Applied General Topology
© Universidad Politécnica de Valencia
Volume 8, No. 2, 2007
pp. 161-170

Finite products of filters that are compact relative to a class of filters

Francis Jordan, Iwo Labuda and Frédéric Mynard

Abstract. Filters whose product with every countable based countably compact filter is countably compact are characterized.

2000 AMS Classification: 54A20, 54B10, 54D30.

Keywords: compact, countably compact, filters, product space, product filters.

1. Introduction

Two families $\mathcal{A}$ and $\mathcal{B}$ of subsets of a topological space X *mesh* (denoted $\mathcal{A}\#\mathcal{B}$), if $A \cap B \neq \varnothing$ whenever $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Given a class $\mathbb{D}$ of filters on X, we say that a filter $\mathcal{F}$ on X is $\mathbb{D}$*-compact at* $A \subset X$ if

$$\mathcal{D} \in \mathbb{D}, \mathcal{D}\#\mathcal{F} \Longrightarrow \operatorname{adh} \mathcal{D} \cap A \neq \varnothing.$$

If $\mathcal{F} = \{X\}$ and $A = X$, we recover the notion of a $\mathbb{D}$-compact space. Instances include compact spaces (when $\mathbb{D}$ is the class $\mathbb{F}$ of all filters), countably compact spaces (when $\mathbb{D}$ is the class $\mathbb{F}_\omega$ of countably based filters), Lindelöf spaces (when $\mathbb{D}$ is the class $\mathbb{F}_{\wedge\omega}$ of countably deep filters) among others. We recall that a filter $\mathcal{F}$ is *countably deep* if $\bigcap \mathcal{A} \in \mathcal{F}$ whenever $\mathcal{A}$ is a countable subfamily of $\mathcal{F}$.

In [11], [12], J. Vaughan investigated stability under product of $\mathbb{D}$-compact spaces under fairly general conditions on the class $\mathbb{D}$. However, even for simple cases like that of $\mathbb{D}$ being the class of countably based filters, no *internal* characterization of *spaces* whose product with every countably compact space is countably compact is known (known characterization involve the Stone-Čech compactification of the space). We investigate the problem in the framework of $\mathbb{D}$-*compact filters.* In Section 3 below, we characterize *filters* whose product with every $\mathbb{D}$-compact filter is $\mathbb{D}$-compact. The result turns out to be interesting for at least two reasons. It leads to a discussion of *exotic filters* in Section 4 which provides a deeper perspective on the productivity problem for variants of compactness like countable compactness. Next, the theorem allows other applications. Indeed, many classes of maps can be characterized as those preserving

$\mathbb{D}$-compact filters [10]. Several local topological properties can be characterized in terms involving $\mathbb{D}$-compact filters. Consequently, our product theorem for $\mathbb{D}$-compact filters can be applied to the investigations of stability under product of global properties ($\mathbb{D}$-compact spaces), local properties (Fréchetness and variants) and maps [9].

2. Terminology and basic facts

Our terminology is standard and compatible with [7]. The word 'space' refers to 'topological space'. A *filter* on a set X is a non-empty family of subsets stable by supersets and finite intersections. The only filter containing the empty set is the *degenerate filter* 2^X. The set of filters on a set X is preordered by inclusion. Denoting $\mathcal{A}^\uparrow = \{B \subset X : \exists A \in \mathcal{A},\, A \subset B\}$, the infimum filter $\mathcal{F} \wedge \mathcal{G}$ is $\{F \cup G : F \in \mathcal{F},\, G \in \mathcal{G}\}^\uparrow$. The supremum $\mathcal{F} \vee \mathcal{G}$ of two filters $\mathcal{F}$ and $\mathcal{G}$ exists whenever $\mathcal{F}\#\mathcal{G}$ and is $\{F \cap G : F \in \mathcal{F},\, G \in \mathcal{G}\}^\uparrow$. We use blackboard fonts to denote classes of filters. For instance $\mathbb{F}$ denotes the class of all filters (with unspecified set) and $\mathbb{F}(X)$ denote the family of filters on X. Analogously, $\mathbb{F}_\omega$ denotes the class of countably based filters (set unspecified) and $\mathbb{F}_\omega(X)$ the family of countably based filters on X. The class of principal filters is denoted $\mathbb{F}_1$. A filter is *free* if its intersection is empty. If $\mathbb{D}$ is a class of filters, we denote by $\mathbb{D}^\varnothing$ the class of free $\mathbb{D}$-filters. We denote by $\mathcal{F}^\varnothing$ the free part $\mathcal{F} \vee \left(\bigcap \mathcal{F}\right)^c$ of a filter $\mathcal{F}$, and by $\mathcal{F}^\bullet$ its principal part $\bigcap \mathcal{F}$. One or the other may be the degenerate filter $\{\varnothing\}^\uparrow = 2^X$. We always have $\mathcal{F} = \mathcal{F}^\varnothing \wedge \mathcal{F}^\bullet$, with the convention that $\mathcal{G} \wedge \{\varnothing\}^\uparrow = \mathcal{G}$.

Let $\mathbb{D}$ be a class of filters and let $\mathcal{A}$ and $\mathcal{B}$ be two families of subsets of a space X. We say that $\mathcal{A}$ is $\mathbb{D}$-*compact at* $\mathcal{B}$ if

$$\mathcal{D} \in \mathbb{D},\ \mathcal{D}\#\mathcal{A} \Longrightarrow \mathrm{adh}_X \mathcal{D}\#\mathcal{B}.$$

If $\mathcal{B} = \{X\}$, we drop 'at $\mathcal{B}$' and if $\mathcal{B} = \mathcal{A}$, we say that $\mathcal{A}$ is $\mathbb{D}$-*selfcompact.* Notice that a subset A of X is compact (resp. countably compact, Lindelöf) if and only if $\{A\}$ is $\mathbb{D}$-selfcompact for the class $\mathbb{D} = \mathbb{F}$ of all (resp. the class $\mathbb{D} = \mathbb{F}_\omega$ of countably based, the class $\mathbb{D} = \mathbb{F}_{\wedge\omega}$ of countably deep) filters. Compactness relative to the class $\mathbb{F}_1$ of principal filters is trivial only for principal filters. It becomes an important concept for general filters. For instance, it is instrumental in characterizing convergence in terms of compactness.

