_n \subseteq coord(U_{if})$ and $coord(U_{if}) \cap B_n = \varnothing$. Put

$$\widetilde{A}_{n+1} = \bigcup\{\bigcup_{i=0}^{n} coord(U_{if}) : f \in \{0,1\}^{A_n}\} \text{ and } a_{n+1} = \min(\omega \setminus (A_n \cup B_n)).$$

We set $A_{n+1} = \widetilde{A}_{n+1} \cup \{a_{n+1}\}$. It is easy to see that A_{n+1} is finite, $A_n \subseteq A_{n+1}$, $A_{n+1} \setminus A_n \neq \varnothing$, and $A_{n+1} \cap B_n = \varnothing$. The choice of a_{n+1} and (vii) together imply that $n+1 \in A_{n+1} \cup B_n$. In addition, for every $f \in \{0,1\}^{A_n}$ and each $i \le n$ we have that $coord(U_{if}) \subseteq A_{n+1}$.

Now we construct B_{n+1}. Take $g \in \{0,1\}^{B_n}$ and let $O(g)$ be the cylinder over g in $\{0,1\}^{\omega\setminus A_{n+1}}$. For each $i = 0,1,\dots,n$, the set $\pi_{\omega\setminus A_{n+1}}(F_i)$ is nowhere dense in $\{0,1\}^{\omega\setminus A_{n+1}}$ (see Lemma 3.4). Hence, for each $i \le n$ there exists an open canonical set $U_{ig} \subseteq \{0,1\}^{\omega\setminus A_{n+1}}$ such that $U_{ig} \subseteq O(g)$ and $U_{ig} \cap \pi_{\omega\setminus A_{n+1}}(F_i) = \varnothing$. Clearly, $\pi^{\omega\setminus A_{n+1}}_{B_n}(U_{ig}) = \{g\}$, $B_n \subseteq coord(U_{ig})$ and $coord(U_{ig}) \cap A_{n+1} = \varnothing$. Let

$$\widetilde{B}_{n+1} = \bigcup\{\bigcup_{i=1}^{n} coord(U_{ig}) : g \in \{0,1\}^{B_n}\} \text{ and } b_{n+1} \in (\omega \setminus (A_{n+1} \cup B_n)).$$

We set $B_{n+1} = \widetilde{B}_{n+1} \cup \{b_{n+1}\}$. It follows from the construction that B_{n+1} is finite, $A_{n+1} \cap B_{n+1} = \varnothing$, $B_n \subseteq B_{n+1}$, and $B_{n+1} \setminus B_n \neq \varnothing$. In addition, $coord(U_{ig}) \subseteq B_{n+1}$, for every $g \in \{0,1\}^{B_n}$ and each $i \le n$. Therefore, A_{n+1} and B_{n+1} satisfy conditions (i)–(vii).

Thus we have constructed two families $\{A_n : n \in \omega\}$ and $\{B_n : n \in \omega\}$. Now we define $A = \bigcup\{A_n : n \in \omega\}$ and $B = \bigcup\{B_n : n \in \omega\}$. It follows from (i)–(iv) that A and B are disjoint infinite subset of ω. Condition (vii) implies that $A \cup B = \omega$. It remains to verify that $\pi_A(S)$ and $\pi_B(S)$ are of the first category in $\{0,1\}^A$ and in $\{0,1\}^B$, respectively.

To prove that $\pi_A(S) = \bigcup_{m=0}^{\infty} \pi_A(F_m)$ is of the first category in $\{0,1\}^A$, it suffices to verify that $\pi_A(F_m)$ is nowhere dense in $\{0,1\}^A$, for each $m \in \omega$. Indeed, fix an open canonical non-empty set $U \subseteq \{0,1\}^A$ and $m \in \omega$. As $coord(U) \subseteq A$ is finite, (iii) implies that there is $k \in \omega$ such that $coord(U) \subseteq A_k$. Let $n = \max\{k,m\}$, then $coord(U) \subseteq A_n$ and $m \le n$. Take an arbitrary $f \in \pi^A_{A_n}(U) \subseteq \{0,1\}^{A_n}$. By (v), there exists an open canonical set U_{mf} in $\{0,1\}^{\omega\setminus B_n}$ such that $\pi^{\omega\setminus B_n}_{A_n}(U_{mf}) = \{f\}$, $U_{mf} \cap \pi_{\omega\setminus B_n}(F_m) = \varnothing$ and $A_n \subseteq coord(U_{mf}) \subseteq A_{n+1} \subseteq A$. Put $V = \pi^{\omega\setminus B_n}_A(U_{mf})$ and let $K = coord(U)$. We have:

$$K \subseteq A_n \subseteq coord(U_{mf}) = coord(V) \tag{1}$$

and

$$\pi^A_K(U) = \{\pi^{A_n}_K(f)\} = \pi^{\omega\setminus B_n}_K(U_{mf}) = \pi^A_K(V). \tag{2}$$

