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CL($\mathbb{R}$) is simply connected under the Vietoris topology

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Abstract. In this paper we present a proof by construction that the hyperspace $\mathbf{CL}(\mathbb{R})$ of closed, nonemtpy subsets of $\mathbb{R}$ is simply connected under the Vietoris topology. This is useful in considering the convergence of time scales. We also present a construction of the (known) fact that this hyperspace is also path connected, as part of the proof.

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1. Introduction

Spaces of all non-empty and closed subsets of a topological space (or *hyperspaces*) are a critical part of the study of time scales. The theory of time scales attempts to organize the solution methods for differential and difference equations which, when considered under the same equation, sometimes have very similar solutions and sometimes have wildly different solutions. The approach is to consider a dynamic equation over an unknown domain, which is a non-empty and closed subset of $\mathbb{R}$, or in other words, a point in $\mathbf{CL}(\mathbb{R})$. Such points of $\mathbf{CL}(\mathbb{R})$ are called *time scales*. For a good introduction to the theory of time scales, see [2]. When studying time scales in this context, there are immediate and interesting questions involving convergence: If a sequence of time scales converges, and we consider solutions of the same dynamic equation over each member of the sequence, will the solutions converge? Of course, a formalized concept for convergence of functions over different domains is needed (in addition to a formalized concept of "sameness"). Results for some of these questions have been given in [12], when the solutions are unique and the dynamic equation is sufficiently continuous.

Central to this discussion is the topology on the space of time scales. There are several well known topologies on hyperspaces, including the Hausdorff metric topology and the Vietoris topology. The Hausdorff metric topology of $\mathbf{CL}(\mathbb{R})$ is, happily, metrizable, but it has the unfortunate property that under it, $[-n, n]$ does not converge to $\mathbb{R}$. Since this convergence would be useful in the context of time scales, we turn instead to the Vietoris topology. This topology is not metrizable on $\mathbf{CL}(\mathbb{R})$; however, on hyperspaces associated to compact metrizable spaces it coincides with the Hausdorff metric topology.

In 1951, it was shown by Michael that $\mathbf{CL}(\mathbb{R})$ was completely regular, seperable, and first countable; see [14]. The statement that it is locally compact turned out to contain an error – a correction to the problematic proposition can be found in [5] – however it is now known that $\mathbf{CL}(\mathbb{R})$ is not locally compact. A result of Ivanova, Keesling and Velichko says that if the Vietoris topology on $\mathbf{CL}(X)$ is normal, then X is compact: see [10], [11], and [16]. It follows that $\mathbf{CL}(\mathbb{R})$ is not a normal space. In 2003, Hola, Pelant and Zsilinszky showed that $\mathbf{CL}(\mathbb{R})$ is not developable and that it is submetrizable; see [8]. It is also known to be strong alpha-favorable; this follows from statements in [19].

More attention has been paid to hyperspaces in the case that X is compact. It was shown as far back as 1931 by Borsuk and Mazurkiewicz that for a metrizable continuum X, both the hyperspace $\mathbf{K}(X)$ of compact subsets of X and $\mathbf{C}(X)$, the hyperspace of subcontinua of X, are path connected [4]. The non-metrizable case was investigated by McWaters [13] and Ward [17]. Local path connectedness of $\mathbf{K}(X)$ and $\mathbf{C}(X)$ was shown to be equivalent to local connectedness of X if X is compact in [18] in 1939. For topological properties on compact Vietoris hyperspaces, see the 1978 book of Nadler [15], or the more recent book by Illanes and Nadler, [9].

In 2002, in [6], Costantini and Kubis showed that under the Vietoris topology, $\mathbf{CL}(\mathbb{R})$ is pathwise connected but not locally connected. They actually showed a stronger statement, applying to a wider class of topologies, and giving conditions for path-wise connectedness. They also give several results for the hyperspace of closed, bounded sets under the Hausdorff metric topolgoy, including that it is an absolute retract.

