APPLIED GENERAL TOPOLOGY

When is an ultracomplete space almost locally compact?

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ABSTRACT.   We study spaces X which have a countable outer base in βX; they are called ultracomplete in the most recent terminology. Ultracompleteness implies Čech-completeness and is implied by almost local compactness ($\equiv$ having all points of non-local compactness inside a compact subset of countable outer character). It turns out that ultracompleteness coincides with almost local compactness in most important classes of isocompact spaces (i.e., in spaces in which every countably compact subspace is compact). We prove that if an isocompact space X is ω-monolithic then any ultracomplete subspace of X is almost locally compact. In particular, any ultracomplete subspace of a compact ω-monolithic space of countable tightness is almost locally compact. Another consequence of this result is that, for any space X such that υX is a Lindelöf Σ-space, a subspace of $C_p(X)$ is ultracomplete if and only if it is almost locally compact. We show that it is consistent with ZFC that not all ultracomplete subspaces of hereditarily separable compact spaces are almost locally compact.

1. Introduction.

In 1987 Ponomarev and Tkachuk introduced in [12] strongly complete spaces as those which have countable outer character in βX. In 1998 Romaguera studied the same class calling its spaces cofinally Čech-complete; he proved in [13] that a metrizable space has a cofinally complete metric if and only if it is cofinally Čech-complete. Buhagiar and Yoshioka gave in [5] an internal characterization of cofinal Čech-completeness and renamed it ultracompleteness; in this paper we will use their term for this class.

It is easy to see that ultracompleteness lies between Čech-completeness and local compactness so, to check whether or not a space X is ultracomplete, it is natural to deal with the set X_0 of points of non-local compactness of X to find out whether X_0 is small in some sense.

The first results in this direction were obtained in [12]: on the one hand, if X is ultracomplete then X_0 has to be a bounded subset of X; on the other hand, if X_0 is contained in a compact subset of countable outer character in X (in this paper we will follow [10] calling such spaces X *almost locally compact*) then the space X is ultracomplete. This places ultracomplete spaces between Čech-complete and almost locally compact ones so the natural question is when an ultracomplete space has to be almost locally compact. It was proved in [12] that ultracompleteness coincides with almost local compactness in the class of paracompact spaces. In [10] the same result was established for the class of Dieudonné complete spaces as well as for Eberlein–Grothendieck ones.

We develop the methods from [10] to find more classes in which ultracompleteness coincides with almost local compactness. The principal object of our considerations is the class of *isocompact spaces*, i.e., the spaces in which every countably compact subset is compact. This class is quite a wide one: it contains all sequential Dieudonné spaces, all spaces with a G_δ-diagonal as well as some spaces dealt with in C_p-theory, such as the splittable spaces and the spaces $C_p(X)$ for which υX is a Lindelöf Σ-space.

We prove that if X is an isocompact ω-monolithic space then a subspace of X is ultracomplete if and only if it is almost locally compact. Consequently, ultracompleteness coincides with almost local compactness in subspaces of the spaces $C_p(X)$ such that υX is Lindelöf Σ. This result gives a positive answer (in a much stronger form) to Problem 3.9 from [10]. Another consequence is coincidence of ultracompleteness and almost local compactness in subspaces of compact ω-monolithic spaces of countable tightness. We give examples of compact spaces (some of them in ZFC and some consistent) which show that neither ω-monolithity nor countable tightness can be omitted here. It is worth mentioning that an easy consequence of results of [10] is that, in any subspace of a first countable compact space, ultracompleteness coincides with almost local compactness (in fact, this coincidence even holds in realcompact spaces of countable pseudocharacter).

We also prove that the coincidence of ultracompleteness and almost local compactness takes place in splittable spaces and give some easy observations which help to solve Problems 3.7 and 3.10 from [10].

2. Notation and terminology.

All spaces under consideration are assumed to be Tychonoff. The space $\mathbb{R}$ is the set of real numbers with its natural topology. For any space X we denote by $C_p(X)$ the space of continuous real-valued functions on X endowed with the topology of pointwise convergence.

