UNIVERSITAT POLITÈCNICA DE VALÈNCIA

DEPARTMENT OF APPLIED MATHEMATICS

Mappings between
Thermodynamics and Quantum Mechanics
that support its interpretation as an emergent theory

PHD THESIS SUBMITTED BY:

Joan Vazquez Molina

UNDER THE ADVISE OF:

Pedro José Fernández de Córdoba Castellá

José María Isidro San Juan

Manuel Monleón Pradas

VALENCIA, APRIL 2017

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Don Pedro José Fernández de Córdoba Castellá, Catedrático de Universidad del departamento de Matemática Aplicada de la Universitat Politècnica de València; Don José María Isidro San Juan, Profesor Titular del Departamento de Matemática Aplicada de la Universitat Politècnica de València; y Don Manuel Monleón Pradas, Catedrático de Universidad del departamento de Termodinámica de la Universitat Politècnica de València

ATESTIGUAN:

> que la presente memoria ‘‘Mappings between Thermodynamics and Quantum Mechanics that support its interpretation as an emergent theory’’ ha sido realizada bajo nuestra dirección por Joan Vazquez Molina y constituye su tesis para optar al grado de Doctor en Ciencias Matemáticas, con mención ‘‘Doctor Internacional’’.
>
> Y para que así conste en cumplimiento de la legislación vigente presentamos y apadrinamos ante la Escuela de Doctorado de la Universitat Politècnica de València la referida tesis firmando el presente certificado.

Valencia, Abril de 2017

Los directores:

Pedro José Fernández de Córdoba Castellá, José María Isidro San Juan y Manuel Monleón Pradas

PhD candidate: Vazquez Molina, Joan

Title: Mappings between Thermodynamics and Quantum Mechanics that support its interpretation as an emergent theory

Adviser: Fernández de Córdoba Castellá, Pedro José
Adviser: Isidro San Juan, José María
Adviser: Monleón Pradas, Manuel

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Título: Mapeos entre la Termodinámica y la Mecánica Cuántica que apoyan su interpretación como teoría emergente

Director: Fernández de Córdoba Castellá, Pedro José
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Institución: Universitat Politècnica de València
Programa: Doctorado en Matemáticas
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Abstract

This PhD thesis is submitted as a *compendium* of the articles [45, 44, 22]. The following has been adapted from their abstracts.

Quantum mechanics has been argued to be a coarse–graining of some underlying deterministic theory. Here we support this view by establishing mappings between non-relativistic quantum mechanics and thermodynamic theories, since the latter are the paradigm of an emergent theory.

First, we map certain solutions of the Schroedinger equation to solutions of the irrotational Navier–Stokes equation for viscous fluid flow. Although this is formally a generalization of Madelung's hydrodynamical interpretation, the presence of a viscous term leads to a novel interpretation. As a physical model for the fluid itself we propose the quantum probability fluid. It turns out that the (state–dependent) viscosity of this fluid is proportional to Planck's constant, while the volume density of entropy is proportional to Boltzmann's constant. Stationary states have zero viscosity and a vanishing time rate of entropy density. On the other hand, the nonzero viscosity of nonstationary states provides an information–loss mechanism whereby a deterministic theory (a classical fluid governed by the Navier–Stokes equation) gives rise to an emergent theory (a quantum particle governed by the Schroedinger equation).

Then, we present a map of standard quantum mechanics onto classical thermodynamics of irreversible processes. In particular, the propagators of the quantum harmonic oscillator are mapped to the conditional probabilities that solve the Chapman-Kolmogorov equation for Markovian Gaussian processes. While no gravity is present in our construction, our map exhibits features that are reminiscent of the holographic principle of quantum gravity.

Finally, the classical thermostatics of equilibrium processes is shown to possess a quantum mechanical dual theory with a finite dimensional Hilbert space of quantum states. Specifically, the kernel of a certain Hamiltonian operator becomes the Hilbert space of quasistatic quantum mechanics. The relation of thermostatics to topological field theory is also discussed in the context of the approach of emergence of quantum theory, where the concept of entropy plays a key role.

Resumen

La presente tesis doctoral se presenta como compendio de las publicaciones [45, 44, 22]. El siguiente resumen es una adaptación de sus resumenes.

Se ha argumentado que la mecánica cuántica podría emerger como promediado de una teoría determinista subyacente. Se apoya dicha visión estableciendo mapeos entre la mecánica cuántica no relativista y teorías termodinámicas, ya que estas constituyen el paradigma de teoría emergente.

Primero, se establece un mapeo entre soluciones de la ecuación de Schroedinger y soluciones de la ecuación de Navier-Stokes irrotacional para fluidos viscosos. Aunque formalmente se trate de una generalización de la interpretación hidrodinámica de Madelung, la presencia del término viscoso sugiere una nueva interpretación. Se propone la probabilidad cuántica como modelo físico del fluido. Se obtiene que la viscosidad (dependiente del estado) es proporcional a la constante de Planck, mientras que la densidad de entropía es proporcional a la constante de Boltzmann. Los estados estacionarios tienen viscosidad y tasa de producción de densidad de entropía nulas. Por otro lado, la viscosidad no nula de los estados no estacionarios proporciona un mecanismo de pérdida de información por el cual una teoría determinista (un fluido clásico gobernado por la ec. de Navier-Stokes) da lugar a una teoría emergente (una partícula cuántica gobernada por la ec. de Schroedinger).

Después, se presenta un mapeo entre la mecánica cuántica y la termodinámica clásica de procesos irreversibles. En particular, los propagadores del oscilador armónico cuántico se mapean a las probabilidades condicionales que resuelven la ecuación de Chapman-Kolmogorov para procesos de Markov Gaussianos. Aunque no hay gravedad, el mapeo exhibe propiedades que recuerdan al principio holográfico de la gravedad cuántica.

