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Lower homomorphisms on additive generalized algebraic lattices

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Abstract. In this paper, with the additivity property ([8]), the generalized way-below relation ([15]) and the maximal system of subsets ([6]) as tools, we prove that all lower homomorphisms between two additive generalized algebraic lattices form an additive generalized algebraic lattice, which yields the classical theorem: the function space between T_0-topological spaces is also a T_0-topological space with respect to the pointwise convergence topology.

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1. Introduction

The notions of a directed set, a way-below relation, a continuous lattice and an algebraic lattice were introduced in [12], and applied in the study of domain theory, topological theory, lattice theory, etc.

As a generalization, D. Novak introduced the notions of a system of subsets, a generalized way-below relation, and defined a generalized continuous lattice (M-continuous lattice) and a generalized algebraic lattice in [15].

In the study of topological theory and lattice theory, many researchers are interested in the topological representation of a complete lattice. For example: suppose (X, T) is a topological space. All open sets T of a topological space may be viewed as a frame and a frame may also be viewed as an open set lattice. About Frame (Locale) theory, see ([13]).

On the other hand, suppose (X, C) is a co-topological space and C the set of all closed subsets of a topological space on X. D. Drake, W. J. Thron, S. Papert

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considered C as a complete lattice $(C, \cup, \cap, \varnothing, X)$([11, 16]). But unfortunately the correspondence between complete lattices and T_0-topological spaces is not one-to-one.

To solve the problem, on the basis of [1, 11, 15, 16], Deng also investigated generalized continuous lattices. He introduced the notions of the maximal system of subsets, additivity property, and homomorphisms in [5, 6, 7, 10]. Finally, the question was settled in [8, 9], He obtained the equivalence between the category of additive generalized algebraic lattices with lower homomorphisms and the category of T_0-topological spaces with continuous mappings.

This paper is a sequel of [2, 3, 4, 8, 9]. In section 2, we begin an overview of generalized continuous lattices, Deng's work, and some separation axioms, which surveys as Preliminaries. In section 3, we prove that all lower homomorphisms between additive generalized algebraic lattices form a additive generalized algebraic lattice, and investigate some results about separation axioms.

2. Preliminaries

We introduce some notions for each area, i.e., generalized continuous lattices and additive generalized algebraic lattices.

2.1. **Generalized Continuous Lattices.**

In [15], D. Novak introduced the notions of a generalized way-below relation and a system of subsets.

Let $(P, \leq)$ be a complete lattice, $\prec$ is said to be a generalized way-below relation if (i) $a \prec b \Rightarrow a \leq b$, (ii) $a \leq b \prec c \leq d \Rightarrow a \prec d$.

Obviously, it is a natural generalization of a way-below relation ([12]).

$M \subseteq 2^P$ is said to be a system of subsets of P, if for $a \in P$, there exists $S \in M$, such that $\downarrow a = \downarrow S$, where $\downarrow a = \{b \mid b \leq a\}$, $\downarrow S = \cup\{\downarrow c \mid c \in S\}$. There are three kinds of common used system of subsets: (i) the system of all finite subsets, (ii) the system of all directed sets and (iii) the system of all subsets.

By means of the notion of a system of subsets, he defined a generalized way-below relation. Suppose M is a system of subsets. For $a, b \in P$, a is said to be way-below b with respect to M, in symbols $a \prec_M b$, if for every $S \in M$ with $b \leq \vee S$, then $a \in \downarrow S$.

Clearly $\prec_M$ is a generalized way-below relation induced by M ([15]). We will denote $\prec_M$ as $\prec$.

$(P, \prec)$ is called a generalized continuous lattice, if for every $a \in P$, we have $a = \vee \Downarrow a$, where $\Downarrow a = \{b \mid b \prec a\}$.

$a \in P$ is called a compact element, if $a \prec a$. Let $K(\prec) = \{a \in P \mid a \prec a\}$. $(P, \prec)$ is called a generalized algebraic lattice, if for every $a \in P$, we have $a = \vee\{\downarrow a \cap K(\prec)\}$. For further study, see [1, 17].

2.2. **Additive Generalized Algebraic Lattices.**

Suppose $(P, \prec)$ is a generalized continuous lattice. Deng introduced the notion of a maximal system of subsets generated by $\prec$, that is,

$M(\prec) = \{S \subseteq P \mid \forall a \in P$ with $a \prec \vee S$, then $a \in \downarrow S\}$.