Lemma 2.1. *Let X be a space and let $\mathcal{F} \in \mathbb{F}(X)$. The following are equivalent:*

(1) $x \in \lim_X \mathcal{F}$;
(2) *$\mathcal{F}$ is compact at $\{x\}$;*
(3) *$\mathcal{F}$ is $\mathbb{F}_1$-compact at $\{x\}$.*

Recall that X is *Fréchet* if, for each $A \subset X$ and each x in the closure of A, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ on A converging to x. A space X is Fréchet if and only if its neighborhood filters are *Fréchet filters* in the following sense: a filter $\mathcal{F}$ is Fréchet if

$$A\#\mathcal{F} \Longrightarrow \exists \mathcal{H} \in \mathbb{F}_\omega, \mathcal{H}\#A \text{ and } \mathcal{H} \geq \mathcal{F}.$$

Similarly, a space X is *strongly Fréchet* if, for each $x \in \bigcap_{n\in\mathbb{N}} \operatorname{cl} A_n$ with a decreasing sequence $(A_n)_{n\in\mathbb{N}}$ of subsets of X, there exists $x_n \in A_n$ such that $x \in \lim(x_n)_{n\in\mathbb{N}}$. A space X is strongly Fréchet if and only if its neighborhood filters are *strongly Fréchet filters* in the following sense: a filter $\mathcal{F}$ is strongly Fréchet if

$$\mathcal{D}\#\mathcal{F}, \mathcal{D} \in \mathbb{F}_\omega \Longrightarrow \exists \mathcal{H} \in \mathbb{F}_\omega, \mathcal{H}\#\mathcal{D} \text{ and } \mathcal{H} \geq \mathcal{F}.$$

Fréchet and strongly Fréchet filters are instances of the following general concept. Let $\mathbb{D}$ and $\mathbb{J}$ be classes of filters.

A filter $\mathcal{F}$ is called $\mathbb{D}$ *to* $\mathbb{J}$ *meshable-refinable, or a* $(\mathbb{D}/\mathbb{J})_{\#\geq}$*-filter,* if

$$\mathcal{D} \in \mathbb{D}, \mathcal{D}\#\mathcal{F} \Longrightarrow \exists \mathcal{J} \in \mathbb{J}, \mathcal{J}\#\mathcal{D} \text{ and } \mathcal{J} \geq \mathcal{F}. \tag{2.1}$$

Similarly, a filter $\mathcal{F}$ is called $\mathbb{D}$ to $\mathbb{J}$ *sup-meshable, or a* $(\mathbb{D}/\mathbb{J})_{\#\vee}$*-filter,* if

$$\mathcal{D} \in \mathbb{D}, \mathcal{D}\#\mathcal{F} \Longrightarrow \exists \mathcal{J} \in \mathbb{J}, \mathcal{J}\#\mathcal{D} \vee \mathcal{F}. \tag{2.2}$$

A filter $\mathcal{F}$ is called $\mathbb{D}$ to $\mathbb{J}$ *meshable, or a* $(\mathbb{D}/\mathbb{J})_{\#}$*-filter,* if

$$\mathcal{D} \in \mathbb{D}, \mathcal{D}\#\mathcal{F} \Longrightarrow \exists \mathcal{J} \in \mathbb{J}, \mathcal{J}\#\mathcal{D}. \tag{2.3}$$

Let $\mathbb{J}$ be a set of filters on a set X. By $X \oplus \mathbb{J}$ we mean the set $X \cup \mathbb{J}$ endowed with the topology in which all points of X are isolated and the neighborhood filters of the points $\mathcal{J}$ of $\mathbb{J}$ are of the form $\mathcal{N}_{X\oplus\mathbb{J}}(\{\mathcal{J}\}) = \mathcal{J} \wedge \{\mathcal{J}\}$. By the very definition of $X \oplus \mathbb{J}$, we have

Proposition 2.2. *A free filter on X is $\mathbb{D}$ to $\mathbb{J}$ meshable if and only if the filter it generates is $\mathbb{D}$-compact in $X \oplus \mathbb{J}$.*

A space X is $\mathbb{D}$-*based* if its neighborhood filters are $\mathbb{D}$-filters; it is $(\mathbb{D}/\mathbb{J})$-*accessible* if it is $(\mathbb{D}/\mathbb{J})_{\#\geq}$-based. With $\mathbb{J}$ being the class of countably based filters, $(\mathbb{D}/\mathbb{J})_{\#\geq}$-filters are *Fréchet* (resp. *strongly Fréchet, productively Fréchet* [5], *weakly bisequential* and *bisequential filters*), whenever $\mathbb{D}$ stands for the class of principal (resp. countably based, strongly Fréchet, countably deep and all) filters [1], [6].

The notion of total countable compactness was first introduced by Z. Frolík [2] for a study of products of countably compact and pseudocompact spaces. The property has been rediscovered under various names by several authors (see [11, p. 212]). A topological space is *totally countably compact* if every countably based filter has a finer (equivalently, meshes a) compact countably based filter. The name comes from *total* nets of Pettis. As one of the possible generalizations of total countable compactness of a set, we say that a filter $\mathcal{F}$ is *compactly countably meshable* (at $\mathcal{A}$) if for every countably based filter $\mathcal{H}\#\mathcal{F}$, there exists a countably based filter $\mathcal{C}\#\mathcal{H}$ which is compact (at $\mathcal{A}$).

