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Completions of pre–uniform spaces

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Abstract. In this paper we prove the existence of a completion of a T_0–pre-uniform space $(X,\mathcal{U})$, with the property that each Cauchy filter in $(X,\mathcal{U})$ contains a weakly round filter.

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1. Introduction

By a pre–uniform space we mean a pair $(X,\mathcal{U})$ where X is a set and $\mathcal{U}$ is a non–empty family of covers of X satisfying certain properties. Every pre-uniform space $(X,\mathcal{U})$ determines a topology $\mathcal{T}_\mathcal{U}$ in X and the convergence properties of filters in $(X,\mathcal{T}_\mathcal{U})$ constitute an important area of study. Pre–uniform spaces generalize the semi–uniform spaces introduced by Morita in [3]. The most important filters to be considered are Cauchy filters in $(X,\mathcal{U})$, *i.e.*, filters in X which intersect every cover of $\mathcal{U}$. In many important examples, for a Cauchy filter η in $(X,\mathcal{U})$ to converge it is necessary and sufficient that η has an adherence point, *i.e.*, a point which belongs to the closure of every member of η.

We consider an increasing chain of four important subclasses of the class of Cauchy filters: strongly round filters, round filters, weakly round filters and minimal filters. Every Cauchy filter in a semi–uniform space contains an strongly round filter and hence, a minimal filter. However, this is not true in more general pre–uniform spaces.

We give all the necessary definitions in next section.

As in the case of uniform or semi–uniform spaces, we say that a pre–uniform space $(X,\mathcal{U})$ is complete if every Cauchy filter in $(X,\mathcal{U})$ converges. We have four less restrictive types of completeness if we require only that the filters in one of the four classes defined above are convergent.

If $(X,\mathcal{U})$ is a pre–uniform space and $A \subseteq X$, the family of cover restrictions $\mathcal{U}_A = \{\alpha|_A : \alpha \in \mathcal{U}\}$ determines a pre–uniform space $(A,\mathcal{U}_A)$ and we say then that $(A,\mathcal{U}_A)$ is a subspace of $(X,\mathcal{U})$ or that $(X,\mathcal{U})$ is an extension of $(A,\mathcal{U}_A)$. It is easy to see that $\mathcal{T}_{\mathcal{U}}|_A = \mathcal{T}_{\mathcal{U}_A}$ and this justifies the use of the words "subspace" and "extension".

In this paper we prove that every T_0–pre–uniform space $(X,\mathcal{U})$ has a canonical T_1 extension $(\widehat{X},\widehat{\mathcal{U}})$, where every weakly round filter converges. We also prove that a necessary and sufficient condition for $(\widehat{X},\widehat{\mathcal{U}})$ to be complete is that every Cauchy filter in $(X,\mathcal{U})$ contains a weakly round filter. As a corollary of this, we deduce the known result that every semi–uniform space is completable. These are the main results of this paper.

2. Pre–uniform spaces and uniform continuity

All the definitions and notation agreements of this paper appear in the doctoral thesis of the second author. For convenience to the reader, we shall include the most important ones.

Note 1. *If* X *is a set,* $F \subseteq X$*,* $p \in X$ *and* α *is a cover of* X*, then:*

$$S_T(p,\alpha) = S_T(\{p\},\alpha) = \cup\{L \in \alpha : p \in L\}$$
$$S_T(F,\alpha) = \cup\{L \in \alpha : L \cap F \neq \varnothing\}.$$

Definition 2.1. *Let* $\mathcal{U}$ *be a non–empty family of covers of a set* X*. The* <u>*topology*</u> $\mathcal{T}_{\mathcal{U}}$ *in* X <u>*induced*</u> *by* $\mathcal{U}$ *is defined as follows:*

$L \subseteq X$ *belongs to* $\mathcal{T}_{\mathcal{U}}$ *iff for every* $x \in L$*, we may find a finite collection* $\{\alpha_1,\alpha_2,\dots,\alpha_n\} \subseteq \mathcal{U}$ *such that* $\bigcap_{i=1}^{n} S_T(x,\alpha_i) \subseteq L$.