Suppose $(P, \prec)$ is a generalized algebraic lattice. Deng defined a new property: $(P, \prec)$ is additivity, if for $a, b, c \in P$ with $a \prec b \vee c$ implies $a \prec b$ or $a \prec c$ ([8, 9]).

He investigated the connection between additive generalized algebraic lattices and T_0-topological spaces as follows.

On one hand, suppose $(P, \prec)$ is a generalized algebraic lattice, let $X = K(\prec)$, and $T : P \to 2^X$, $T(a) = \downarrow a \cap K(\prec)$. If $(P, \prec)$ is additive, then T satisfies: (1) $T(0) = \varnothing$, (2) $T(1) = X$, (3) for $S \in M(\prec) = M(K(\prec))$, $T(\vee S) = \cup T(S)$, (4) for $S \subseteq P$, $T(\wedge S) = \cap T(S)$, (5) $T(a \vee b) = T(a) \cup T(b)$.

If $C = T(P)$, then (X, C) is a T_0 co-topological space, and $(P, \prec)$ is isomorphic to (X, C) (see [8, 9]).

On the other hand, assume (X, C) is a co-topological space and let $Q = \{\{x\}^- \mid x \in X\}$ be the collection of closure of all singletons. Clearly Q is a $\vee-$base for C, i.e., $a \in C$, a is a closed subset, and we have $a = \vee \downarrow a$.

$M(Q) = \{S \mid S \subseteq X$, for $a \in Q$, $a \leq \vee S$ we have $a \in \downarrow S\}$ is a system of subsets induced by Q, then $(C, \prec_{M(Q)})$ is a additive generalized algebraic lattice with $K(\prec_{M(Q)}) = Q$. In this case, $a \prec_{M(Q)} b$, for $a, b \in C$ if and only if $a \subseteq \{x\}^-$ for some $x \in b$. It is clear that $\prec_{M(Q)}$ is the specialization order ([12]) which is essentially in topological theory and domain theory.

Furthermore, $(C, \prec_{M(Q)})$ is an example of additive generalized algebraic lattice. For another example in commutative ring, see [9].

Suppose $(P_1, \prec_1)$, $(P_2, \prec_2)$ are two generalized continuous lattices. $h : P_1 \to P_2$ is said to be a lower homomorphism if it preserves arbitrary joins and the generalized way-below relations. Thus a lower homomorphism h is residuated. If g be its upper adjoint, we have (g, h) is a Galois connection ([7]).

The lower homomorphism also corresponds to the closed mapping. So he obtained the equivalence between the category of additive generalized algebraic lattices with lower homomorphisms and the category of T_0-topological spaces with continuous mappings in [8, 9].

From the point of view of Deng's work ([8, 9]), an additive generalized algebraic lattice is an algebraic abstraction of a topological space. Thus topological theory may be directly constructed on it. The work will benefit the study of the theory of topological algebra and the possible application on additive generalized algebraic lattices. In [2, 3, 4], we constructed Stone compactification, Tietze extension theorem, Separation axioms.

In this paper, we will prove that all lower homomorphisms between additive generalized algebraic lattices form an additive generalized algebraic lattice.

In [2, 3, 4], we defined some separation axioms.

Definition 2.1. *$(P, \prec)$ is said to be regular, if for $x \in K(\prec)$, $b \in P$, $x \not\prec b$, then $x \wedge b = 0$.*

Definition 2.2. *A family of elements $\langle c_\alpha \mid \alpha \in [0,1]$ & α is a rational number $\rangle$ is called a scale of $(P, \prec)$, if it satisfies: for $\alpha < \beta$, we have $c_\alpha \prec c_\beta$.*

For $a, b \in P$, if there exists a scale $\langle c_\alpha \rangle$, such that $a \leq c_0$, $c_1 \leq b$. We use the symbol $a \lhd b$ to indicate the relation.

$(P, \prec)$ is said to be completely regular, if $\forall a \in L$, $a = \wedge\{b \mid a \lhd b\}$.