The Stone-Čech compactification of a space X is denoted by βX. The outer character of a subspace $A \subset X$, denoted by $\chi(A, X)$, is the minimal of the cardinalities of all outer bases of A in X. A space X is Čech-complete if it is a G_δ-set in βX. A topological space X is called *ultracomplete* if $\chi(X, \beta X) \leq \omega$. It is clear that any ultracomplete space is also Čech-complete. The space X is of (pointwise) countable type if for any compact $F \subset X$ ($x \in X$) there exists a compact $K \subset X$ such that $F \subset K$ ($x \in K$) and $\chi(K, X) \leq \omega$.

A space X is called *hemicompact* if there is a countable family $\{F_n : n < \omega\}$ of compact subsets of X such that for any compact $K \subset X$ there exists $n \in \omega$ for which $K \subset F_n$. A space X is called *scattered* if any $Y \subset X$ has an isolated point. A space X is ω-*monolithic* if for any $Y \subset X$ with $|Y| \leq \omega$ we have $nw(\overline{Y}) \leq \omega$, where $nw(Y)$ is the network weight of the space Y.

Given a space X the family $\tau(X)$ is its topology and $\tau^*(X) = \tau(X) \setminus \{\varnothing\}$; if $x \in X$ then $\tau(x, X) = \{U \in \tau(X) : x \in U\}$. The *tightness* $t(X)$ of a space X is the smallest cardinal κ such that for any $A \subset X$ and $x \in \overline{A}$ there exists $B \subset A$ with $|B| \leq \kappa$ such that $x \in \overline{B}$. A subset A of a space X is *bounded in* X if every $f \in C_p(X)$ is bounded on the set A.

A space X is called Lindelöf Σ if it has a compact cover $\mathcal{C}$ and a sequence of closed sets $\mathcal{F} = \{F_n : n \in \omega\}$ such that for each set $C \in \mathcal{C}$ and any $U \in \tau(X)$ with $C \subset U$ there is an $F \in \mathcal{F}$ such that $C \subset F \subset U$. A space X is called splittable if, for each $f \in \mathbb{R}^X$, there exists a countable $N \subset C_p(X)$ such that $f \in \overline{N}$ (the closure is taken in $\mathbb{R}^X$).

The rest of our terminology is standard and follows [8].

3. Ultracomplete subspaces of isocompact spaces.

The following fact, (see [12]), is useful for working with ultracomplete spaces.

Theorem 3.1. *For any space X, the following conditions are equivalent:*
(i) $\chi(X, cX) \leq \omega$ for some compactification cX of the space X;
(ii) $\chi(X, kX) \leq \omega$ for every compactification kX of the space X;
(iii) $\chi(X, \beta X) \leq \omega$ for the Stone–Čech compactification βX of the space X;
(iv) $cX \setminus X$ is hemicompact for some compactification cX of the space X;
(v) $kX \setminus X$ is hemicompact for every compactificaction kX of the space X;
(vi) $\beta X \setminus X$ is hemicompact.

A space X is called *ultracomplete* if it satisfies one of the conditions of Theorem 3.1.

Definition 3.2. *Call a space X almost locally compact if there is a compact $K \subset X$ such that $\chi(K, X) \leq \omega$ and $X_0 = \{x \in X : X$ is not locally compact at $x\} \subset K$.*

Given a space X let $\mathcal{N}(X)$ denote the family of all countably infinite closed and discrete subspaces of X.

Definition 3.3. *A countable family $\mathcal{U} \subset \mathcal{N}(X)$ marks a point $x \in X$, if for any $W \in \tau(x, X)$ there exists $D \in \mathcal{U}$ such that the set $D \cap W$ is infinite. A point $x \in X$ is called marked in X if it is marked by some countable $\mathcal{U} \subset \mathcal{N}(X)$.*

It is easy to see that if x is marked by a family $\mathcal{U}$, then $Y = \{x\} \cup (\bigcup \mathcal{U})$ is a countable set which reflects the non-local countable compactness of the space X at the point x. The proof of the following statement is an easy exercise.