Note 2. *Suppose* $\alpha_1,\alpha_2,\dots,\alpha_n,\alpha,\beta$ *are covers of a set* X*. We denote:*

$$\alpha_1 \wedge \alpha \wedge \cdots \wedge \alpha_n = \{L_1 \cap L_2 \cap \cdots \cap L_n : L_1 \in \alpha_1, L_2 \in \alpha_2, \cdots, L_n \in \alpha_n\}.$$

$\alpha \leq \beta$ *means that* α <u>*refines*</u> β *i.e., there exists a map* $\lambda : \alpha \to \beta$ *such that* $A \subseteq \lambda(A)$ *for every* $A \in \alpha$*. Clearly* $\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n \leq \alpha_i$ *for every* $i \in \{1,2,\dots,n\}$.

If $\mathcal{U}_1$, $\mathcal{U}_2$ are collections of covers of a set X, we write $\mathcal{U}_1 \leq \mathcal{U}_2$ if for every $\alpha \in \mathcal{U}_1$, there exists a finite collection $\{\beta_1,\beta_2,\dots,\beta_n\} \subseteq \mathcal{U}_2$ such that $\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_n \leq \alpha$. $\mathcal{U}_1$ and $\mathcal{U}_2$ are <u>equivalent</u> if $\mathcal{U}_1 \leq \mathcal{U}_2$ and $\mathcal{U}_2 \leq \mathcal{U}_1$.

Clearly $\mathcal{U}_1 \subseteq \mathcal{U}_2 \Rightarrow \mathcal{U}_1 \leq \mathcal{U}_2 \Rightarrow \overline{\mathcal{T}_{\mathcal{U}_1} \subseteq \mathcal{T}_{\mathcal{U}_2}}$.

If $\mathcal{U}$ is a collection of covers a set X, $\mathcal{U}^+$ denotes the family of covers γ of X such that for some $\alpha \in \mathcal{U}$, we have $\alpha \leq \gamma$. Clearly, the families $\mathcal{U}$ and $\mathcal{U}^+$ are equivalent and hence they generate the same topology on X.

Definition 2.2. *A non-empty collection $\mathcal{U}$ of covers of a set* X *is a* <u>*pre–uniformity basis*</u> *on* X *if* $\mathcal{U}$ *satisfies the following two conditions:*

1) *Whenever* $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathcal{U}$ *there exists a cover* $\beta \in \mathcal{U}$ *such that* $\beta \leq \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n$.
2) *For each* $\alpha \in \mathcal{U}$, *there exists a cover* $\beta \in \mathcal{U}$ *such that* $\beta \leq \overset{\circ}{\alpha}$, *where* $\overset{\circ}{\alpha} = \{\operatorname{int}_{\mathcal{T}_{\mathcal{U}}} \mathrm{L} : \mathrm{L} \in \alpha\}$.

Hence, every pre–uniformity basis $\mathcal{U}$ is equivalent to a pre–uniformity basis $\mathcal{U}'$ where each cover $\alpha \in \mathcal{U}'$ is open with respect to the topology $\mathcal{T}_{\mathcal{U}}$.

Definition 2.3. *A* <u>*pre–uniform space*</u> *is a pair* $(\mathrm{X}, \mathcal{U})$, *where* X *is a set and* $\mathcal{U}$ *is a pre–uniformity basis on* X.

Remark 2.4. If $(\mathrm{X}, \mathcal{U})$ is a pre–uniform space and $\mathrm{A} \subseteq \mathrm{X}$, then $\mathcal{U}|_{\mathrm{A}} = \{\alpha|_{\mathrm{A}} : \alpha \in \mathcal{U}\}$ is a pre–uniformity basis on A and the topologies $\mathcal{T}_{\mathcal{U}}|_{\mathrm{A}}$ and $\mathcal{T}_{\mathcal{U}|_{\mathrm{A}}}$ coincide. We say then that $(\mathrm{A}, \mathcal{U}|_{\mathrm{A}})$ is a pre–uniform subspace of $(\mathrm{X}, \mathcal{U})$.

Remark 2.5. Let $(\mathrm{X}, \mathcal{U})$ be a pre–uniform space. Then, for every $\mathrm{A} \subseteq \mathrm{X}$, we have:

$$\operatorname{int}_{\mathcal{T}_{\mathcal{U}}} \mathrm{A} = \{x \in \mathrm{A} : \text{ there exists } \alpha_x \in \mathcal{U} \text{ such that } \mathrm{S_T}(x, \alpha_x) \subseteq \mathrm{A}\}. \tag{2.1}$$

Definition 2.6.