More generally, in order to maximize the number of cases handled by the result, we introduce the following key notion:

Let $\mathbb{D}$ and $\mathbb{J}$ be classes of filters. A filter $\mathcal{F}$ is called *compactly* $\mathbb{D}$ *to* $\mathbb{J}$ *meshable (at* $\mathcal{A}$), or $\mathcal{F}$ is a *compactly* $(\mathbb{D}/\mathbb{J})_{\#}$-filter, if for every $\mathbb{D}$-filter $\mathcal{D}\#\mathcal{F}$ there exists a $\mathbb{J}$-filter $\mathcal{J}\#\mathcal{D}$ which is compact (at $\mathcal{A}$).

It turns out to be the notion permitting characterizations of filters whose product with every $\mathbb{D}$-compact filter (of a certain class) is $\mathbb{D}$-compact . With various instances of $\mathbb{D}$ and $\mathbb{J}$, the concept is also instrumental in the characterization of a wide variety of notions, from total countable compactness, total Lindelöfness and total pseudocompactness [12] to Fréchetness, strong Fréchetness, productive Fréchetness and bisequentiality [4], and properties of maps and their range (see [10] for details). In particular, we have

Proposition 2.3. *Let $\mathbb{D}$ and $\mathbb{J}$ be classes of filters on a space X. The following are equivalent.*

1. *X is $(\mathbb{D}/\mathbb{J})$-accessible;*
2. *for every $\mathcal{D} \in \mathbb{D}$, $\operatorname{adh}\mathcal{D} \subset \bigcup\{\lim \mathcal{J} : \mathcal{J}\#\mathcal{D}, \mathcal{J} \in \mathbb{J}\}$;*
3. *$x \in \lim \mathcal{F} \Longrightarrow \mathcal{F}$ is compactly $\mathbb{D}$ to $\mathbb{J}$ meshable at $\{x\}$.*

We also note that

$$\mathcal{F} \text{ compactly } (\mathbb{D}/\mathbb{J})_{\#} \Longrightarrow \mathcal{F}\ \mathbb{D}\text{-compact} \underset{X \text{ is } (\mathbb{D}/\mathbb{J})\text{-accessible}}{\Longrightarrow} \mathcal{F} \text{ compactly } (\mathbb{D}/\mathbb{J})_{\#}$$

$$\mathcal{F} \text{ compact} \underset{\mathcal{F}\in(\mathbb{D}/\mathbb{J})_{\#\geq}}{\Longrightarrow} \mathcal{F} \text{ compactly } (\mathbb{D}/\mathbb{J})_{\#}.$$

In particular, a countably compact filter on a strongly Fréchet space is compactly countably meshable and a bisequential filter is compactly $\mathbb{F}$ to $\mathbb{F}_\omega$ meshable if and only if it is compact.

If $\mathbb{D}$ and $\mathbb{J}$ are two classes of filters, we say that $\mathbb{J}$ is $\mathbb{D}$-*composable* if for any X and Y, the (possibly degenerate) filter $\mathcal{HF}$ generated by $\{HF : H \in \mathcal{H}, F \in \mathcal{F}\}$ belongs to $\mathbb{J}(Y)$ whenever $\mathcal{F} \in \mathbb{J}(X)$ and $\mathcal{H} \in \mathbb{D}(X \times Y)$, with the convention that every class of filters contains the degenerate filter. If a class $\mathbb{D}$ is $\mathbb{D}$-composable, we simply say that $\mathbb{D}$ is *composable.* Notice that

$$\mathcal{H}\#(\mathcal{F} \times \mathcal{G}) \Longleftrightarrow \mathcal{HF}\#\mathcal{G} \Longleftrightarrow \mathcal{H}^{-}\mathcal{G}\#\mathcal{F}, \tag{2.4}$$

where $\mathcal{H}^{-}\mathcal{G} = \{H^{-}G = \{x \in X : (x,y) \in H \text{ and } y \in G\} : H \in \mathcal{H}, G \in \mathcal{G}\}^{\uparrow}$.

Observe also [6] that a composable class of filters that contains principal filters is stable under finite products.

3. A Product Theorem

Theorem 3.1. *Let $\mathbb{D}$ be a composable class of filters containing all principal filters and let $\mathbb{J}$ be a $\mathbb{D}$-composable class of filters. Let $A \subset X$. The following are equivalent.*

1. *$\mathcal{F}$ is a compactly $(\mathbb{J}/\mathbb{D})_{\#}$- filter at $A \subset X$;*
2. *for every space Y, every $B \subset Y$ and every $\mathbb{J}$-filter $\mathcal{G}$ which is $\mathbb{D}$-compact at B, the filter $\mathcal{F} \times \mathcal{G}$ is $\mathbb{D}$-compact at $A \times B$;*
3. *for every space Y, every $B \subset Y$ and every $\mathbb{J}$-filter $\mathcal{G}$ which is $\mathbb{D}$-compact at B, the filter $\mathcal{F} \times \mathcal{G}$ is $\mathbb{F}_1$-compact at $A \times B$.*

Moreover, if $\mathcal{F} \in (\mathbb{J}^{\varnothing}/\mathbb{D}^{\varnothing})_{\#\vee}$, then the above are also equivalent to:

4. *for every space Y and for every $\mathbb{D}$-compact $\mathbb{J}$-filter $\mathcal{G}$, the filter $\mathcal{F} \times \mathcal{G}$ is $\mathbb{D}$-compact at $A \times Y$.*

