Extension of Compact Operators from DF-spaces to C(K) spaces

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ABSTRACT. It is proved that every compact operator from a DF-space, closed subspace of another DF-space, into the space C(K) of continuous functions on a compact Hausdorff space K can be extended to a compact operator of the total DF-space.

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1. INTRODUCTION

Let E and X be topological vector spaces with E a closed subspace of X. We are interested in finding out when a continuous operator $T : E \rightarrow C(K)$ has an extension $\tilde{T} : X \rightarrow C(K)$, where $C(K)$ is the space of continuous real functions on a compact Hausdorff space K and $C(K)$ has the norm of the supremum. When this is the case we will say that $(E, X)$ has the extension property. Several advances have been made in this direction, a basic resume and bibliography for this problem can be found in [5]. In this work we will focus in the case when the operator T is a compact operator. In [4], p.23, it is proved that $(E, X)$ has the extension property when E and X are Banach spaces and $T : E \rightarrow C(K)$ is a compact operator. In this paper we extend this result to the case when E and X are DF-spaces (to be defined below), for this, we use basic tools from topological vector spaces.

2. NOTATION AND BASIC RESULTS IN DF-SPACES.

We will use basic duality theory of topological vector spaces. For concepts in topological vector spaces see [3] or [2]. All the topological vector spaces in this work are Hausdorff and locally convex.

Let $(X, t)$ be a topological vector space and $E < X$ be a closed vector subspace. Let $X' = (X, t)'$, $E' = (E, t)'$ be the topological duals of X and E respectively.
A topological vector space \((X, t)\) possesses a a fundamental sequence of bounded sets if there exists a sequence \(B_1 \subset B_2 \subset \cdots\) of bounded sets in \((X, t)\), such that every bounded set \(B\) is contained in some \(B_k\).

We take the following definition from [3], p. 396.

**Definition 2.1.** A locally convex topological vector space \((X, t)\) is said to be a DF-space if

1. it has a fundamental sequence of bounded sets, and
2. every strongly bounded subset \(M\) of \(X'\) which is the union of countably many equicontinuous sets is also equicontinuous

A quasi-barrelled locally convex topological vector space with a fundamental sequence of bounded set is always a DF-space. Thus every normed space is a DF-space. Later we will mention topological vector spaces which are DF-spaces but they are not normed spaces.

First, we state some theorems to be used in the proof of the main result.

If \(K\) is a compact Hausdorff topological space, we define, for each \(k \in K\) the injective evaluation map \(\hat{k} : C(K) \to \mathbb{R}\), \(\hat{k}(f) = f(k)\) which is linear and continuous, that is \(\hat{k} \in C(K)\). Let \(\hat{K} = \{\hat{k} | k \in K\} \subset C(K)'\) and \(cch(\hat{K})\) the balanced, closed and convex hull of \(\hat{K}\) (which is bounded).

**Theorem 2.2.** With the notation above we have

1. \(\hat{K}\) is \(\sigma(C(K)', C(K))\)-compact and \(K\) is homeomorphic to \((\hat{K}, \sigma(C(K)', C(K)))\). Here \(\sigma(C(K)', C(K))\) denotes the weak-* topology on \(C(K)\).

2. If \(T : E \to C(K)\) is a compact operator then \(A = T'(cch(\hat{K}))^\beta\) is \(\beta(E', E)\)-compact. Here \(\beta(E', E)\) is the strong topology on \(E'\), this topology is generated by the polars sets of all bounded sets of \((E, t)\).

*Proof. See [1], p. 490.*

**Theorem 2.3.** If \((X, t)\) is a DF-space then \((X', \beta(X', X))\) is a Frechet space.

*Proof. See [3], p. 397*

**Theorem 2.4.** Let \(M\) be paracompact, \(Z\) a Banach space, \(N \subset Z\) convex and closed, and \(\varphi : M \to F(N)\) lower semicontinuous (l.s.c.) Then \(\varphi\) has a selection.

*Proof. See [6]*

In the above theorem, \(F(N) = \{S \subset N : S \neq \emptyset, S\) closed in \(N\) and convex\}; \(\varphi : M \to F(N)\) is l.s.c. if \(\{m \in M : \varphi(m) \cap V \neq \emptyset\}\) is open in \(M\) for every open \(V\) in \(N\), and \(f : M \to N\) is a selection for \(\varphi\) if \(f\) is continuous and \(f(m) \in \varphi(m)\) for every \(m \in M\).

Theorem above remains true if \(Z\) is only a complete, metrizable, locally convex topological vector space (see [7]).
3. Main Results

**Lemma 3.1.** Let $A \subset E'$. If there is a continuous map
\[ f : (A, \sigma(E', E)) \to (X', \tau'), \quad \sigma(X', X) \leq \tau' \leq \beta(X', X) \]
such that
1. $f(a)|_E = a$ and
2. $f(A)$ is an equicontinuous subset of $X'$.
Then every linear and continuous map $T : E \to C(K)$ has a linear and continuous extension $\tilde{T} : X \to C(K)$.