1) *A map* $f : (\mathrm{X}, \mathcal{U}) \to (\mathrm{Y}, \mathcal{V})$ *between pre–uniform spaces is* <u>*uniformly continuous*</u> *if for every* $\beta \in \mathcal{V}$, *we may find a cover* $\alpha \in \mathcal{U}$ *such that* $\alpha \leq \{f^{-1}(\mathrm{B}) : \mathrm{B} \in \beta\}$.
2) *A bijection* $\varphi : (\mathrm{X}, \mathcal{U}) \to (\mathrm{Y}, \mathcal{V})$ *between pre–uniform spaces is said to be a* <u>*unimorphism*</u> *if* φ *and* φ^{-1} *are both uniformly continuous maps and we say, in this case, that* $(\mathrm{X}, \mathcal{U})$ *and* $(\mathrm{Y}, \mathcal{V})$ *are* <u>*unimorphic spaces*</u>.
3) *A map* $g : (\mathrm{X}, \mathcal{U}) \to (\mathrm{Y}, \mathcal{V})$ *from the pre–uniform space* $(\mathrm{X}, \mathcal{U})$ *into the pre–uniform space* $(\mathrm{Y}, \mathcal{V})$ *is a* <u>*unimorphic embedding*</u> *if* $g(\mathrm{X})$ *is dense in* Y *and* g *is a unimorphism from* $(\mathrm{X}, \mathcal{U})$ *onto* $\left(g(\mathrm{X}), \mathcal{V}|_{g(\mathrm{X})}\right)$.

Remark 2.7.

1) If $f : (\mathrm{X}, \mathcal{U}) \to (\mathrm{Y}, \mathcal{V})$ is uniformly continuous, then $f : (\mathrm{X}, \mathcal{T}_{\mathcal{U}}) \to (\mathrm{Y}, \mathcal{T}_{\mathcal{V}})$ is continuous.
2) Si $\mathcal{U}$, $\mathcal{V}$ are pre–uniformity bases in the same set X, then the identity map $id : (\mathrm{X}, \mathcal{U}) \to (\mathrm{X}, \mathcal{V})$ is uniformly continuous iff $\mathcal{V} \leq \mathcal{U}$. Therefore, id is a unimorphism iff $\mathcal{U}$ and $\mathcal{V}$ are equivalent.

Definition 2.8 (See [2], Definition 1.2.2, page 10)**.**

1) *A non–empty family* $\mathcal{U}$ *of covers of a set* X *is a* <u>*semi–uniformity basis*</u> *on* X *if* $\mathcal{U}$ *satisfies 2.2.1 and:*

   SU) *For every* $\alpha \in \mathcal{U}$*, there exists a cover* $\beta \in \mathcal{U}$ *such that for every* $\mathrm{B} \in \beta$*, we may find a cover* $\gamma_{\mathrm{B}} \in \mathcal{U}$ *and a set* $\mathrm{A_B} \in \alpha$ *such that* $\mathrm{S_T}(\mathrm{B}, \gamma_{\mathrm{B}}) \subseteq \mathrm{A_B}$.

2) *A* <u>*semi–uniform space*</u> *is a pair* $(\mathrm{X}, \mathcal{U})$*, where* X *is a set and* $\mathcal{U}$ *is a semi–uniformity basis on* X.

3) $\mathcal{U}$ *is a* <u>*uniformity basis*</u> *on* X *if* $\mathcal{U}$ *satisfies condition 2.2.1 and the stronger condition:*

   U) *For every* $\alpha \in \mathcal{U}$*, there exists a cover* $\beta \in \mathcal{U}$ *such that:*

$$\{\mathrm{S_T}(\mathrm{B}, \beta) : \mathrm{B} \in \beta\} \leq \alpha.$$

The following facts are well known:

Theorem 2.9.

1) *Let* $\mathcal{U}$ *be a semi–uniformity basis on a set* X*. Then* $\mathcal{T}_{\mathcal{U}}$ *is a a regular topology on* X*. Conversely, if* $(\mathrm{X}, \mathcal{T})$ *is a regular topological space, there exists a semi–uniformity basis* $\mathcal{U}$ *on* X *such that* $\mathcal{T} = \mathcal{T}_{\mathcal{U}}$.