Proof. (1) $\Longrightarrow$ (2).
Let $\mathcal{D}$ be a $\mathbb{D}$-filter such that $\mathcal{D}\#\mathcal{F}\times\mathcal{G}$. As $\mathbb{J}$ is $\mathbb{D}$-composable, $\mathcal{D}^{-}\mathcal{G}$ is a $\mathbb{J}$-filter and $\mathcal{D}^{-}\mathcal{G}\#\mathcal{F}$. Since $\mathcal{F}$ is a compactly $(\mathbb{J}/\mathbb{D})_{\#}$-filter at A, there exists a $\mathbb{D}$-filter $\mathcal{C}\#\mathcal{D}^{-}\mathcal{G}$ which is compact at A. Now $\mathcal{DC}\#\mathcal{G}$ and $\mathcal{DC}$ is a $\mathbb{D}$-filter, so that there exists a filter $\mathcal{M}\#\mathcal{DC}$ which is convergent to some y in B and meshes with $\mathcal{DC}$. The filter $\mathcal{D}^{-}\mathcal{M}$ meshes with $\mathcal{C}$, which is compact at A. Therefore, there exists $\mathcal{U}\#\mathcal{D}^{-}\mathcal{M}$ which is convergent to some point x in A. The filter $\mathcal{U}\times\mathcal{M}$ meshes with $\mathcal{D}$ and converges to $(x,y)\in A\times B$.

(2) $\Longrightarrow$ (3) is clear, as $\mathbb{F}_1\subset\mathbb{D}$ and (3) $\Longrightarrow$ (4) is clear.

(3) $\Longrightarrow$ (1).
If $\mathcal{F}$ is not compactly $(\mathbb{J}/\mathbb{D})_{\#}$ at A, then there exists a $\mathbb{J}$-filter $\mathcal{J}\#\mathcal{F}$ such that for every $\mathbb{D}$-filter $\mathcal{D}\#\mathcal{J}$, there exists an ultrafilter $\mathcal{U}_{\mathcal{D}}\geq\mathcal{D}$ such that $\lim\mathcal{U}_{\mathcal{D}}\cap A=\varnothing$. Consider the topological space $Y{=}X\oplus\{\mathcal{U}_{\mathcal{D}}:\mathcal{D}\#\mathcal{J},\,\mathcal{D}\in\mathbb{D}\}$. Then the filter $\widehat{\mathcal{J}}$ generated by $\mathcal{J}$ on Y is $\mathbb{D}$-compact at $B=\{\mathcal{U}_{\mathcal{D}}:\mathcal{D}\#\mathcal{J},\,\mathcal{D}\in\mathbb{D}\}$ but $\mathcal{F}\times\widehat{\mathcal{J}}$ is not $\mathbb{F}_1$-compact at $A\times B$. Indeed, $\Delta=\{(x,x):x\in X\}\subset X\times Y$ is in $\mathbb{F}_1$ and $\Delta\#\mathcal{F}\times\widehat{\mathcal{J}}$ because $\mathcal{F}\#\mathcal{J}$. But $\mathrm{adh}\Delta\cap A\times B=\varnothing$. Indeed, if $\mathcal{H}$ is a filter on Δ, then there exists a filter $\mathcal{H}_0$ on X such that $\mathcal{H}$ is generated by $\{\{(x,x):x\in H\}:H\in\mathcal{H}_0\}$. If $(x,\mathcal{U}_{\mathcal{D}})\in\lim_{X\times Y}\mathcal{H}\cap(A\times B)$ then $\mathcal{U}_{\mathcal{D}}\in\lim_Y\mathcal{H}_0$ (and $x\in\lim_X\mathcal{H}_0$). Hence as a filter on X, $\mathcal{H}_0{=}\,\mathcal{U}_{\mathcal{D}}$, so that $\lim_X\mathcal{H}_0\cap A=\varnothing$. Thus $x\notin A$.

(4) $\Longrightarrow$ (1) if $\mathcal{F}\in(\mathbb{J}^{\varnothing}/\mathbb{D}^{\varnothing})_{\#\vee}$. Indeed, if $\mathcal{F}$ is not compactly $(\mathbb{J}/\mathbb{D})_{\#}$ at A, then the filter generated by $\mathcal{J}$ on $Y=X\oplus\{\mathcal{U}_{\mathcal{D}}:\mathcal{D}\#\mathcal{J},\,\mathcal{D}\in\mathbb{D}\}$ (constructed as in (3) $\Longrightarrow$ (1)) is $\mathbb{D}$-compact . Since $\mathcal{F}\in(\mathbb{J}^{\varnothing}/\mathbb{D}^{\varnothing})_{\#\vee}$, there exists a *free* $\mathbb{D}$-filter $\mathcal{D}_0\#\mathcal{J}\vee\mathcal{F}$. The filter $\widehat{\mathcal{D}_0\times\mathcal{D}_0}$ generated on $X\times Y$ by $\mathcal{D}_0\times\mathcal{D}_0$ is a $\mathbb{D}$-filter by composability of $\mathbb{D}$ [6]. Moreover $\left(\widehat{\mathcal{D}_0\times\mathcal{D}_0}\right)\#\left(\mathcal{F}\times\mathcal{J}\right)$. But $\mathrm{adh}_{X\times Y}\left(\widehat{\mathcal{D}_0\times\mathcal{D}_0}\right)\cap(A\times Y)=\varnothing$. Indeed, consider an ultrafilter finer than $\widehat{\mathcal{D}_0\times\mathcal{D}_0}$, which can be written $\mathcal{U}\times\mathcal{U}$. As $\mathcal{D}_0$ is free, $\mathcal{U}$ converges in Y only if $\mathcal{U}=\mathcal{U}_{\mathcal{D}}$ for some $\mathcal{D}\in\mathbb{D}$ such that $\mathcal{D}\#\mathcal{J}$. But in this case, $\lim_X\mathcal{U}\cap A=\varnothing$. $\square$

Observe that (3) $\Longrightarrow$ (1) only uses $\mathbb{F}_1\subset\mathbb{D}$ and no other composability assumption.