Definition 2.3. *$(P, \prec)$ is said to be normal, if for $a, b \in P$, $a \wedge b = 0$, then there exist $c, d \in P$, such that $a \wedge c = 0$, $b \wedge d = 0$ and $c \vee d = 1$.*

For other notions and results cited in this paper, see [2, 3, 4, 8, 9, 15].

3. Lower homomorphisms

Definition 3.1. *Suppose P_1 and P_2 are two additive generalized algebraic lattices. $\forall p \in K(\prec_1)$, $q \in K(\prec_2)$, we define*

$$\langle p, q\rangle(a) = \begin{cases} q & \text{if} \quad p \prec_1 a \\ 0 & \text{if} \quad p \not\prec_1 a \end{cases} \quad \forall a \in P_1.$$

Lemma 3.2. *$\langle p, q\rangle$ is a lower homomorphism.*

Proof. First, we have to show that $\langle p, q\rangle$ preserves arbitrary join.

Suppose $\{a_\alpha\} \subseteq P_1$. If $p \prec_1 \bigvee a_\alpha$, we obtain $\langle p, q\rangle(\bigvee a_\alpha) = q$. Since P_1 is additive, by $p \prec_1 \bigvee a_\alpha$, there exists a_{α_0}, such that $p \prec_1 a_{\alpha_0}$. So $\langle p, q\rangle(a_{\alpha_0}) = q$, thus $\langle p, q\rangle(\bigvee a_\alpha) = q = \bigvee \langle p, q\rangle(a_\alpha)$.

If $p \not\prec_1 \bigvee a_\alpha$, then $\langle p, q\rangle \bigvee a_\alpha = 0$. By this, we have $p \not\prec_1 a_\alpha$ for every α. So $\langle p, q\rangle(a_\alpha) = 0$, which implies that $\langle p, q\rangle(\bigvee a_\alpha) = 0 = \bigvee \langle p, q\rangle(a_\alpha)$.

Second, we have to prove that $\langle p, q\rangle$ preserves the generalized way-below relation.

Given $a, c \in P_1$, and $a \prec_1 c$, if $p \prec_1 a$, then $p \prec_1 c$, we have $\langle p, q\rangle(a) = q$, $\langle p, q\rangle(c) = q$, thus $\langle p, q\rangle(a) \prec_2 \langle p, q\rangle(c)$; if $p \not\prec_1 a$, $p \prec_1 c$, then $\langle p, q\rangle(a) = 0$, $\langle p, q\rangle(c) = q$, thus $\langle p, q\rangle(a) \prec_2 \langle p, q\rangle(c)$; if $p \not\prec_1 a$, $p \not\prec_1 c$, then $\langle p, q\rangle(a) = 0$, $\langle p, q\rangle(c) = 0$, thus $\langle p, q\rangle(a) \prec_2 \langle p, q\rangle(c)$.

By the above proof, we obtain that $\langle p, q\rangle$ also preserves the generalized way below relation. □

By Lemma 3.2, $\langle p, q\rangle$ is a lower homomorphism. Let g_{pq} be its upper adjoint. Then $(\langle p, q\rangle, g_{pq})$ is a Galois connection.

Let $[P_1 \to P_2]$ be the set of all lower homomorphisms from P_1 to P_2 and suppose $h_1, h_2 \in [P_1 \to P_2]$. Then we may define $h_1 \vee h_2 : P_1 \to P_2$, for every $p \in K(\prec_1), (h_1 \vee h_2)(p) = h_1(p) \vee h_2(p)$. Similarly, $(h_1 \wedge h_2)(p) = h_1(p) \wedge h_2(p)$. So $[P_1 \to P_2]$ is a complete lattice with the minimal element 0 and the maximal element 1, where $0(p) = 0, 1(p) = 1$ for every $p \in K(\prec_1)$.

We also define $h_1 \leq h_2$, if for every $p \in K(\prec_1)$, we have $h_1(p) \leq_2 h_2(p)$, where $\leq_2$ is the partial order on P_2. Similarly, $h_1 \prec^* h_2$, if for every $p \in K(\prec_1)$, we have $h_1(p) \prec_2 h_2(p)$.