Proposition 3.4. *If every point of a countable subspace A of a space X is marked then all points of the set $\overline{A}$ are marked as well. In particular, if $t(X) = \omega$ then the set of all marked points of X is closed in X.*

It is clear that, in any space, only the points of non-local countable compactness can be marked. If X is a space and a point $x \in X$ has a countable local base $\{B_n : n \in \omega\}$ such that, for every $n \in \omega$ the set $\overline{B}_n$ is not countably compact then, choosing a countably infinite closed discrete $D_n \subset \overline{B}_n$ for each $n \in \omega$ we obtain a family $\mathcal{U} = \{D_n : n \in \omega\}$ which marks the point x. We push further this idea in the following theorem to characterize the points of non-local countable compactness in a reasonably general class of spaces.

Theorem 3.5. *Suppose that X is a space of pointwise countable type such that $t(K) \leq \omega$ for any compact $K \subset X$. Then a point $x \in X$ is marked in X if and only if x is not a point of local countable compactness of X.*

Proof. We already saw that only sufficiency must be proved so assume that $x \in X$ is not a point of local countable compactness in X. By Proposition 3.4, the set M of all marked points of X is ω-closed in X, i.e., $\overline{A} \subset M$ for any countable $A \subset M$; suppose that $x \in X \setminus M$. The space X being of pointwise countable type, there is a compact $K \subset X$ such that $x \in K$ and $\chi(K, X) \leq \omega$. The set $F = M \cap K$ is ω-closed in K; since $t(K) \leq \omega$, the set F is closed in K and hence in X.

Therefore $K \setminus F$ is an open neighbourhood of the point x in the space K; this makes it possible to find a closed $K' \subset K$ such that K' is a G_δ-subset of K and $x \in K' \subset K \setminus F$. It is straightforward that $\chi(K', X) \leq \chi(K', K) \cdot \chi(K, X) \leq \omega$ so we can find an outer base $\{B_n : n \in \omega\}$ of the set K' in X such that $\overline{B}_{n+1} \subset B_n$ for all $n \in \omega$. Since $x \in B_n$, it is possible to choose a countably infinite closed discrete $D_n \subset \overline{B}_n$ for any $n \in \omega$.

We claim that the family $\mathcal{U} = \{D_n : n \in \omega\} \subset \mathcal{N}(X)$ marks some point of K'. Indeed, if this is not so, then every $y \in K'$ has a neighbourhood U_y

such that $U_y \cap D_n$ is finite for any $n \in \omega$. Since K' is compact, there exist $y_1, \ldots, y_k \in K'$ such that $K' \subset U = U_{y_1} \cup \ldots \cup U_{y_k}$. There is $n \in \omega$ such that $B_n \subset U$ and hence $\overline{B}_{n+1} \subset U$. Therefore $D_{n+1} \subset U$ while $U_{y_i} \cap D_{n+1}$ is finite for every $i \leq k$; this contradiction shows that some $y \in K'$ is marked by $\mathcal{U}$. However, all marked points are in M which does not meet K'; this final contradiction proves that x is marked in X. $\qquad\square$

Example 3.6. Let ξ be a free ultrafilter on ω; the space $X = \omega \cup \{\xi\}$ (with the topology induced from $\beta\omega$) is countable and non-locally countably compact at ξ. However, if ξ is a P-point of $\beta\omega \setminus \omega$ then ξ is not marked in X. Therefore, under CH, there is a countable space whose set of points of non-local (countable) compactness does not coincides with the set of marked points of X.