Notice that in the special case where $\mathbb{J}=\mathbb{D}$, the condition $\mathcal{F}\in(\mathbb{J}^{\varnothing}/\mathbb{D}^{\varnothing})_{\#\vee}$ is always verified. Therefore, in the case $\mathbb{J}=\mathbb{D}=\mathbb{F}_{\kappa}$, and $A=X$, we get

Corollary 3.2. *The product filter $\mathcal{F}\times\mathcal{G}$ is κ-compact for every κ-compact and κ-based filter $\mathcal{G}$ if and only if $\mathcal{F}$ is a compactly $(\mathbb{F}_{\kappa}/\mathbb{F}_{\kappa})_{\#}$-filter.*

In particular, if $\kappa=\omega$, we obtain the announced result on the productivity of countable compactness at the level of filters.

Corollary 3.3.

(1) *The product filter $\mathcal{F} \times \mathcal{G}$ is countably compact for every countably based and countably compact filter $\mathcal{G}$ if and only if $\mathcal{F}$ is compactly countably meshable.*

(2) *Let $\mathcal{F}$ be a countably based filter. The product filter $\mathcal{F} \times \mathcal{G}$ is countably compact for every strongly Fréchet and countably compact filter $\mathcal{G}$ if and only if $\mathcal{F}$ is compactly countably meshable.*

Proof. (2) follows from the simple observation that if $\mathcal{F}$ is a countably based and compactly $(\mathbb{F}_\omega/\mathbb{F}_\omega)_\#$-filter, then it is also a compactly $((\mathbb{F}_\omega/\mathbb{F}_\omega)_{\#\geq}/\mathbb{F}_\omega)_\#$-filter. $\square$

In particular, if $\mathcal{F} = \{X\}$, we obtain that the product of a totally countably compact space with not only countably compact spaces but also strongly Fréchet countably compact filters is countably compact, generalizing [11, Theorem 2]. More generally, the part ($1 \Longrightarrow 3$) of Theorem 3.1 applied to principal filters $\mathcal{F} = \{X\}$ and $\mathcal{G} = \{Y\}$, for various instances of $\mathbb{D} = \mathbb{J}$ allows to recover results of J. Vaughan [11], and also to provide new variants. For instance:

Corollary 3.4. *The product of a Lindelöf space and a compactly $(\mathbb{F}_{\wedge\omega}/\mathbb{F}_{\wedge\omega})$-meshable space is Lindelöf.*

Denote by $\mathcal{O}(\mathcal{A})$ the family $\{O \text{ open: } \exists A \in \mathcal{A},\ A \subset O\}$ whenever $\mathcal{A}$ is a family of subsets of a space X. Further, $\mathcal{O}(\mathbb{D})$ will denote the class of $\mathbb{D}$-filters $\mathcal{D}$ such that $\mathcal{D} = \mathcal{O}(\mathcal{D})^{\uparrow}$. A topological space X is *feebly compact* if and only if $\{X\}$ is $\mathcal{O}(\mathbb{F}_\omega)$-compact . Completely regular feebly compact spaces are called pseudocompact.

Corollary 3.5. *The product of a compactly $(\mathcal{O}(\mathbb{F}_\omega)/\mathcal{O}(\mathbb{F}_\omega))$-meshable space and a feebly compact space is feebly compact.*

Other applications of Theorem 3.1 to results of stability under product of local topological properties (like Fréchetness and its variants) and of maps (variants of perfect maps and of quotient maps) are presented in [9].

4. $\mathbb{D}$-exotic filters

This section is devoted to a discussion of the additional assumption that $\mathcal{F} \in (\mathbb{J}^{\varnothing}/\mathbb{D}^{\varnothing})_{\#\vee}$ in point (4) of Theorem 3.1. The very fact that such an additional condition is needed is surprising. In our experience with $\mathbb{D}$-compact filters, this is the only instance we know of, where it makes a significant difference to consider $\mathbb{D}$-compact filters at a proper subset of X or simply $\mathbb{D}$-compact filters (that is, at X). As noticed before, this condition is always fulfilled if $\mathbb{D} = \mathbb{J}$.

Let $\mathbb{D}$ be a class of filters. A filter $\mathcal{F}$ is $\mathbb{D}$-*exotic* if every ultrafilter $\mathcal{U}$ finer than $\mathcal{F}$ is $\mathbb{D}$-*deep*, that is, $\mathcal{D} \in \mathbb{D}$ and $\mathcal{D} \leq \mathcal{U}$ implies that $\mathcal{D}^\bullet \in \mathcal{U}$. Notice that every fixed ultrafilter is $\mathbb{D}$-exotic, whatever the class $\mathbb{D}$ is. A $\mathbb{D}$-exotic filter is *non-trivial* if it has a finer *free* $\mathbb{D}$-deep ultrafilter. Recall [6] that a class $\mathbb{J}$ of filters is $\mathbb{D}$-*steady* if $\mathcal{J} \vee \mathcal{D} \in \mathbb{J}$ whenever $\mathcal{J} \in \mathbb{J}$ and $\mathcal{D} \in \mathbb{D}$.