Lemma 3.3. *$\prec^*$ is a generalized way below relation on $[P_1 \to P_2]$.*

Proof. We have to show (1) and (2),
(1) $h_1 \prec^* h_2 \Rightarrow h_1 \le h_2$,
(2) $h_1 \le h_2 \prec^* h_3 \le h_4 \Rightarrow h_1 \prec^* h_4$.
The proof is trivial.
□

Lemma 3.4. *$\langle p, q\rangle$ is a compact element of $[P_1 \to P_2]$.*

Proof. By the definition of $\prec^*$, the proof is trivial. □

Clearly, $K(\prec^*) = \{\langle p, q\rangle \mid p \in K(\prec_1), q \in K(\prec_2)\}$.

Lemma 3.5. *If h is a lower homomorphism, then $\forall p \in K(\prec_1),\ h(p) \in K(\prec_2)$.*

Proof. See [8, 9]. □

Lemma 3.6. *If $h \in [P_1 \to P_2]$, $q \le_2 h(p)$, we have $\langle p, q\rangle \prec^* h$.*

Proof. $\forall a \in P_1, \langle p, q, \rangle(a) = \begin{cases} q & \text{if} \quad p \prec_1 a \\ 0 & \text{if} \quad p \not\prec_1 a \end{cases}$

If $p \prec_1 a$, $\langle p, q\rangle(a) = q \le_2 h(p) \le_2 h(a)$; if $p \not\prec_1 a, \langle p, q\rangle(a) = 0 \le_2 h(a)$. Thus we have $\langle p, q\rangle(a) \le_2 h(a)$ for all $a \in P_1$, thus $\langle p, q\rangle \le h$. By Lemma 3.4, since $\langle p, q\rangle$ is a compact element, we obtain $\langle p, q\rangle \prec^* h$. □

Lemma 3.7. *If $h \in [P_1 \to P_2]$, if $p \in K(\prec_1)$, then $h(p) = \langle p, h(p)\rangle(p)$.*

Proof. For $p \in K(\prec_1)$, since h is a lower homomorphism, $h(p) \in K(\prec_2)$ (see [8]). Thus we have $h(p) = \langle p, h(p)\rangle(p)$. □

Note 1. *$\bigvee\langle p_\alpha, q_\alpha\rangle$ does not preserve the way below relation in general.*

Example 3.8. Without the assumption of additive property, Lemma 3.7 does not hold.

Suppose P_1, P_2 are two classical algebraic lattices [12]. If $K(\prec_2) \ne P_2$, there exists $e \in P_2, e \notin K(\prec_2)$. Since P_2 is algebraic, there exists a directed set $\{q_\alpha\} \subseteq K(\prec_2)$, such that $e = \vee q_\alpha$. We define
$\langle 0, q_\alpha\rangle : P_1 \to P_2, \forall x \in P_1,\ \langle 0, q_\alpha\rangle(x) = q_\alpha$.
$c_e : P_1 \to P_2, \forall x \in P_1,\quad c_e(x) = e$.
It is easy to show that $\{\langle 0, q_\alpha\rangle\}$ is also a directed set in $[P_1 \to P_2]$, which preserves the way-below relation, but $c_e = \vee\langle 0, q_\alpha\rangle$ does not hold.

Proposition 3.9. $\forall h \in [P_1 \to P_2]$, $h = \bigvee\limits_{p \in K(\prec_1)} \bigvee\limits_{q \le_2 h(p)} \langle p, q\rangle$.

Proof. For every $a \in P_1$ and since P_1 is generalized algebraic, we have $a = \vee\{p \mid p \in K(\prec_1)\}$, and h preserves arbitrary joins. Thus it suffices to prove that for every $p \in K(\prec_1)$, $h(p) = \bigvee\limits_{p \in K(\prec_1)} \bigvee\limits_{q \le_2 h(p)} \langle p, q\rangle(p) = \bigvee\limits_{q \le_2 h(p)} \langle p, q\rangle(p)$.