2) *Let* $\mathcal{U}$ *be a uniformity basis on a set* X*. Then* $\mathcal{T}_{\mathcal{U}}$ *is a a completely regular topology on* X*. Conversely, if* $(\mathrm{X}, \mathcal{T})$ *is a completely regular topological space, there exists a uniformity basis* $\mathcal{U}$ *on* X *such that* $\mathcal{T} = \mathcal{T}_{\mathcal{U}}$.

3. Filters and completeness

We recall now some definitions about filters.

Let X be a set. A <u>filter</u> in X is a non–empty subfamily $\mathcal{F}$ of the power set $\mathcal{P}(\mathrm{X})$ which satisfies the following properties:

i) $\varnothing \notin \mathcal{F}$.
ii) $\mathrm{F_1}, \mathrm{F_2} \in \mathcal{F} \Rightarrow \mathrm{F_1} \cap \mathrm{F_2} \in \mathcal{F}$.
iii) $\mathrm{F} \in \mathcal{F}$, $\mathrm{F} \subseteq \mathrm{L} \subseteq \mathrm{X} \Rightarrow \mathrm{L} \in \mathcal{F}$.

A <u>filter base</u> in X is a non–empty subfamily η of $\mathcal{P}(\mathrm{X})$ satisfying the properties:

i) $\varnothing \notin \eta$.
ii) $\mathrm{N_1}, \mathrm{N_2} \in \eta \Rightarrow \exists\, \mathrm{N_3} \in \eta$ such that $\mathrm{N_3} \subseteq \mathrm{N_1} \cap \mathrm{N_2}$.

For any subfamily $\mathcal{G} \subseteq \mathcal{P}(\mathrm{X})$, we denote:

$$\mathcal{G}^+ = \{\mathrm{A} \in \mathcal{P}(\mathrm{X}) : \mathrm{G} \subseteq \mathrm{A} \text{ for some } \mathrm{G} \in \mathcal{G}\}.$$

If $\mathcal{G}_1, \mathcal{G}_2 \subseteq \mathcal{P}(\mathrm{X})$, $\mathcal{G}_1 \leq \mathcal{G}_2$ means that $\mathcal{G}_2^+ \subseteq \mathcal{G}_1^+$, *i.e.*, $\mathcal{G}_1 \leq \mathcal{G}_2$ iff every element of $\mathcal{G}_2$ contains an element of $\mathcal{G}_1$. $\mathcal{G}_1 \sim \mathcal{G}_2$ ($\mathcal{G}_1$ is <u>equivalent</u> to $\mathcal{G}_2$) means that $\mathcal{G}_1 \leq \mathcal{G}_2$ and $\mathcal{G}_2 \leq \mathcal{G}_1$, *i.e.*, $\mathcal{G}_1 \sim \mathcal{G}_2$ iff $\mathcal{G}_1^+ = \mathcal{G}_2^+$. This is clearly an equivalence relation in $\mathcal{P}(\mathcal{P}(\mathrm{X}))$. If we restrict ourselves to filter basis in X, $\sim$ is still an equivalence relation. It is easy to prove that every equivalence class contains exactly one filter, namely, if η is a filter base in X, η^+ is the only filter in X which satisfies the relation $\eta \sim \eta^+$.

Let $(\mathrm{X}, \mathcal{T})$ be a topological space and let η be a filter in X. A point $x \in \mathrm{X}$ is an adherence point of η (in symbols, $\eta \mapsto x$) if every neighborhood of x intersects every element of η. Equivalently, $\eta \mapsto x$ iff $x \in C\ell\,(\mathrm{N})$ (= the $\mathcal{T}$–closure of N) for every $\mathrm{N} \in \eta$. x is a convergence point of η (in symbols, $\eta \to x$) if every neighborhood of x belongs to η.

A filter η in a set X is an ultrafilter in X if η is not properly contained in any other filter in X. Two filters η_1, η_2 in X mingle (in symbols, $\eta_1 \leftrightarrow \eta_2$) if every element of η_1 intersects every element of η_2.

Remark 3.1. Two filters η_1, η_2 in X mingle iff there exists a filter η in X such that $\eta_1 \cup \eta_2 \subseteq \eta$.

A filter η in a topological space $(\mathrm{X}, \mathcal{T})$ is balanced if every adherence point of η is also a convergence point of η. For any $x \in \mathrm{X}$, η_x denotes the filter of $\mathcal{T}$–neighborhoods of x.