Proof. Since P-points in $\beta\omega \setminus \omega$ exist under CH, it suffices to show that ξ is not marked in X if ξ is a P-point. Suppose that $\mathcal{D} = \{D_n : n \in \omega\} \subset \mathcal{N}(X)$ marks the point ξ. Then $D_n \notin \xi$ for any $n \in \omega$ and hence $U_n = (\beta\omega \setminus \omega) \setminus \overline{D}_n$ is an open neighbourhood of ξ in $\beta\omega \setminus \omega$ (the bar denotes the closure in the space $\beta\omega$). Since ξ is a P-point, there is a clopen $W \subset \beta\omega \setminus \omega$ with $\xi \in W \subset \bigcap_{n \in \omega} U_n$. Choose a set $V \in \xi$ such that $\overline{V} \cap (\beta\omega \setminus \omega) \subset W$; it is straightforward that $V \cup \{\xi\}$ is a clopen neighbourhood of ξ in X such that $D_n \cap V$ is finite for any $n \in \omega$. This contradiction shows that the point ξ is not marked in X. $\qquad\square$

Recall that a space X is called *isocompact* if every countably compact subspace of X is compact.

Corollary 3.7. *If X is an isocompact space of pointwise countable type then a point $x \in X$ is marked if and only if x is not a point of local compactness of X.*

Proof. By isocompactness of X, if x is not a point of local compactness of X then x is not a point of local countable compactness of X. Besides, any compact $K \subset X$ has countable tightness—it is an easy exercise that any isocompact compact space has countable tightness. Thus we can apply Theorem 3.5 to conclude that x is marked. $\qquad\square$

Theorem 3.8. *If X is an isocompact ω-monolithic space then a subspace $Y \subset X$ is ultracomplete if and only if it is almost locally compact.*

Proof. We only must prove necessity so assume that $Y \subset X$ is ultracomplete. The space Y is of pointwise countable type being Čech-complete so the set M of all points at which Y is not locally compact coincides with the set of marked points of Y by Corollary 3.7. If M is not countably compact then it has a countably infinite closed discrete subspace S. Any point $s \in S$ is marked by a countable family of $\mathcal{D}_s \subset \mathcal{N}(Y)$.

The family $\mathcal{D} = \bigcup\{\mathcal{D}_s : s \in S\}$ is countable; since any closed subspace of an ultracomplete space is ultracomplete, the space $E = \mathrm{cl}_Y(\bigcup \mathcal{D})$ is ultracomplete; besides, $nw(E) \leq \omega$ because Y is ω-monolithic. Now, even in Čech-complete spaces the weight and the network weight coincide so $w(E) = nw(E) = \omega$.

Therefore E is a metrizable space and hence the subspace E_0 of points at which E is not locally compact, is a compact space by [12, Corollary 5].

Observe that any $s \in S$ is marked in Y by the family $\mathcal{D}_s$ and it is clear that the family $\mathcal{D}_s$ marks the point s in the space E as well. Therefore $s \in E_0$ for all $s \in S$ which shows that $S \subset E_0$ is an infinite closed discrete subset of E_0, a contradiction with compactness of E_0. Thus M is countably compact and hence compact by isocompactness of Y. Finally observe that Y is of countable type so there is a compact $K \subset Y$ for which $M \subset K$ and $\chi(K, Y) \leq \omega$, i.e., Y is almost locally compact. $\qquad\square$

To apply Theorem 3.8, let us look at some well-known classes of isocompact spaces. The following fact gives a positive answer to Problem 3.9 of [10].

Corollary 3.9. *Suppose that X is a space such that υX is Lindelöf Σ. Then a subspace $Y \subset C_p(X)$ is ultracomplete if and only if it is almost locally compact. In particular, this is true if $C_p(X)$ is a Lindelöf Σ-space or X is pseudocompact.*

Proof. It is known that, for such X, the space $C_p(X)$ is ω-monolithic and isocompact (see [2, Proposition IV.9.10] and [2, Theorem II.6.34]). $\qquad\square$

Corollary 3.10. *If X is a compact ω-monolithic space of countable tightness then any ultracomplete $Y \subset X$ is almost locally compact. In particular, a subspace of a Corson compact space is ultracomplete if and only if it is almost locally compact.*

Proof. The space X is Fréchet–Urysohn and hence isocompact so Theorem 3.8 does the rest. $\qquad\square$

Corollary 3.10 shows that it is natural to ask whether the same result can be proved if we omit ω-monolithity of the space X. We will show later that it is impossible, at least, consistently. However, the conclusion of Corollary 3.10 remains valid if we strengthen countable tightness of X to first countability. In fact, the following much more general statement is true.