**Proof.** Let us define $\tilde{T} : X \to C(K)$ in the following way: for each $x \in X$, $\tilde{T}(x) : K \to \mathbb{R}$ is given by $\tilde{T}(x)(k) = f(T(k))(x)$. Here, $\tilde{k}$ is the injective evaluation map defined before Theorem 2.2. It is easy to check that $\tilde{T}$ is linear and extends $T$.

First, let us show that $\tilde{T}(x) \in C(K)$ for each $x \in X$. For this let $O \subset \mathbb{R}$ be an open set. We have that $\tilde{T}(x)^{-1}(O) = T^{-1}(f^{-1}(O))$. Since $x : X'[\sigma(X', X)] \to \mathbb{R}$, $f$ and $T'$ are all continuous maps with the weak* topology, $\tilde{T}(x)^{-1}(O)$ is open in $K$. This proves that $\tilde{T}(x) \in C(K)$.

Let us check that $\tilde{T}$ is continuous. Let $\{x_\lambda\} \xrightarrow{\ell} 0$ in $X$, we need to show that $\{\tilde{T}(x_\lambda)\} \xrightarrow{\|\cdot\|_{C(K)}} 0$.

For this, let $\epsilon > 0$. By hypothesis $f(A)$ is a equicontinuous subset of $X'$, so that, $\epsilon f(A)^o \subset X$ is a $t$-neighborhood of $0$. Here $f(A)^o$ denotes the polar set of $f(A)$. Hence, there is $\lambda_0 \in \Lambda$ such that $x_\lambda \in \epsilon f(A)^o$ for all $\lambda \geq \lambda_0$. From part 2 of Theorem 2.2 we have $T'(K) \subset A$, hence
\[ |\tilde{T}(x_\lambda)(k)| = |f(T'(k))(x_\lambda)| \leq \epsilon \text{ for all } \lambda \geq \lambda_0 \]
This implies that
\[ \|\tilde{T}(x_\lambda)\|_{C(K)} = \sup \{ |f(T'(k))(x_\lambda)| / k \in K \} \leq \epsilon \text{ for all } \lambda \geq \lambda_0 \]
This proves that $\{\tilde{T}(x_\lambda)\} \xrightarrow{\|\cdot\|_{C(K)}} 0$.

Let $i : E \to X$ be the inclusion map and $i' : X' \to E'$ the dual map of $i$, that is, if $y \in X'$, $i'(y) = y|_E$.

Let $\mathcal{P}(X') = \{ Y \mid Y \neq \emptyset, Y \subset X' \}$ and define $\psi : E' \to \mathcal{P}(X')$ by $\psi(e') = \{ \text{extensions of } e' \text{ to } X \}$. Notice that $y \in \psi(i'(y))$ for all $y \in X'$ and $\psi(e') \in \mathcal{S}(X')$.

With this notation, we have

**Proposition 3.2.** Let $(E, t)$ and $(X, t)$ be DF-spaces, with $E \subset X$ a closed subspace. If $O \subset X'$ is a $\beta(X', X)$-open set then the set $U_O = \{ z \in E' \mid \psi(z) \cap O \neq \emptyset \}$ is an open set in $(E', \beta(E', E))$.

**Proof.** Notice that $U_O = \{ z \in E' \mid \psi(z) \cap O \neq \emptyset \} = i'(O)$. By Theorem 2.3 $(X', \beta(X', X))$ and $(E', \beta(E', E))$ are Frechet spaces. By the Banach-Schauder theorem (see [3], p. 166), the map $i' : (X', \beta(X', X)) \to (E', \beta(E', E))$ is an open map. Since $i'(O)$ is open in $E'$, $U_O$ is also open.
Corollary 3.3. Let \((E, t)\) and \((X, t)\) be DF-spaces, with \(E < X\) a closed subspace. Let \(A = T'(cch(K))\) be as in part 2 of Theorem (2.2) Then \(\varphi : (A, \beta(E', E)) \to \mathcal{P}(X')\) given by \(\varphi = \psi|_A\) is a lower semicontinuous function, \(X'\) provided with the strong topology \(\beta(X', X)\).

Proof. It follows from
\[
\{ z \in A \mid \varphi(z) \cap O \neq \emptyset \} = \{ z \in E' \mid \psi(z) \cap O \neq \emptyset \} \cap A
\]
and Proposition 3.2. □

With the notation in Corollary 3.3, we have

Proposition 3.4. If \((X, t)\) is a DF-space then \(\varphi : (A, \beta(E', E)) \to \mathcal{P}(X')\) admits a selection, that is, there is a continuous function \(f : (A, \beta(E', E)) \to (X', \beta(X', X))\) such that \(f(a) \in \varphi(a)\).