It follows from (1) and (2) that $V \subseteq U$. To verify that $V \cap \pi_A(F_m) = \varnothing$, we assume the contrary and choose an element $g \in V \cap \pi_A(F_m)$. Since $coord(V) \subseteq A_{n+1} \subseteq A$, we have that $\pi^A_{A_{n+1}}(g) \in \pi^A_{A_{n+1}}(V) \cap \pi_{A_{n+1}}(F_m)$. It follows from the definition of V that $\pi^A_{A_{n+1}}(V) = \pi^{\omega\setminus B_n}_{A_{n+1}}(U_{mf})$. Hence,

$\pi^{A}_{A_{n+1}}(g) \in \pi^{\omega\setminus B_n}_{A_{n+1}}(U_{mf}) \cap \pi_{A_{n+1}}(F_m)$. Take an element $\widetilde{g} \in \pi_{\omega\setminus B_n}(F_m)$ such that $\pi^{\omega\setminus B_n}_{A_{n+1}}(\widetilde{g}) = \pi^{A}_{A_{n+1}}(g)$. Then $\widetilde{g} \in U_{mf}$, since $coord(U_{mf}) \subseteq A_{n+1}$. We obtain a contradiction, as $U_{mf} \cap \pi_{\omega\setminus B_n}(F_m) = \varnothing$. We have thus proved that $\pi_A(F_m)$ is nowhere dense in $\{0,1\}^A$ for each $m \in \omega$ and, hence, $\pi_A(S) = \bigcup_{m=1}^{\infty} \pi_A(F_m)$ is of the first category in $\{0,1\}^A$. The verification that $\pi_B(S) = \bigcup_{m=1}^{\infty} \pi_B(F_m)$ is a first category set in $\{0,1\}^B$ is similar. □

Lemma 3.6. *Let S be a set of the first category in the Cantor set C. There is a decomposition $\omega = \bigcup_{n=0}^{\infty} A_n$ such that each A_n is infinite, $A_n \cap A_m = \varnothing$ if $n \neq m$, and $\pi_{A_n}(S)$ is a first category set in $\{0,1\}^{A_n}$ for every $n \in \omega$.*

Proof. The proof consists of applying ω times the previous lemma. □

If $f\colon X \to Y$ is a continuous mapping and $U \subseteq X$, then we put $f^{\#}(U) = Y \setminus f(X \setminus U)$. In the following lemma we collect some well-known facts that will be frequently used in the sequel.

Lemma 3.7.

(a) *Let $f\colon X \to Y$ be a continuous closed and irreducible mapping. If $U \subseteq X$ is open, then $V = f^{-1}(f^{\#}(U))$ is an open dense subset of U (see* [1, Problem 109, Chapter VI]*).*
(b) *The inverse image of a closed nowhere dense set under a continuous closed irreducible mapping is a closed nowhere dense set.*
(c) *If X is a G_δ-set in a compact space and X does not have isolated points, then $|X| \geq \mathfrak{c}$ (see* [2, 3.12.11 (b)]*).*
(d) *A non-empty open subset of the Cantor set can be represented as the union of countably many pairwise disjoint copies of the Cantor set.*

Theorem 3.8. *Every compact metrizable space without isolated points can be represented as the union of 2^{ω} pairwise disjoint copies of the Cantor set.*

Proof. We consider the Cantor set C and a compact metrizable space without isolated points Y. We can assume that $Y \subseteq I^{\omega}$, where I is the closed unit interval.

Let $f\colon C \to I$ be the continuous mapping onto I considered in the proof of Theorem 3.3. Now define $g\colon C^{\omega} \to I^{\omega}$ by $g = \prod\{g_i : i \in \omega\}$ where $g_i = f$ for each $i \in \omega$. Since g is a perfect mapping, there exists a closed subset $C' \subseteq C^{\omega}$ such that $g(C') = Y$ and $g\restriction_{C'}\colon C' \to Y$ is irreducible (see [2, 3.1.C (a)]). Clearly, C' does not have isolated points and hence is homeomorphic to C. Therefore, we can assume from the very beginning that there exists a continuous irreducible mapping of C onto Y which is denoted by the same letter f. Let $S = \{x \in C : |f^{-1}(f(x))| > 1\}$.