Proposition 3.11. *If X is a hereditarily realcompact space (in particular, if X is a realcompact space of countable pseudocharacter) then a subspace $Y \subset X$ is ultracomplete if and only if it is almost locally compact.*

Proof. It suffices to observe that any ultracomplete realcompact space is almost locally compact by [10, Theorem 2.4]. $\qquad\square$

Another important class of isocompact spaces is given by *splittable spaces*. Recall that a space X is splittable if, for any $A \subset X$ there is a continuous map $f : X \to \mathbb{R}^\omega$ such that $A = f^{-1}f(A)$. The class of splittable spaces is isocompact because every pseudocompact splittable space is compact and metrizable (see [4, Theorem 3.2]). However, a splittable space need not be ω-monolithic so we cannot apply Theorem 3.8 directly. Theorem 3.4 of [3] shows that if X is a splittable space of non-measurable cardinality then X is hereditarily realcompact so Proposition 3.11 works to establish that any

ultracomplete subspace of X is almost locally compact. However, the same result can be proved without assuming anything about the cardinality of X.

Proposition 3.12. *For any splittable space X, a subspace $Y \subset X$ is ultracomplete if and only if Y is almost locally compact.*

Proof. Since splittability is hereditary, it suffices to prove that any splittable ultracomplete space Y is almost locally compact. It follows from [4, Corollary 2.15] that Y is first countable. Let M be the set of points of non-local compactness of Y. If M is not countably compact then fix a countably infinite closed discrete $D \subset M$ and apply Corollary 3.7 to find a family $\mathcal{D}_s \subset \mathcal{N}(Y)$ which marks the point s for any $s \in D$; consider the family $\mathcal{D} = \bigcup_{s \in D} \mathcal{D}_s$. The set $Z = \bigcup \mathcal{D}$ is countable; since Y is first countable, the set $F = \overline{Z}$ has cardinality at most $\mathfrak{c}$. Splittable spaces of cardinality at most $\mathfrak{c}$ have a weaker second countable topology [4, Corollary 2.19]. Therefore F is an ultracomplete realcompact space which implies that F is almost locally compact by [10, Theorem 2.4]. In particular, the set F_0 of the points of non-local compactness of F is compact. However, $\mathcal{D}$ marks all points of D in F; this shows that $D \subset F_0$ is a closed and discrete subspace of F_0 which is a contradiction. Therefore M is countably compact and hence compact (and metrizable). The space Y being of countable type, there is a compact $K \subset Y$ such that $M \subset K$ and $\chi(K, Y) \leq \omega$, i.e., Y is almost locally compact. $\qquad\square$

Example 3.13. It was proved in [6] that the space $Y = \omega_1^\omega$ is ultracomplete and has no points of local compactness. This example disproves many hypothesis showing, in particular, that

(1) a first countable ultracomplete space need not be almost locally compact (compare with Proposition 3.11);
(2) the restriction on tightness cannot be omitted in Corollary 3.10;
(3) an ultracomplete subspace of a Σ-product of real lines need not be almost locally compact (this draws a limit for possible generalizations of the statement on Corson compact spaces in Corollary 3.10);
(4) there is a space X such that X^ω is Lindelöf while some ultracomplete subspace of $C_p(X)$ is not almost locally compact (therefore the Σ-property cannot be omitted in Corollary 3.9);
(5) there exists a homogeneous non-locally compact ultracomplete space. This gives a negative answer to Problem 3.10 of [10] being of interest also because any ultracomplete topological group has to be locally compact—this was proved in [11, Corollary 2.13].