Remark 3.2. Let η_1, η_2 be filters in a topological space $(\mathrm{X}, \mathcal{T})$ and let $x \in \mathrm{X}$. Then:

1) $\eta_1 \to x$ implies $\eta_1 \mapsto x$.
2) $\eta_1 \to x$ and $\eta_1 \leftrightarrow \eta_2$ imply that $\eta_2 \mapsto x$.
3) $\eta_1 \mapsto x$ iff $\eta_1 \leftrightarrow \eta_x$.
4) $\eta_1 \to x$ iff $\eta_1 \supseteq \eta_x$.
5) $\eta_1 \leq \eta_2$ and $\eta_1 \mapsto x$ imply that $\eta_2 \mapsto x$.
6) $\eta_1 \leq \eta_2$ and $\eta_2 \to x$ imply that $\eta_1 \to x$.
7) Every ultrafilter in X is balanced.
8) Every filter in X without adherence points is balanced.
9) Every convergent filter in a Hausdorff space is balanced.

Definition 3.3. *A filter η in a pre–uniform space $(\mathrm{X}, \mathcal{U})$ is $\mathcal{U}$–Cauchy (or Cauchy in $(\mathrm{X}, \mathcal{U})$) if for every $\alpha \in \mathcal{U}$, we have $\eta \cap \alpha \neq \varnothing$.*

Remark 3.4. If $\varphi : (\mathrm{X}, \mathcal{U}) \to (\mathrm{Y}, \mathcal{V})$ is a uniformly continuous map and if $\mathcal{F}$ is a $\mathcal{U}$–Cauchy filter, then $\varphi(\mathcal{F})^+$ is a $\mathcal{V}$–Cauchy filter. Hence, if $\mathcal{U}$ and $\mathcal{V}$ are equivalent pre–uniform bases on X, then $(\mathrm{X}, \mathcal{U})$ and $(\mathrm{X}, \mathcal{V})$ have the same Cauchy filters.

Definition 3.5. *For every Cauchy filter $\mathcal{F}$ in a pre–uniform space $(\mathrm{X}, \mathcal{U})$, define:*

$$\begin{aligned} \mathcal{F}' &= \{\mathrm{S_T}(\mathrm{F}, \alpha) : \mathrm{F} \in \mathcal{F},\ \alpha \in \mathcal{U}\}^+ \\ \mathcal{F}^r &= \left\{\mathrm{S^*_T}(\mathcal{F}, \alpha) : \alpha \in \mathcal{U}\right\}^+ \\ \mathcal{F}^{rr} &= \left\{\mathrm{S^{**}_T}(\mathcal{F}, \alpha) : \alpha \in \mathcal{U}\right\}^+, \end{aligned}$$

where

$$\mathrm{S^*_T}(\mathcal{F}, \alpha) = \cup\{\mathrm{A} \in \alpha : \mathrm{A} \cap \mathrm{F} \neq \varnothing \textit{ for every } \mathrm{F} \in \mathcal{F}\}$$

and

$$\mathrm{S}_{\mathrm{T}}^{**}(\mathcal{F},\alpha)=\cup\{\mathrm{A}:\mathrm{A}\in\mathcal{F}\cap\alpha\}.$$

Remark 3.6. If $\mathcal{F}$ is a Cauchy filter in $(\mathrm{X},\mathcal{F})$, then $\mathcal{F}'$, $\mathcal{F}^r$, $\mathcal{F}^{rr}$ are also filters in X and we have:

$$\mathcal{F}'\subseteq\mathcal{F}^r\subseteq\mathcal{F}^{rr}\subseteq\mathcal{F}.$$

Definition 3.7. *Let $\mathcal{F}$ be a Cauchy filter in a pre–uniform space $(\mathrm{X},\mathcal{U})$.*

1) *$\mathcal{F}$ es minimal if $\mathcal{F}$ does not properly contain any other Cauchy filter in $(\mathrm{X},\mathcal{U})$.*
2) *$\mathcal{F}$ is weakly round if $\mathcal{F}=\mathcal{F}^{rr}$.*
3) *$\mathcal{F}$ is round if $\mathcal{F}=\mathcal{F}^{r}$.*
4) *$\mathcal{F}$ is strongly round if $\mathcal{F}=\mathcal{F}'$.*

We summarize in a Theorem the most important relations among these different kinds of filters. The proofs can be found in [2].