Since $q \le_2 h(p)$, we have $\langle p, q\rangle(p) = q \le_2 h(p)$. By Lemma 3.7, $\langle p, h(p)\rangle(p) = h(p)$, we obtain $h(p) = \bigvee\limits_{q \le_2 h(p)} \langle p, q\rangle(p)$. □

Proposition 3.10. *Suppose P_1 and P_2 are two generalized algebraic lattices. Then $[P_1 \to P_2]$ is a generalized algebraic lattice.*

Proof. By Proposition 3.9, we have $h = \vee(\downarrow h \cup K(\prec^*))$ for $h \in [P_1 \to P_2]$. So $[P_1 \to P_2]$ is a generalized algebraic lattice. □

Proposition 3.11. $[P_1 \to P_2]$ *is additive.*

Proof. Suppose $\langle p, q\rangle \in K(\prec^*)$, $h_1, h_2 \in [P_1 \to P_2]$, and $\langle p, q\rangle \prec^* h_1 \vee h_2$. We have $\langle p, q\rangle(p) = q \prec_2 (h_1 \vee h_2)(p) = h_1(p) \vee h_2(p)$. Since P_2 is additive, we have $q \prec^* h_1(p)$, or $q \prec^* h_2(p)$. By this, we obtain $\langle p, q\rangle \prec^* h_1$, or $\langle p, q\rangle \prec^* h_2$. Thus $[P_1 \to P_2]$ is additive. □

By Propositions 3.10 and 3.11, we obtain $[P_1 \to P_2]$ is an additive generalized algebraic lattice. From the point of of view of topological theory, the result corresponds to the classical theorem: the function space between two T_0-topological spaces is also T_0-topological space with respect to the pointwise convergence topology.

Proposition 3.12. *If $(P_2, \prec_2)$ is regular, then $[P_1 \to P_2]$ is also regular.*

Proof. For $\langle p, q\rangle \in K(\prec^*)$, $h \in [P_1 \to P_2]$, if $\langle p, q\rangle \not\prec^* h$, by the definition of $\langle p, q\rangle$, we have $\langle p, q\rangle(p) \not\prec_2 h(p)$, so $q \not\prec_2 h(p)$. Since $(P_2, \prec_2)$ is regular, we obtain $q \wedge h(p) = 0$, which implies that $\langle p, q\rangle \wedge h = 0$, thus $[P_1 \to P_2]$ is regular. □

Proposition 3.13. *If $(P_2, \prec_2)$ is completely regular, then $[P_1 \to P_2]$ is also completely regular.*

Proof. For $h_1 \lhd h_2$, by the definition of $\prec^*$, it is equivalent to: for every $p \in K(\prec_1)$, $h_1(p) \lhd h_2(p)$. Since $(P_2, \prec_2)$ is completely regular, so $h_1(p) = \wedge\{h_2(p) \mid h_1(p) \lhd h_2(p)\}$, which implies that $h_1 = \wedge\{h_2 \mid h_1 \lhd h_2\}$. Proposition 3.13 holds. □

Proposition 3.14. *If $(P_2, \prec_2)$ is normal, then $[P_1 \to P_2]$ is also normal.*

Proof. If $h_1, h_2 \in [P_1 \to P_2]$, and $h_1 \wedge h_2 = 0$, then for any $p \in K(\prec_1)$, $h_1(p) \wedge h_2(p) = 0$.

For $h_1(p) \wedge h_2(p) = 0$, since $(P_2, \prec_2)$ is normal, there exist $c_p, d_p \in P_2$, such that $h_1(p) \prec_2 c_p$, $h_2(p) \prec_2 d_p$, and $c_p \vee d_p = 1$. Let $h_{c_p} = \bigvee_{q \prec_2 c_p} \langle p, q\rangle$, $h_{d_p} = \bigvee_{q \prec_2 d_p} \langle p, q\rangle$, so $h_1 \prec^* h_{c_p}$ and $h_2 \prec^* h_{d_p}$.

Let $h_c = \wedge_{p \in K(\prec_1)} h_{c_p}$, $h_d = \bigvee p \in K(\prec_1) h_{d_p}$. It is easy to prove $h_1 \prec^* h_c$, $h_2 \prec^* h_d$, and $h_c \vee h_d = 1$. Thus $[P_1 \to P_2]$ is also normal. □

Based on the above work, we constructed Tietze extension theorem in [3].

Proposition 3.15 (Tietze extension theorem).
$(P, \prec)$ is normal iff for every closed lower sublattice $(Q, \prec_Q)$ of $(P, \prec)$, and a lower homomorphism $h : (Q, \prec_Q) \to (C_J, \prec_J)$, there exists a lower homomorphism $H : (P, \prec) \to (C_J, \prec_J)$, such that $H|_Q = h$

The proof can be seen in [3].

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