Proof. It is evident that the space Y is first countable and not almost locally compact; besides, Y is a subspace of $(\omega_1+1)^\omega$ which is an ω-monolithic compact space. Since ω_1 embeds in a Σ-product of real lines, so does ω_1^ω. This proves (1)–(3). It is known that any Σ-product of real lines is homeomorphic to a space $C_p(X)$ where X is the Lindelöfication of an uncountable discrete space. Since X^ω is Lindelöf (see [2, Proposition IV.2.21]), we also have (4). Finally, a famous theorem of Dow and Pearl [7, Theorem 2] says that the countably

infinite power of any first countable zero-dimensional space is homogeneous. Since the space ω_1 is zero-dimensional and first countable, we conclude that ω_1^ω is homogeneous; this settles (5). $\square$

Example 3.14. It is consistent with ZFC that there is a hereditarily separable compact space X such that some ultracomplete $Y \subset X$ is not almost locally compact. This shows, in a much stronger form, that ω-monolithity cannot be omitted in Corollary 3.10.

Proof. Fedorchuk proved in [9] that it is consistent with ZFC that there is a hereditarily separable compact space X of cardinality $2^{\mathfrak{c}}$ without non-trivial convergent sequences. It is easy to see that any infinite scattered compact space has a convergent sequence so X is not scattered; fix a non-empty closed dense-in-itself set $P \subset X$. If D is a countable dense subset of P then $Y = P \setminus D$ is dense in P so it does not have points of local compactness. Another easy fact is that any countable space without convergent sequences is hemicompact; thus D is hemicompact so we can apply Theorem 3.1 to see that Y is an ultracomplete space without points of local compactness. Therefore Y is not almost locally compact. $\square$

It follows from [12, Lemma 4] that any ultracomplete space without points of local compactness is pseudocompact. So far, all examples of such ultracomplete spaces were countably compact. We will see that the following statement (which seems to be of interest in itself) implies that there are ZFC examples of ultracomplete non-countably compact spaces without points of local compactness.

Theorem 3.15. *The space $\{0,1\}^{\mathfrak{c}}$ has a dense countable subspace without non-trivial convergent sequences.*

Proof. It was proved in [1, Theorem 2.3] that $\{0,1\}^{\mathfrak{c}}$ has a dense countable irresolvable subspace D. Consider the set $E = \{x \in D:$ there is a non-trivial sequence in D which converges to $x\}$. If E is dense in D then, enumerating the relevant countable family of convergent sequences, it is standard to construct disjoint $A, B \subset D$ such that the sets $A \cap S$ and $B \cap S$ are infinite for any sequence S from this family. An immediate consequence is that both A and B are dense in D which is a contradiction.

Thus there is a non-empty open set U in the space D with $U \cap E = \varnothing$; therefore U has no non-trivial convergent sequences. If V is open in $\{0,1\}^{\mathfrak{c}}$ and $V \cap D = U$ then U is dense in V. It is easy to find an open set W in the space $\{0,1\}^{\mathfrak{c}}$ such that $W \subset V$ and W is homeomorphic to $\{0,1\}^{\mathfrak{c}}$. It is immediate that $U' = U \cap W$ is dense in W; identifying W with $\{0,1\}^{\mathfrak{c}}$, we conclude that U' is the promised countable dense subspace of $\{0,1\}^{\mathfrak{c}}$. $\square$

Example 3.16. There exists a dense subspace X of the space $\{0,1\}^{\mathfrak{c}}$ which is ultracomplete, non-countably compact and has no points of local compactness.

Proof. Apply Theorem 3.15 to fix a countable dense set $D \subset \{0,1\}^{\mathfrak{c}}$ which has no convergent sequences; we can assume, without loss of generality, that the

point $u \in \{0,1\}^{\mathfrak{c}}$ whose all coordinates are equal to zero, belongs to D. It is easy to see that D is hemicompact so the space $X = \{0,1\}^{\mathfrak{c}} \setminus D$ is ultracomplete by Theorem 3.1. The density of X in $\{0,1\}^{\mathfrak{c}}$ is evident; it follows from density of D in $\{0,1\}^{\mathfrak{c}}$ that X has no points of local compactness.