Theorem 4.1. *Let $\mathbb{D}$ be an $\mathbb{F}_1$-steady class of filters and let $\mathcal{F}$ be a filter on X. The following are equivalent.*

(1) *$\mathcal{F}$ is $\mathbb{D}$-exotic;*
(2) *$\mathcal{D}^\bullet \neq \varnothing$ whenever $\mathcal{D}$ is a $\mathbb{D}$-filter $\mathcal{D}$ meshing with $\mathcal{F}$;*
(3) *$\mathcal{F}$ is $\mathbb{D}$-compact on X endowed with the discrete topology.*

If moreover $\mathbb{D}$ is composable and contains fixed ultrafilters, then the above conditions are equivalent to

(4) *for every space Y, and for every $\mathbb{D}$-compact $\mathbb{D}$-filter $\mathcal{G}$, the filter $\mathcal{F} \times \mathcal{G}$ is $\mathbb{D}$-compact in $X \times Y$ with discrete X.*

Proof. (1) $\Longrightarrow$ (2).
Let $\mathcal{D}\#\mathcal{F}$ be a $\mathbb{D}$-filter. Then there exists an ultrafilter $\mathcal{U}$ finer than both $\mathcal{F}$ and $\mathcal{D}$. Since $\mathcal{F}$ is $\mathbb{D}$-exotic, $\mathcal{U}$ is $\mathbb{D}$-deep so that $\mathcal{D}^\bullet \in \mathcal{U}$ and consequently $\mathcal{D}^\bullet \neq \varnothing$.

(2) $\Longrightarrow$ (3) because $\mathrm{adh}\mathcal{D} = \mathcal{D}^\bullet$ in the discrete topology of X.

(3) $\Longrightarrow$ (1).
Let $\mathcal{U}$ be an ultrafilter finer than $\mathcal{F}$ and let $\mathcal{D} \in \mathbb{D}$ with $\mathcal{D} \leq \mathcal{U}$. Then $\mathcal{D}\#\mathcal{F}$ so that $\mathrm{adh}\mathcal{D} = \mathcal{D}^\bullet \neq \varnothing$. Observe that $\mathcal{D}^\bullet \in \mathcal{U}$. Otherwise, $(\mathcal{D}^\bullet)^c \in \mathcal{U}$ and we would have $(\mathcal{D}^\bullet)^c \#\mathcal{D}$. Hence $\mathcal{D}^{\varnothing}$ would be a free $\mathbb{D}$-filter meshing with $\mathcal{F}$ and so having empty adherence in the discrete topology of X. Hence $\mathcal{U}$ is $\mathbb{D}$-deep.

(2) $\Longrightarrow$ (4).

Let X carry the discrete topology, let $\mathcal{G}$ be a $\mathbb{D}$-compact $\mathbb{D}$-filter on Y and let $\mathcal{D} \in \mathbb{D}$ such that $\mathcal{D}\#(\mathcal{F} \times \mathcal{G})$. If $\mathbb{D}$ is composable, then $\mathcal{D}^{-}\mathcal{G} \in \mathbb{D}$. Moreover $\mathcal{D}^{-}\mathcal{G}\#\mathcal{F}$. By (2), $(\mathcal{D}^{-}\mathcal{G})^\bullet \neq \varnothing$. Let $x \in (\mathcal{D}^{-}\mathcal{G})^\bullet$. Then $\mathcal{D}\{x\}\#\mathcal{G}$ and $\mathcal{D}\{x\} \in \mathbb{D}$, so that there exists a convergent filter $\mathcal{L}$ meshing with $\mathcal{D}\{x\}$. Then $(\{x\}^\uparrow \times \mathcal{L})\#\mathcal{D}$ and $\{x\}^\uparrow \times \mathcal{L}$ is $X \times Y$ convergent. Hence $\mathrm{adh}_{X\times Y}\mathcal{D} \neq \varnothing$.

(4) $\Longrightarrow$ (2).
If $\mathcal{F}$ is not $\mathbb{D}$-exotic, then there exists a free $\mathbb{D}$-filter $\mathcal{D}\#\mathcal{F}$. The filter $\mathcal{D}$ is convergent, hence compact in $Y = X \oplus \{\mathcal{D}\}$. However, $\mathcal{F} \times \mathcal{D}$ is not $\mathbb{D}$-compact. Indeed, the filter generated by $\{(x,x) : x \in D\}_{D\in\mathcal{D}}$ on $X \times Y$ is a $\mathbb{D}$-filter meshing with $\mathcal{F} \times \mathcal{D}$, but its adherence in $X \times Y$ is empty if X carries the discrete topology, since any filter finer than $\mathcal{D}$ is free, hence does not converge in X. ☐

Corollary 4.2. *Let $\mathbb{D}$ be an $\mathbb{F}_1$-steady class of filters. A filter $\mathcal{F} \notin (\mathbb{J}^{\varnothing}/\mathbb{D}^{\varnothing})_{\#\vee}$ if and only if there exists a free $\mathbb{J}$-filter $\mathcal{J}$ meshing with $\mathcal{F}$ such that $\mathcal{F} \vee \mathcal{J}$ is $\mathbb{D}$-exotic.*

So the assumption that $\mathcal{F} \in (\mathbb{J}^{\varnothing}/\mathbb{D}^{\varnothing})_{\#\vee}$ in Theorem 3.1 is empty, if non-trivial $\mathbb{D}$-exotic filters do not exist. This is often the case, as shown by the following observation.