For the reader who is not familiar with it, in Section 2 we will briefly discuss the Vietoris topology on a general topological space X. Then in Section 4 we shall prove:

Theorem 1.1. *Under the Vietoris topology, $\boldsymbol{CL}(\mathbb{R})$ is simply connected.*

For the purposes of the proof, we will also present an alternate proof that $\mathbf{CL}(\mathbb{R})$ is path connected, by constructing an explicit path from any point in $\mathbf{CL}(\mathbb{R})$ to $\mathbb{R}$; this will be done in Section 3. This will assist in the construction of a nullhomotopy of an arbitrary loop in Section 4.

2. The Vietoris topology

Suppose that you have a topological space (X, τ). The Vietoris topology is one of a group of topologies called "hit and miss" topologies. The name is indicative of the fact that open sets in the space $\mathbf{CL}(X)$ are given by those subsets of X which "hit" certain specific open sets of X and "miss" their complements. For a full discussion of hit and miss topologies, see [1].

Let $U_1, \ldots, U_n$ be a finite collection of open sets in X, i.e. members of τ. We define an open set in $\mathbf{CL}(X)$, denoted $B =< U_1, \ldots, U_n >$, to be all those non-empty and closed subsets A of X satisfying the following two properties:

(1) $A \cap U_i \neq \varnothing$, for $i = 1, \ldots, n$. ("hit")
(2) $A \subset \bigcup_{i=1}^{n} U_i$ ("miss")

The collection of all such sets, for any finite collection of U_i, forms a basis for the Vietoris topology on $\mathbf{CL}(X)$. When $X = \mathbb{R}$, it is not hard to see that under this topology, the sequence of time scales $\mathbb{T}_n = [-n, n]$ does in fact converge to $\mathbb{R}$.

An alternative way of looking at the Vietoris topology is to use the fact that it is the supremum of the upper and lower Vietoris topologies, the first of which is generated by all sets of the form $U^+ = \{A \in \mathbf{CL}(X) : A \subset U\}$, and the second of which is generated by sets of the form $U^- = \{A \in \mathbf{CL}(X) : A \cap U \neq \varnothing\}$, where U is a τ-open set. Subbase elements of the Vietoris topology are of the form U^+ with $U \in \tau$ and $\bigcap_{U \in \mathcal{U}} U^-$, with $\mathcal{U} \subset \tau$ finite.

3. Path connected

In the following, we will consider $\mathbf{CL}(\mathbb{R})$ endowed with the Vietoris topology.

Theorem 3.1. ***CL***(ℝ) *is path connected.*

Proof. Let $\mathbb{T} \in \mathbf{CL}(\mathbb{R})$ be an arbitrary point of the hyperspace. In the future, to distinguish between points of $\mathbf{CL}(\mathbb{R})$ and points of $\mathbb{R}$, we will refer to the former as time scales. We construct a path from $\mathbb{T}$ to $\mathbb{R}$. As $\mathbb{T} \neq \varnothing$, choose $t_0 \in \mathbb{T}$.

Define $\gamma : [0,1] \to \mathbf{CL}(\mathbb{R})$ by $\gamma(1) = \mathbb{R}$, and for $s \in [0,1)$,

$$\gamma(s) = \mathbb{T} \cup [t_0 - \frac{s}{1-s}, t_0 + \frac{s}{1-s}]$$

We denote by $A(s)$ the closed interval $[t_0 - \frac{s}{1-s}, t_0 + \frac{s}{1-s}]$.

Note that $\gamma(s)$ is clearly nonempty and closed, as it is the finite union of closed sets, the first of which is always nonempty. Note also that $\lim_{s \to 1} \frac{s}{1-s}$ diverges to infinity.

First we must show that γ is continuous. It is enough to show γ is continuous with respect to the upper and lower Vietoris topologies.

First let $\gamma(s_0) \in U^+$, where U^+ is a basic open set in the upper Vietoris topology. Then $\gamma(s_0) \subset U$. If $s_0 = 1$, then $\gamma(s_0) = \mathbb{R} \subset U = \mathbb{R}$, so clearly for all $s \in [0,1]$, $\gamma(s) \subset U$ and $\gamma(s) \in U^+$. Assume $s_0 \neq 1$. As U is open and $\gamma(s_0)$ is compact, there exists some $\epsilon > 0$ such that $B(\gamma(s_0), \epsilon) \subset U$. By continuity of $f(x) = \frac{x}{1-x}$, there exists some $\delta > 0$ such that if $|s - s_0| < \delta$, then $|f(s) - f(s_0)| < \epsilon$, and therefore $A(s) \subset U$, so $\gamma(s) \in U^+$.