To see that X is not countably compact, consider, for any $\alpha < \mathfrak{c}$, the point $u_\alpha \in \{0,1\}^{\mathfrak{c}}$ defined by $u_\alpha(\alpha) = 1$ and $u_\alpha(\beta) = 0$ for any $\beta \in \mathfrak{c} \setminus \{\alpha\}$. It is easy to see that the set $A = \{u_\alpha : \alpha < \mathfrak{c}\} \cup \{u\}$ is homeomorphic to the one-point compactification of a discrete space of cardinality $\mathfrak{c}$. In particular, any countably infinite subset of $A \setminus \{u\}$ is a sequence which converges to u. Since the set D has no convergent sequences, the set $D' = D \cap A$ is finite; it is straightforward that the set $A \setminus D'$ is an infinite (even uncountable) closed discrete subspace of X so X is not countably compact. $\qquad \square$

We will finish this paper with a couple of observations on the set X_1 of points of first countability of a compact space X. It was proved in [10, Example 2.17] that, in a countable Eberlein–Grothendieck space X, the set of points of non-local compactness of X can be compact without X being ultracomplete. Since little is known yet on ultracompleteness of X_1 even in scattered Eberlein compact spaces, it is worth mentioning that such a situation is not possible in the space X_1 whenever X is a scattered compact space.

Proposition 3.17. *If X is a scattered compact space and the set M of all points of non-local compactness of X_1 is compact, then X_1 is ultracomplete.*

Proof. Since $X \setminus X_1$ is Lindelöf by [10, Proposition 2.19], the set X_1 is of countable type, i.e., every compact subset of X_1 is contained in a compact subspace of countable outer character in X_1. In particular, this is true for M so X_1 is almost locally compact and hence ultracomplete by [12, Lemma 9]. $\quad \square$

The following result gives a positive answer to Problem 3.7 from [10].

Proposition 3.18. *If X is a compact space then $l(X \setminus Y) \leq \mathfrak{c}$ for every $Y \subset X_1$.*

Proof. If $\mathcal{U} \subset \tau(X)$ is a cover of $Z = X \setminus Y$ then $K = X \setminus (\bigcup \mathcal{U})$ is a compact set contained in X_1. We have $\chi(K) \leq \omega$ so $|K| \leq \mathfrak{c}$; an easy consequence is that K is the intersection of $\leq \mathfrak{c}$-many open subsets of the space X. Hence the set $U = \bigcup \mathcal{U}$ is a union of $\leq \mathfrak{c}$-many compact subsets of X. Therefore we can find $\mathcal{U}' \subset \mathcal{U}$ such that $|\mathcal{U}'| \leq \mathfrak{c}$ and $\bigcup \mathcal{U}' = U \supset Z$. Thus the family $\mathcal{U}'$ is a subcover of Z of cardinality at most $\mathfrak{c}$. $\qquad \square$

4. Open problems.

The are quite a few interesting open questions on coincidence of ultracompleteness and almost local compactness. The list below shows that the topic of this paper still has a strong potential for development.

Problem 4.1. *Does there exist in ZFC a compact space X of countable tightness such that some ultracomplete $Y \subset X$ is not almost locally compact?*

Problem 4.2. *Does there exist an isocompact space X such that some ultracomplete subspace of X is not almost locally compact?*

Problem 4.3. *Must any non-empty isocompact ultracomplete space have points of local compactness?*

Problem 4.4. *Let X be a sequential compact space. Is it true that every ultracomplete $Y \subset X$ is almost locally compact?*

Problem 4.5. *Let X be a Fréchet–Urysohn compact space. Is it true that every ultracomplete $Y \subset X$ is almost locally compact?*

Problem 4.6. *Let X be a space with a G_δ-diagonal. Is it true that any ultracomplete subspace of X is almost locally compact?*

Problem 4.7. *Is there a ZFC example of a countable space X with some point of non-local compactness which is not marked in X?*

Problem 4.8. *Let X be a homogeneous ultracomplete space without points of local compactness. Must X be countably compact?*

Problem 4.9. *Let X be a scattered compact space. Is it true that every ultracomplete $Y \subset X$ is almost locally compact?*

Problem 4.10. *Let X be a scattered compact space of countable tightness. Is it true that every ultracomplete $Y \subset X$ is almost locally compact? What happens if X is Fréchet–Urysohn or sequential?*

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