Theorem 3.8.

1) *Every strongly round filter is round.*
2) *Every round filter is weakly round.*
3) *Every weakly round filter is minimal.*
4) *Two round filters $\mathcal{F}_1$, $\mathcal{F}_2$ in a pre–uniform space mingle iff $\mathcal{F}_1=\mathcal{F}_2$.*
5) *Every neighborhood filter is weakly round.*
6) *If $(\mathrm{X},\mathcal{U})$ is a semi–uniform space and if $\mathcal{F}$ is a Cauchy filter in $(\mathrm{X},\mathcal{U})$, then $\mathcal{F}'$ is also a Cauchy filter in $(\mathrm{X},\mathcal{U})$ and $\mathcal{F}'$ is contained in any Cauchy filter $\mathcal{G}$ contained in $\mathcal{F}$. In fact, $\mathcal{F}'$ is a strongly round filter in $(\mathrm{X},\mathcal{U})$ and $\mathcal{F}''=\mathcal{F}'$.*

Definition 3.9. *A pre–uniform space $(\mathrm{X},\mathcal{U})$ is complete if every Cauchy filter in $(\mathrm{X},\mathcal{U})$ converges.*

Lemma 3.10. *Let $\varphi:(\mathrm{X},\mathcal{U})\to(\mathrm{Y},\mathcal{V})$ be a unimorphism between pre–uniform spaces and let η be a weakly round filter in* X. *Then $\varphi(\eta)=\{\varphi(\mathrm{N}):\mathrm{N}\in\eta\}$ is a weakly round filter in $(\mathrm{Y},\mathcal{V})$.*

Proof. Fix an element $\mathrm{N}\in\eta$ and let $\alpha\in\mathcal{U}$ be such that:

$$\mathrm{H}(\alpha)=\cup\{\mathrm{L}:\mathrm{L}\in\eta\cap\alpha\}\subseteq\mathrm{N}.$$

Let $\beta\in\mathcal{V}$ be such that $\beta\leq\varphi(\alpha)=\{\varphi(\mathrm{A}):\mathrm{A}\in\alpha\}$. We shall prove that $\mathrm{K}(\beta)=\cup\{\mathrm{B}:\mathrm{B}\in\varphi(\eta)\cap\beta\}\subseteq\varphi(\mathrm{N})$. For each $\mathrm{B}\in\beta$ select an element $\mathrm{A}_{\mathrm{B}}\in\alpha$ such that $\mathrm{B}\subseteq\varphi(\mathrm{A}_{\mathrm{B}})$. Therefore, if also $\mathrm{B}\in\varphi(\eta)$, we have $\varphi(\mathrm{A}_{\mathrm{B}})\in\varphi(\eta)$ and $\mathrm{A}_{\mathrm{B}}\in\eta$ (recall φ is a bijective map). Therefore,

$$\mathrm{K}(\beta)\subseteq\cup\{\varphi(\mathrm{A}):\mathrm{A}\in\eta\cap\alpha\}=\varphi(\mathrm{H}(\alpha))\subseteq\varphi(\mathrm{N}).$$

□

Lemma 3.11. *Let η be a weakly round filter in a pre–uniform space* $(\mathrm{X},\mathcal{U})$ *and let* $\mathrm{A} \subseteq \mathrm{X}$*. Then* $\eta|_{\mathrm{A}}$ *is a weakly round filter in* $(\mathrm{A},\mathcal{U}_{\mathrm{A}})$*.*

Proof. Obvious. □

Definition 3.12. *A complete pre–uniform space* $(\mathrm{Y},\mathcal{V})$ *is a* <u>*completion*</u> *of a pre–uniform space* $(\mathrm{X},\mathcal{U})$ *if there exists a unimorphic embedding* $\varphi : (\mathrm{X},\mathcal{U}) \to (\mathrm{Y},\mathcal{V})$*.*

Definition 3.13. *A pre–uniform space* $(\mathrm{X},\mathcal{U})$ *is* <u>*proper*</u> *if every Cauchy filter in* $(\mathrm{X},\mathcal{U})$ *contains a weakly round filter.*