Next suppose $\gamma(s_0) \in U_1^- \cap \cdots \cap U_n^-$, a basic open set in the lower Vietoris topology. If $s_0 = 1$, then there is some $\epsilon > 0$ such that if $s \in (1-\epsilon, 1]$, $A(s) \cap U_i \neq \varnothing$ for all i. In addition, if $\mathbb{T} \in U_i^-$, then $\gamma(s) \in U_i^-$ for all $s \in [0,1]$. So assume that $\mathbb{T} \notin U_i^-$ and $s_0 \neq 1$. Therefore $A(s_0) \in U_1^- \cap \cdots \cap U_n^-$. Choose $t_i \in A(s_0) \cap U_i$, and let $d_i > 0$ be such that $B(t_i, d_i) \subset U_i \cap A(s_0)$ for all $i \in \{1, \ldots, n\}$. As f is continuous, we can find $\delta > 0$ such that if $|s - s_0| < \delta$, then $|f(s) - f(s_0)| < \min\{d_1, \ldots, d_n\}$. Then $t_i \in \gamma(s)$ for all i and therefore $\gamma(s) \in U_1^- \cap \cdots \cap U_n^-$. □

4. Simply connected

It is enough to show that all loops with a particular base point $\mathbb{T}$ are null-homotopic. We choose the base point to be $\mathbb{R}$, and assume that we are given an arbitrary loop based at $\mathbb{R}$, i.e. a continuous map $f : [0,1] \to \mathbf{CL}(\mathbb{R})$ with $f(0) = f(1) = \mathbb{R}$.

Lemma 4.1. *There exists a continuous map $x : [0,1] \to \mathbb{R}$ such that $x(s) \in f(s)$.*

Proof. We define $x : [0,1] \to \mathbb{R}$ by letting $x(s)$ be the point of $f(s)$ which is closest to the origin, choosing the positive point in the case of a tie. This map is well-defined because each $f(s) \in \mathbf{CL}(\mathbb{R})$, meaning it is a nonempty and closed subset of the real line, so such a point exists. We claim that this map is in fact a loop in $\mathbb{R}$. It is easy to see that $x(0) = x(1) = 0$, so we need only check continuity. Notationally we will sometimes write x_s for $x(s)$.

Fix $s_0 \in [0,1]$ and fix $\epsilon > 0$. Consider $f(s_0)$. We know that $x(s_0) \in f(s_0)$ by definition of x. Consider the ball around $f(s_0)$ in $\mathbf{CL}(\mathbb{R})$

$$B_1 = < \mathbb{R}, (x_{s_0} - \epsilon/2, x_{s_0} + \epsilon/2) >$$

Continuity of f implies there exists a $\delta_1 > 0$ such that if $s \in (s_0 - \delta_1, s_0 + \delta_1)$, then $f(s) \in B_1$. In particular, $f(s) \cap (x_{s_0} - \epsilon/2, x_{s_0} + \epsilon/2) \neq \varnothing$. Therefore the closest point of $f(s)$ to the origin can have distance from the origin no greater than $|x_{s_0}| + \epsilon/2$.

Should x_{s_0} be within $\epsilon/2$ of the origin, we can let $\delta = \delta_1$ at this point. If not, consider

$$B_2 =< (-\infty, -|x_{s_0}| + \epsilon/2), (|x_{s_0}| - \epsilon/2, \infty) >$$

Because $f(s_0)$ contains no points closer to the origin than x_{s_0}, $f(s_0) \in B_2$. By continuity of f, there exists some $\delta_2 > 0$ such that $s \in (s_0 - \delta_2, s_0 + \delta_2)$ implies $f(s) \in B_2$. But this means that the closest point of $f(s)$ to the origin can have distance from the origin no less than $|x_{s_0}| - \epsilon/2$.