Proposition 4.3. *Let X be an infinite set. If the class $\mathbb{D}(X)$ of filters contains the cofinite filter $\mathcal{C}$ on X, then there is no free $\mathbb{D}$-deep ultrafilter on X, and therefore, no non trivial $\mathbb{D}$-exotic filter on X.*

Proof. Assume that $\mathcal{U}$ is a free $\mathbb{D}$-deep ultrafilter. Then $\mathcal{U} \geq \mathcal{C}$ because $\mathcal{U}$ is free. Since $\mathcal{C} \in \mathbb{D}$, $\mathcal{C}^\bullet \in \mathcal{U}$ and therefore $\mathcal{C}^\bullet \neq \varnothing$; a contradiction. ☐

Notice that if $\mathbb{D} \subset \mathbb{J}$ are two classes of filters, then every $\mathbb{J}$-exotic filter is also $\mathbb{D}$-exotic. In other words, if there is no non-trivial $\mathbb{D}$-exotic filter, then there is no non-trivial $\mathbb{J}$-exotic filter. As the cofinite filter on an infinite set is almost principal, hence productively Fréchet [6], the class of $\mathbb{D}$-exotic filters is trivial when $\mathbb{D}$ is the class of almost principal, of productively Fréchet, of strongly Fréchet, of Fréchet, of countably tight, or of countably fan-tight filters, among others (see [6] for details). Recall that a filter $\mathcal{F}$ is *almost principal* [6] if there exists $F_0 \in \mathcal{F}$ such that $|F \backslash F_0| < \omega$ for every $F \in \mathcal{F}$.

This, however, does not take care of the case $\mathbb{D} = \mathbb{F}_\omega$. In fact, the existence of $\mathbb{F}_\omega$-exotic filters depends on the cardinality of the underlying set. Recall that the cardinality of X is *measurable* if there exists a countably additive $\{0,1\}$-measure on 2^X. The free ultrafilter formed by the sets of measure 1 is then $\mathbb{F}_\omega$-deep.

Theorem 4.4. *The following are equivalent:*

(1) *card* (X) *is measurable;*
(2) *the cofinite filter on* X *is* not *a bisequential filter;*
(3) *there exists a non trivial* $\mathbb{F}_\omega$*-exotic (ultra)filter on* X*;*
(4) *there exists an* $\mathbb{F}_\omega$*-compact (ultra)filter on* X *with the discrete topology;*
(5) X *endowed with the discrete topology is* not *realcompact.*

Proof. (1) $\iff$ (2) is proven in [8, Example 10.15] though in the following different language: the one point compactification of a discrete space X is bisequential if and only if $card(X)$ is not measurable. As the one point compactification of X is $X \oplus \mathcal{C}$, its bisequentiality amounts to that of the filter $\mathcal{C}$. (1) $\iff$ (3) is clear and (3) $\iff$ (4) follows from Theorem 4.1. Finally (1) $\iff$ (5) is [3, Theorem 12.2]. $\square$

Consequently, if the cardinality of the set is non measurable, then the class of $(\mathbb{F}/\mathbb{F}_\omega)_{\#\geq}$-exotic filters, and therefore that of $\mathbb{F}_\omega$-exotic filters is trivial.

Corollary 4.5. *Let* X *be a set of non measurable cardinality and let* $\mathcal{F}$ *be a bisequential filter on* X*. Then* $\mathcal{F} \times \mathcal{G}$ *is countably compact for every countably compact filter* $\mathcal{G}$ *if and only if* $\mathcal{F}$ *is compact.*

Proof. In view of Theorem 3.1 and Theorem 4.4, $\mathcal{F} \times \mathcal{G}$ is countably compact for every countably compact filter $\mathcal{G}$ if and only if $\mathcal{F}$ is a compactly $(\mathbb{F}/\mathbb{F}_\omega)_\#$-filter. As observed before, this condition is equivalent to compactness for a bisequential filter. $\square$

Corollary 4.6. *Let* X *be a set of non measurable cardinality and let* $\mathcal{F}$ *be a filter on* X*. Let* $\widehat{\mathcal{F}}$ *denote the filter generated by* $\mathcal{F}$ *on the space* $X \oplus \{\mathcal{F}\}$*. Then* $\widehat{\mathcal{F}} \times \mathcal{G}$ *is countably compact for every countably compact filter* $\mathcal{G}$ *if and only if* $\mathcal{F}$ *is bisequential.*

Proof. If $\mathcal{F}$ is bisequential, then $\widehat{\mathcal{F}}$ is both compact and bisequential on $X \oplus \{\mathcal{F}\}$. In view of Corollary 4.5, $\widehat{\mathcal{F}} \times \mathcal{G}$ is countably compact for every countably compact filter $\mathcal{G}$.

Conversely, if $\mathcal{F}$ is not bisequential, then there exists an ultrafilter $\mathcal{U}_0$ finer than $\mathcal{F}$ such that every countably based filter $\mathcal{D}$ coarser than $\mathcal{U}$ is not finer than $\mathcal{F}$. In other words, for every countably based $\mathcal{D} \leq \mathcal{U}$, there exists $F_{\mathcal{D}} \in \mathcal{F}$ such that $F^c_{\mathcal{D}} \# \mathcal{D}$. Let $\mathcal{U}_{\mathcal{D}}$ denote an ultrafilter of $\mathcal{D} \vee F^c_{\mathcal{D}}$. The filter $\widehat{\mathcal{U}_0}$ generated by $\mathcal{U}_0$ on $Y = X \oplus \{\mathcal{U}_{\mathcal{D}} : \mathcal{D} \in \mathbb{F}_\omega, \mathcal{D} \leq \mathcal{U}_0\}$ is countably compact. But $\widehat{\mathcal{F}} \times \widehat{\mathcal{U}_0}$ is not countably compact. Indeed, there exists a free countably based filter $\mathcal{D}$ coarser than $\mathcal{U}_0$ on X, because the cardinality of X is non measurable. The filter $\widetilde{\mathcal{D}}$ generated on $(X \oplus \{\mathcal{F}\}) \times Y$ by $\{(x,x) : x \in D, D \in \mathcal{D}\}$ is countably based and meshes with $\widehat{\mathcal{F}} \times \widehat{\mathcal{U}_0}$. But $\mathrm{adh}_{(X \oplus \{\mathcal{F}\}) \times Y} \widetilde{\mathcal{D}} = \varnothing$. Indeed, if a filter finer than $\widetilde{\mathcal{D}}$ has a convergent Y-projection then its Y-projection is one of the ultrafilters $\mathcal{U}_{\mathcal{D}}$. Then its $(X \oplus \{\mathcal{F}\})$-projection is also $\mathcal{U}_{\mathcal{D}}$ and therefore does not mesh with $\mathcal{F}$. Consequently, the $(X \oplus \{\mathcal{F}\})$-projection does not converge in $X \oplus \{\mathcal{F}\}$. □