Our first step is to verify that S is of the first category in C. For each $m \in \mathbb{N}$, we choose a finite covering $\{U_{m1}, \ldots, U_{mk_m}\}$ of C by clopen sets of diameter less than or equal to $1/m$ with respect to a given metric on C. For each $m \in \mathbb{N}$ and each $i = 1, 2, \ldots, k_m$, the set $V_{mi} = f^{-1}(f^{\#}(U_{mi}))$ is open and dense in U_{mi} by (a) of Lemma 3.7 and, hence, $F_{mi} = U_{mi} \setminus V_{mi}$ is closed and nowhere dense in C. Therefore, for each $m \in \mathbb{N}$ the set $F_m = \bigcup_{i=1}^{k_m} F_{mi}$ is closed and nowhere dense in C. Put $F = \bigcup_{m=1}^{\infty} F_m$. We claim that $S = F$. Indeed, for $x \in C \setminus S$ we have that $|f^{-1}(f(x))| = 1$. From this, if $x \in U_{mi}$ for some $m \in \mathbb{N}$ and $i \in \{1, 2, \ldots, k_m\}$, then $x \in f^{-1}(f^{\#}(U_{ni})) = V_{mi}$, that is, $x \notin F_{mi} = U_{mi} \setminus V_{mi}$. Now, it is evident that $x \notin F$. Thus, $F \subseteq S$. For $x \in S$, let $z = f(x)$. There exists $y \in C \setminus \{x\}$ such that $z = f(y)$. We choose $m \in \mathbb{N}$ and $i \in \{1, 2, \ldots, k_m\}$ such that $x \in U_{mi}$ and $y \notin U_{mi}$. Then $z = f(y) \in f(Y \setminus U_{mi})$, that is, $z \notin f^{\#}(U_{mi})$. We have that $x \in f^{-1}(z) \cap U_{mi} \subseteq U_{mi} \setminus f^{-1}(f^{\#}(U_{mi})) = F_{mi} \subseteq F_m \subseteq F$. Therefore, $S \subseteq F$. This implies that $S = F = \bigcup_{m=1}^{\infty} F_m$, where every F_m is a closed nowhere dense subset of C.

Apply Lemma 3.6 to find a family $\{A_n : n \in \omega\}$ of infinite 1 disjoint subsets of ω whose union is equal to ω and such that $\pi_{A_n}(S)$ is a first category set in $C_n = \{0,1\}^{A_n}$ for each $n \in \omega$. It follows that the complement $C_n \setminus \pi_{A_n}(S)$ is a dense subset of the Cantor set C_n. In addition, $C_n \setminus \pi_{A_n}(S)$ is a G_δ-set in C_n, since every set $\pi_{A_n}(F_m)$ is closed in C_n. Then by Lemma 3.7(c), we obtain that $|C_n \setminus \pi_{A_n}(S)| = \mathfrak{c}$ for each $n \in \omega$.

Thus the family $\{A_n : n \in \omega\}$ and the set $S \subseteq C$ satisfy the conditions of Lemma 3.1 and, hence, we can represent C in the form $C = \bigcup\{C_\gamma : \gamma \in \Gamma\}$, where $|\Gamma| = 2^{\omega}$ and every C_γ is a copy of C. In addition, $C_\gamma \cap C_\lambda = \varnothing$ if $\gamma \neq \lambda$ and $|S \cap C_\gamma| \leq 1$ for each $\gamma \in \Gamma$.

Let $K_\gamma = f(C_\gamma)$ for each $\gamma \in \Gamma$. Since $f{\restriction}_{C_\gamma}$ is a homeomorphism, K_γ is a copy of the Cantor set for each $\gamma \in \Gamma$ and we have that $Y = \bigcup\{K_\gamma : \gamma \in \Gamma\}$.

Note that the family $\{K_\gamma : \gamma \in \Gamma\}$ has the following properties:

(i) if $C_\gamma \cap S = \varnothing$, then $K_\gamma \cap K_\lambda = \varnothing$ for each $\lambda \in \Gamma \setminus \{\gamma\}$;

(ii) if $C_\gamma \cap S \neq \varnothing$, then $C_\gamma \cap S = \{s\}$ for some $s \in S$; in this case for each $\lambda \in \Gamma \setminus \{\gamma\}$ we have:
   - (ii_1) $K_\gamma \cap K_\lambda = \varnothing$ if $C_\lambda \cap f^{-1}(f(s)) = \varnothing$;
   - (ii_2) $K_\gamma \cap K_\lambda = \{f(s)\}$ if $C_\lambda \cap f^{-1}(f(s)) \neq \varnothing$.