Note 3. *Let* $(\mathrm{X},\mathcal{U}_1)$ *be a* T_0*–pre–uniform space and choose an open pre–uniformity basis* $\mathcal{U}$ *equivalent to* $\mathcal{U}_1$*. Let* $\widehat{\mathrm{X}} = \{\xi : \ \xi \text{ is a weakly round filter in } (\mathrm{X},\mathcal{U})\}$*. For every* $\mathrm{A} \subseteq \mathrm{X}$ *and every* $\alpha \in \mathcal{U}$*, define:*

$$\widehat{\mathrm{A}} = \left\{\xi \in \widehat{\mathrm{X}} : \mathrm{A} \in \xi\right\}$$

$$\widehat{\alpha} = \left\{\widehat{\mathrm{A}} : \mathrm{A} \in \alpha\right\}$$

and let $\widehat{\mathcal{U}} = \{\widehat{\alpha} : \alpha \in \mathcal{U}\}$*. For every* $x \in \mathrm{X}$*, let* $\varphi(x) = \eta_x =$ *filter of* $\mathcal{T}_{\mathcal{U}}$*–neighborhoods of* x*.*

Theorem 3.14. *Keep the notation of 3 and suppose the topology* $\mathcal{T}_{\mathcal{U}}$ *is* T_0*. Then* $\widehat{\mathcal{U}}$ *is an open pre–uniformity basis in* $\widehat{\mathrm{X}}$ *and* φ *is a unimorphic embedding of* $(\mathrm{X},\mathcal{U})$ *into* $\left(\widehat{\mathrm{X}},\widehat{\mathcal{U}}\right)$*. Besides, every weakly round filter in* $\left(\widehat{\mathrm{X}},\widehat{\mathcal{U}}\right)$ *is convergent.*

Proof. Everything is proved in [2], except the last part. Let $\mathcal{F}$ be a weakly round filter in $\left(\widehat{\mathrm{X}},\widehat{\mathcal{U}}\right)$. Define:

$$\eta = \left\{\mathrm{A} \in \cup\{\alpha : \alpha \in \mathcal{U}\} : \widehat{\mathrm{A}} \in \mathcal{F}\right\}^{+}.$$

It is easy to prove that η is a Cauchy filter in $(\mathrm{X},\mathcal{U})$. We shall prove that η is weakly round. Choose an element $\mathrm{L} \in \eta$. By the definition of η, there exists a cover $\alpha \in \mathcal{U}$ and an element $\mathrm{A} \in \alpha$ such that $\mathrm{A} \subseteq \mathrm{L}$ and $\widehat{\mathrm{A}} \in \mathcal{F}$. Since $\mathcal{F}$ is weakly round in $\left(\widehat{\mathrm{X}},\widehat{\mathcal{U}}\right)$, there exists a cover $\beta \in \mathcal{U}$ such that $\cup\left\{\widehat{\mathrm{B}} : \widehat{\mathrm{B}} \in \mathcal{F} \cap \widehat{\beta}\right\} \subseteq \widehat{\mathrm{A}}$. Take an element $\mathrm{B} \in \eta \cap \beta$. Therefore, $\widehat{\mathrm{B}} \in \mathcal{F} \cap \widehat{\beta}$ and $\widehat{\mathrm{B}} \subseteq \widehat{\mathrm{A}}$. Conversely, $\mathrm{B} = \varphi^{-1}\left(\widehat{\mathrm{B}}\right) \subseteq \varphi^{-1}\left(\widehat{\mathrm{A}}\right) = \mathrm{A} \subseteq \mathrm{L}$. Therefore, η is weakly round in $(\mathrm{X},\mathcal{U})$ and $\eta \in \widehat{\mathrm{X}}$. It remains to prove that $\mathcal{F} \to \eta$. Take a set $\mathrm{W} \in \mathcal{T}_{\widehat{\mathcal{U}}}$ containing η. Since $\cup\{\widehat{\alpha} : \alpha \in \mathcal{U}\}$ is a basis for the topology $\mathcal{T}_{\widehat{\mathcal{U}}}$, we may assume that W coincides with $\widehat{\mathrm{A}}$, where $\mathrm{A} \in \alpha$ for some $\alpha \in \mathcal{U}$. This implies that $\mathrm{A} \in \eta$ and, therefore, $\widehat{\mathrm{A}} \in \mathcal{F}$. Hence, the filter of $\mathcal{T}_{\widehat{\mathcal{U}}}$–neighborhoods of η is contained in $\mathcal{F}$ and $\mathcal{F} \to \eta$. □

Theorem 3.15. *Let* $(\mathrm{X},\mathcal{U})$ *be a* T_0*–pre–uniform space. Then* $(\mathrm{X},\mathcal{U})$ *admits at least one completion iff* $(\mathrm{X},\mathcal{U})$ *is proper.*

Proof.