On the other hand, if the cardinality of X is measurable, an $\mathbb{F}_\omega$-deep ultrafilter on X defines a P-point in the Čech-Stone compactification βX of the discrete topological space X. Therefore, the set of ultrafilters finer than an $\mathbb{F}_\omega$-exotic filter is a closed subset of P-points of the compact set βX, hence a compact set of P-points. But every countably compact space of P-points is finite [3, 4.K.1]. Therefore:

Proposition 4.7. *If the cardinality of X is measurable, the $\mathbb{F}_\omega$-exotic filters on X are the infima of finitely many $\mathbb{F}_\omega$-deep ultrafilters on X.*

Neither Theorem 4.4 nor Proposition 4.3 apply to the question of existence of non-trivial $\mathbb{F}_{\wedge\omega}$-exotic filters. However, we observe that $\mathbb{F}_{\wedge\omega}$-exotic ultrafilters can exist only on a countable set. Indeed,

Proposition 4.8. *Let X be an uncountable set. If the class $\mathbb{D}(X)$ contains the cocountable filter $\mathcal{C}_\omega$ then there is no non-trivial $\mathbb{D}$-exotic filter that does not contain any countable set.*

Proof. Let $\mathcal{F}$ be a $\mathbb{D}$-exotic filter that does not contain any countable set. There is a free ultrafilter $\mathcal{U}$ finer than $\mathcal{F}$ which does not contain any countable set. Therefore, $\mathcal{U} \geq \mathcal{C}_\omega$. Moreover, as an ultrafilter of $\mathcal{F}$, it is $\mathbb{D}$-deep, and $\mathcal{C}_\omega \in \mathbb{D}$, so that $\mathcal{C}^\bullet_\omega \in \mathcal{U}$. A contradiction, because $\mathcal{C}^\bullet_\omega = \varnothing$. □

In particular, the cocountable filter on an uncountable set is countably deep. Therefore $\mathbb{F}_{\wedge\omega}$-exotic filters must contain a countable set. Moreover, no $\mathbb{F}_{\wedge\omega}$-filter on a countable set is free, so that every filter containing a countable set is $\mathbb{F}_{\wedge\omega}$-exotic.

Corollary 4.9. *A free filter is $\mathbb{F}_{\wedge\omega}$-exotic if and only if it contains a countable set.*

As a final remark on $\mathbb{D}$-exotic filters, notice that a class $\mathbb{D}$ of filters generating many $\mathbb{D}$-exotic filters cannot contain the cofinite filter or the cocountable filter. Roughly speaking, it seems that a naturally defined class of filters that does not contain the cofinite filter would have to be defined either in terms of cardinality of a base — in which case the existence of exotic filters implies the existence of measurable cardinals — or in terms of depth of the filter, in which case the class would contain the cocountable filter. So, in the latter case, the question of existence of exotic filters is, in a sense, reduced to sets of small cardinality.

References

1. S. Dolecki, *Active boundaries of upper semicontinuous and compactoid relations; closed and inductively perfect maps*, Rostock. Math. Coll. **54** (2000), 51–68.
2. Z. Frolík, *The topological product of two pseudocompact spaces*, Czech. Math. J. **85** (1960), no. 10, 339–349.
3. L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, 1960.
4. F. Jordan and F. Mynard, *Espaces productivement de Fréchet*, C. R. Acad. Sci. Paris, Ser I **335** (2002), 259–262.
5. ———, *Productively Fréchet spaces*, Czech. Math. J. **54** (2004), no. 129, 981–990.
6. ———, *Compatible relations on filters and stability of local topological properties under supremum and product*, Top. Appl. **153** (2006), 2386–2412.
7. J. Kelley, *General topology*, Van Nostrand, 1955.
8. E. Michael, *A quintuple quotient quest*, Gen. Topology Appl. **2** (1972), 91–138.
9. F. Mynard, *Products of compact filters and applications to classical product theorems*, Top. Appl. **154** (2007), no. 4, 953–958.
10. ———, *Relations that preserve compact filters*, Applied Gen. Top. **8** (2007),171–185.
11. J. Vaughan, *Products of topological spaces*, Gen. Topology Appl. **8** (1978), 207–217.
12. J. E. Vaughan, *Countably compact and sequentially compact spaces*, vol. Handbook of set-theoretic Topology, pp. 569–602, Elsevier Science, 1984.

Received September 2005

Accepted January 2006

Francis Jordan (fjordan@GeorgiaSouthern.edu)
Dept. Mathematical Sciences, Georgia Southern University, PO Box 8093, Statesboro, Ga 30460-8093, U.S.A.

Iwo Labuda (mmlabuda@olemiss.edu)
Department of Mathematics, University of Mississippi,University, MS 38677, U.S.A.

Frédéric Mynard (fmynard@georgiasouthern.edu)
Dept. Mathematical Sciences, Georgia Southern University, PO Box 8093, Statesboro, Ga 30460-8093, U.S.A.