Choose $\delta = \mathbf{min}\{\delta_1, \delta_2\}$. Then for all $s \in (s_0 - \delta, s_0 + \delta)$, the closest point of $f(s)$ to the origin lies within $\epsilon/2$ of either x_{s_0} or $-x_{s_0}$. By choosing the positive one in all tie cases, we ensure that in fact x_s is within $\epsilon/2$ of x_{s_0}. Therefore x is a continuous function. $\square$

Given an arbitrary point $p_0 \in \mathbb{T}$, let $\gamma_{\mathbb{T}} : [0,1] \to \mathbf{CL}(\mathbb{R})$ be the map defined by $\gamma_{\mathbb{T}}(s) = \mathbb{T} \cup [p_0 - \frac{s}{1-s}, p_0 + \frac{s}{1-s}]$ for $s \in [0,1)$ and $\gamma_{\mathbb{T}}(1) = \mathbb{R}$. We know from the proof of Theorem 3.1 in Section 3 that this map is a continuous path from $\mathbb{T}$ to $\mathbb{R}$. Since γ depends on the point p_0, we will sometimes write $\gamma_{\mathbb{T}}(p_0, s)$.

We wish to find a homotopy from f to the constant loop $c(s) = \mathbb{R}$, $s \in [0,1]$. In other words, we require a continuous map $F : [0,1] \times [0,1] \to \mathbf{CL}(\mathbb{R})$ with the following properties:

(1) For all $s \in [0,1]$, $F(s,0) = f(s)$, i.e., at time zero we have the original loop f.
(2) For all $s \in [0,1]$, $F(s,1) = \mathbb{R}$, i.e., at time one we have the constant loop c.
(3) For all $t \in [0,1]$, $F(0,t) = F(1,t) = \mathbb{R}$, i.e., at all other times we do, in fact, have loops based at $\mathbb{R}$.

Theorem 4.2. *$F(s,t) = \gamma_{f(s)}(x(s),t)$ is a homotopy from f to the constant loop.*

Proof. It is easy to see that F has the three properties listed. Continuity is all that remains to check.

Fix a particular point (s_0, t_0). It is clear that F is continuous in t because the path $\gamma_{f(s_0)}$ is continuous. Let us check continuity in s. Note that if $t = 1$, then $F(s,t) = \mathbb{R}$ for all s. Therefore we need only consider the case $t_0 \neq 1$. Again, we check continuity with respect to the upper and lower Vietoris topologies.

We know that $F(s_0, t_0) = f_{s_0} \cup [x_{s_0} - \frac{t_0}{1-t_0}, x_{s_0} + \frac{t_0}{1-t_0}]$ For brevity, we will refer to that closed interval as I_{s_0}.

Let $F(s_0, t_0) \in U^+$, a basic open set in the upper Vietoris topology. As f is continuous, there exists some $\delta_1 > 0$ such that if $|s - s_0| < \delta_1$, then $f_s \in U^+$. Because I_{s_0} is compact, there is some $\epsilon > 0$ such that $B(I_{s_0}, \epsilon) \subset U$. By continuity of $x(s)$, there exists some δ_2 such that if $|s - s_0| < \delta_2$, then $|x_s - x_{s_0}| < \epsilon$. Then it is clear that $I_s \subset B(I_{s_0}, \epsilon)$, and so $I_s \in U^+$. Let $|s - s_0| < \min\{\delta_1, \delta_2\}$, and we have that $F(s, t_0) \in U^+$.

Next let $F(s_0, t_0) \in U_1^- \cap \cdots \cap U_n^-$, a basic open set in the lower Vietoris topology. If $f_{s_0} \in U_i^-$, then by continuity of f, there exists some $\delta > 0$ such that if $|s - s_0| < \delta$, $f_s \in U_i^-$, and so $F(s, t_0) \in U_i^-$. So we can suppose without loss of generality that $f_{s_0} \notin U_i^-$ for all i. Therefore $I_{s_0} \in U_1^- \cap \cdots \cap U_n^-$. We use a reasoning similar to that in the proof of Theorem 3.1. Take $t_i \in I_{s_0} \cap U_i$, and let d_i such that $B(t_i, d_i) \subset I_{s_0} \cap U_i$. There exists some $\delta > 0$ such that when $|s - s_0| < \delta$, $|x_s - x_{s_0}| < \min\{d_1, \ldots, d_n\}$ and therefore $t_i \in I_s$ for all i. Thus we have that $F(s, t_0) \in U_1^- \cap \cdots \cap U_n^-$. □

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