Now for every $y \in f(S)$, we define $\mathcal{K}_y = \{K_\gamma : \gamma \in \Gamma,\ C_\gamma \cap f^{-1}(y) \neq \varnothing\}$. It is easy to see that $\bigcup \mathcal{K}_y \cap \bigcup \mathcal{K}_z = \varnothing$ if $y \neq z$ and

$$Y = \bigcup\{K_\gamma : \gamma \in \Gamma, C_\gamma \cap S = \varnothing\} \cup \bigcup\{\bigcup \mathcal{K}_y : y \in f(S)\}. \qquad (*)$$

Consider the family $\mathcal{K}_y$ for some $y \in f(S)$. Choose an element $\gamma_y \in \Gamma$ such that $K_{\gamma_y} \in \mathcal{K}_y$. For every $\lambda \in \Gamma \setminus \{\gamma_y\}$ such that $K_\lambda \in \mathcal{K}_y$, put $L_\lambda = K_\lambda \setminus \{y\}$. Every L_λ is an open subset of the Cantor set, so it can be represented as the union of countably many disjoint copies of the Cantor set (item (d) of Lemma 3.7). Since

$$\bigcup \mathcal{K}_y = \bigcup\{L_\lambda :\ K_\lambda \in \mathcal{K}_y,\ \lambda \neq \gamma_y\} \cup K_{\gamma_y},$$

for every $y \in f(S)$, the set $\bigcup \mathcal{K}_y$ can be represented as the union of some number κ_y of disjoint copies of the Cantor set, where $|\mathcal{K}_y| \leq \kappa_y \leq |\mathcal{K}_y| \cdot \omega$. Taking into account the equality $(*)$, we obtain the conclusion of the theorem. □

Remark 3.9. Let Y be a compact metrizable space without isolated points, C the Cantor set and $f\colon C \to Y$ a continuous irreducible mapping. If S is a set of the first category in C and $\{x \in C : |f^{-1}(f(x))| > 1\} \subseteq S$, then repeating the argument in the proof of Theorem 3.8 one can represent Y as the union of 2^ω pairwise disjoint copies of the Cantor set such that every copy meets the set $f(S)$ in at most one point.

Due to the above observation it is possible to extend Theorem 3.8 as follows:

Theorem 3.10. *Every separable completely metrizable space without isolated points can be represented as the union of 2^ω pairwise disjoint copies of the Cantor set.*

Proof. Let X be a separable completely metrizable space without isolated points. We can assume that $X \subseteq I^\omega$. Then $Y = cl_{I^\omega}(X)$ is a metrizable compactification of X. Since X has no isolated points and is dense in Y, it follows that Y is a compact metrizable space without isolated points. The space X is Čech-complete, therefore, the remainder $R = Y \setminus X$ is a F_σ-set in Y. Hence, $R = \bigcup_{n=1}^\infty F_n$, where every F_n is a closed nowhere dense subset of Y.

Consider a continuous irreducible mapping $f\colon C \to Y$, where C is the Cantor set (see the proof of Theorem 3.8). Then each $f^{-1}(F_n)$ is a closed nowhere dense subset of C, by (b) of Lemma 3.7, and $S' = f^{-1}(R) = \bigcup_{n=1}^\infty f^{-1}(F_n)$. In the proof of Theorem 3.8 we have established that $S'' = \{x \in C : |f^{-1}(f(x))| > 1\}$ is the union of countably many closed nowhere dense subsets of C. Put $S = S' \cup S''$ and apply Remark 3.9 to conclude that $Y = \bigcup\{K_\gamma : \gamma \in \Gamma\}$, where $|\Gamma| = 2^\omega$, every K_γ is a copy of the Cantor set, $K_\gamma \cap K_\lambda = \varnothing$ if $\gamma \neq \lambda$ and $|K_\gamma \cap f(S)| \leq 1$ for each $\gamma \in \Gamma$.

Since $R \subseteq f(S)$, the set $R \cap K_\gamma$ is empty or consists of one point. For each $\gamma \in \Gamma$, we define $\widetilde{K}_\gamma = K_\gamma \setminus (K_\gamma \cap R)$. It is evident that $X = \bigcup\{\widetilde{K}_\gamma : \gamma \in \Gamma\}$. Every $\widetilde{K}_\gamma$ is a copy the Cantor set or an open subset of this copy. In the last case $\widetilde{K}_\gamma$ can be represented as a union of ω pairwise disjoint copies of the Cantor set, by (d) of Lemma 3.7. This finishes the proof. □

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