Proof. From Theorem 2.3, \((X, t)\) DF-space implies \((X', \beta(X', X))\) Fréchet. From Theorem 2.2, part 2, \(A\) is \(\beta(E', E)\)-compact, hence \(A\) is a paracompact set. By Corollary 3.3, \(\varphi\) is a lower semi continuous function, therefore, by Theorem 2.4, \(\varphi\) admits a selection. □

Theorem 3.5. If \((X, t)\) and the closed subspace \(E\) are DF-spaces then every compact operator \(T : E \to C(K)\) has a compact extension \(\hat{T} : X \to C(K)\).

Proof. Let \(A\) be as in Proposition 3.4 and \(f : (A, \beta(E', E)) \to (X', \beta(X', X))\) a selection function. Since \(A\) is \(\beta(E', E)\)-compact and \(f\) is continuous, \(f(A)\) is compact, hence \(f(A)\) is an equicontinuous set. Let \(\hat{T}\) be the linear extension of \(T\) given in Lemma 3.1.

Let us prove that \(\hat{T}\) is a compact operator. For this, we need to show that there is a \(t\)-neighborhood \(V\) such that \(\hat{T}(V)\) is a relatively compact set.

Since \(f(A) \subset X'\) is an equicontinuous set and \(X\) is a DF space, [2] (p. 260 and p. 214) tells us that there is \(V \subset X\) a balanced, closed and convex \(t\)-zero-neighborhood such that \(f(A) \subset V^0\) and the topologies \(\beta(X', X)\) and \(\rho_{V^0}\) coincide on \(f(A)\). Here \(\rho_{V^0}\) is the Minkowski functional of \(V^0\). In this case \(\rho_{V^0}\) is a norm and \((X'_{V^0}, \rho_{V^0})\) is a Banach space.

By using the Arzela-Ascoli Theorem, we will show that \(\hat{T}(V) \subset C(K)\) is relatively compact.

First, \(\hat{T}(V)\) is pointwise bounded because, for each \(x \in V\) and \(k \in K\),
\[
|\hat{T}(x)(k)| = |f(T'(\hat{k}))(x)| \leq 1 \text{ since } f(A) \subset V^0.
\]

Now let us prove that \(\hat{T}(V)\) is equicontinuous in \(C(K)\).

Choose and fix \(k_0 \in K\) and \(\epsilon > 0\). Since the chain of functions
\[
K \xrightarrow{\sim} K \xrightarrow{T} (A, \beta(X', X)) \xrightarrow{f} (f(A), \beta(X', X))
\]
is continuous, given a \(\beta\)-neighborhood \(W\) of \(f(T'(\hat{k}_0))\) on \(f(A)\), there exists \(O \subset K\) neighborhood of \(k_0\) such that \(k \in O \Rightarrow f(T'(\hat{k})) \in W\). Since \(\rho_{V^0}|_{f(A)} = \beta(X', X)|_{f(A)}\), we can say that
\[
k \in O \Rightarrow \rho_{V^0} \left(f(T'(\hat{k})) - f(T'(\hat{k}_0))\right) < \epsilon
\]
For each $x \in X$, \( x : (X'_\rho \circ V) \to \mathbb{R} \) is linear and continuous, moreover, 
\[ |x'(x)| \leq \|x\|_\rho \circ V \circ (x') \]
for all \( x' \in X' \), where
\[ \|x\|_\rho \circ V = \sup\{|x'(x)| \mid x' \in V'\} \]

If \( x \in V \), \( \|x\|_\rho \circ V \leq 1 \). Therefore, for every \( k \in O \) and every \( x \in V \)
\[ \left| f(T'(\hat{k})) - f(T'(\hat{k}_0))(x) \right| \leq \|x\|_\rho \circ V \left( f(T'(\hat{k})) - f(T'(\hat{k}_0)) \right) \leq (1)\epsilon \]
This proves that \( \tilde{T}(V) \) is equicontinuous in \( C(K) \) and, by the Arzela-Ascoli Theorem, \( \tilde{T}(V) \) is relatively compact which means that \( \tilde{T} \) is a compact operator. \( \square \)

In [3] (p. 402) it is shown that the topological inductive limit of a sequence of DF-spaces is a DF-space. In particular, if \((E_n)\) is a sequence of Banach spaces such that \( E_n \) is a proper subspace of \( E_{n+1} \), its inductive limit is DF-space. This inductive limit is not metrizable (see [8] p. 291). For this kind of spaces, Theorem 3.5 can be applied, i.e., given a fixed \( n \), a compact operator \( T : E_n \to C(K) \) can be extended to a compact operator of the inductive limit.

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References


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