($\Rightarrow$) Let $(\mathrm{Y},\mathcal{V})$ be a completion of $(\mathrm{X},\mathcal{U})$ and let $\varphi : (\mathrm{X},\mathcal{U}) \to (\mathrm{Y},\mathcal{V})$ be a unimorphic embedding. Let η be a Cauchy filter in $(\mathrm{X},\mathcal{U})$. Then $\mathcal{F} = \varphi(\eta)^+$ is a Cauchy filter in $(\mathrm{Y},\mathcal{V})$ and hence, there exists an element $y \in \mathrm{Y}$ such that $\mathcal{F} \to y$. Let $\mathcal{F}_y$ be the filter of $\mathcal{T}_{\mathcal{V}}$–neighborhoods of y. By 3.8.5, $\mathcal{F}_y$ is a weakly round filter in $(\mathrm{Y},\mathcal{V})$. Besides, $\mathcal{F}_y \subseteq \mathcal{F}$ because $\mathcal{F} \to y$. Using lemmas 3.10 and 3.11, we deduce that $\eta_0 = \varphi^{-1}\left(\mathcal{F}_y|_{\varphi(\mathrm{X})}\right)$ is a weakly round filter in $(\mathrm{X},\mathcal{U})$ contained in η.

($\Leftarrow$) We shall prove that $\left(\widehat{\mathrm{X}},\widehat{\mathcal{U}}\right)$ is a completion of $(\mathrm{X},\mathcal{U})$. Let $\mathcal{F}$ be a Cauchy filter in $\left(\widehat{\mathrm{X}},\widehat{\mathcal{U}}\right)$ and let:

$$\eta = \left\{ \mathrm{A} \in \bigcup_{\alpha\in\mathcal{U}} \alpha : \widehat{\mathrm{A}} \in \mathcal{F} \right\}^+ .$$

It is easy to prove that η is a Cauchy filter in $(\mathrm{X},\mathcal{U})$. Since $(\mathrm{X},\mathcal{U})$ is proper, there exists a weakly round filter η_0 in $(\mathrm{X},\mathcal{U})$ contained in η. Let us prove that $\mathcal{F} \to \eta_0$. Let $\alpha \in \mathcal{U}$ and $\mathrm{A} \in \alpha$ be such that $\eta_0 \in \widehat{\mathrm{A}}$. Therefore, $\mathrm{A} \in \eta_0 \subseteq \eta$ and $\widehat{\mathrm{A}} \in \mathcal{F}$. Hence, every $\mathcal{T}_{\widehat{\mathcal{U}}}$–neighborhood of η_0 belongs to $\mathcal{F}$, *i.e.*, $\mathcal{F} \to \eta_0$.

□

Corollary 3.16. *Let* $(\mathrm{X},\mathcal{U})$ *be a* T_0*–pre–uniform space where every Cauchy filter in* $(\mathrm{X},\mathcal{U})$ *contains a round filter. Then* $\left(\widehat{\mathrm{X}},\widehat{\mathcal{U}}\right)$ *is a Hausdorff completion of* $(\mathrm{X},\mathcal{U})$ *and the only round filters in* $\left(\widehat{\mathrm{X}},\widehat{\mathcal{U}}\right)$ *are the neighborhood filters.*

Proof. See [2]. □

Example 3.17. Let $(\mathrm{X},\mathcal{U})$ be a Hausdorff topological space and let $\mathcal{U}$ be the family of open covers α of X such that some finite subfamily $\lambda \subseteq \alpha$ covers a dense subset of X. Then $\mathcal{U}$ is an open pre–uniformity basis on X satisfying the condition of 3.16. In this case, $\mathcal{T} = \mathcal{T}_{\mathcal{U}}$ and the Hausdorff completion $\left(\widehat{\mathrm{X}},\widehat{\mathcal{U}}\right)$ is a Hausdorff closed extension of $(\mathrm{X},\mathcal{U})$.

Proof. See [2] and [5]. □

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