Finalmente, se muestra cómo la termoestática clásica de procesos de equilibrio posee una teoría cuántica dual con un espacio de Hilbert finito - dimensional de estados cuánticos. Concretamente, el núcleo de cierto operador Hamiltoniano se convierte en el espacio de Hilbert de una mecánica cuántica cuasiestática. La relación de la termoestática a la teoría topológica de campos se discute en el contexto de la mecánica cuántica emergente, donde el concepto de entropía juega un papel clave.

Resum

Aquesta tesi doctoral es presenta com a compilació de les publicacions [45, 44, 22]. El següent resum es una adaptació dels seus resums.

S'ha argumentat que la mecànica quàntica podria emergir com a granulat gros d'una teoria determinista subjacent. Es dóna suport a aquesta visió mitjaçant uns mapes entre la mecànica quàntica no relativista i teories termodinàmiques, ja que les darreres són el paradigma de teoria emergent.

Primer, s'estableix un mapa entre certes solucions de l'equació de Schroedinger i solucions de l'equació de Navier-Stokes irrotacional per a fluids viscosos. Tot i que formalment es tracte d'una generalització de la interpretació hidrodinàmica de Madelung, la presència del terme viscós ens porta a una nova interpretación. Es proposa la probabilitat quàntica com a model físic del fluid. S'obté que la viscositat del fluid (que depén de l'estat) es proporcional a la constant de Planck, mentre que la densitat d'entropía es proporcional a la constant de Boltzmann. Els estats estacionaris tenen viscositat nul·la i taxa de producció d'entropia nul·la. Per alta banda, la viscositat no nul·la dels estats estacionaris proporciona un mecanisme de pèrdua d'informació pel qual una teoria determinista (un fluid clàssic governat per l'equació de Navier-Stokes) dóna lloc a una teoria emergent (una partícula quàntica governada per l'equació de Schroedinger).

Després, es presenta un mapa entre la mecànica quàntica i la termodinàmica clàssica de processos irreversibles. En particular, els propagadors de l'oscil·lador harmònic quàntic es mapejen a les probabilitats condicional que resolen l'ecuació de Chapman-Kolmogorov per a processos de Markov Gaussians. Tot i que no hi ha gravetat present a la nostra construcció, el mapa exhibeix propietats que recorden al principi hologràfic de la gravetat quàntica.

Finalment, es mostra cóm la termoestàtica clàssica de processos d'equilibri té una teoria quàntica dual amb un espai de Hilbert de dimensió finita d'estats quàntics. En concret, el nucli de cert operador Hamiltonià es converteix en l'espai de Hilbert d'una mecànica quàntica quasiestàtica. La relació de la termoestàtica a la teoria topològica de camps es dicuteix en el context de la mecànica quàntica emergent, on el concepte d'entropia té un paper clau.

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Above all, I had a wonderful team of advisers to which I am extremely grateful. To Pedro, for providing professional support during critical steps of my career and for making this PhD possible. To José María, for taking the time and effort to teach me the well-known Physics, answering all my tedious questions, and for counting on me for all his research activities from the very beginning. And to Manolo, for his seminars, for always pushing me out of my comfort zone, and for his scientific and moral example.

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Contents

Chapter 1

Introduction

The goal of physics is not only to predict the outcome of experiments but to describe the world, to give us an ontology. If we keep building huge and expensive experiments that test the unseen it is because we want to be able to describe *what there is* and provide a coherent and complete account of reality. At this respect, even an opposing "*extreme **shut-up-and-calculate** approach to physics*" ends up making ontological claims, such that "*our universe is not just described by mathematics — it **is** mathematics*" [114][1].

"*1. What is matter? What is space and time? 2. What are the laws of nature? 3. How does matter in space and time, being subject to certain laws, explain the observable phenomena?*" are some of the traditional questions of natural philosophy and of central interest to physics [40]. What are the most basic constituents of matter and how do they give rise to the reality that we observe?

Current physics does not give a single definite answer to these questions. To employ a topological language, it consists of an *atlas* of different physical theories whose interrelationship is not free from conceptual problems. Simplifying in extreme, we have Classical Mechanics (CM), which applies -at least- at human scales; Quantum Mechanics (QM), which -in general- becomes relevant at microscopic scales; General Relativity (GR), which -usually- becomes relevant at cosmological scales; and Thermodynamics or

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[1] Boldface not present in the original

Classical Field Theories, which give adequate descriptions of phenomena for many-body problems at very different scales. Of course, the boundaries are not definite at all and we observe, for example, macroscopic quantum phenomena (see [121] for a recent review). The problem is that the ontologies provided by these different theories, or *charts* of reality, are not always consistent. This is precisely one of the motivations for the *unification* trend that has dominated theoretical physics during the last century.

In any case, with or without unification, the *intertheoretical relationships* and *emergence* are hot research topics in physics [74] and philosophy of science [29, 10]. The most widely known intertheoretical problem, which is considered by many to be the biggest open problem in current physics, is how to make compatible GR with QM, or the quest for *quantum gravity*. But this is not the only one. Another relevant intertheoretical problem will be the relationship of many-body or continuum field theories to their underlying particle mechanics, which raises questions about novelty, autonomy, emergence, phase transitions, etc.

Quantum mechanics has been seen as the ultimate microscopic theory, partly because of its tremendous success in the *predictive* goal of physics. But its inner problems (implications of the measurement problem, mysteries of nonlocality), its problems with respect to other theories (no quantum gravity), and its plurality of interpretations gave rise to research lines embraced under the umbrella of *foundations of quantum mechanics* since its very beginning. *"Quantum mechanics is unique in that its equations are known but not its principles"* [119].

Nowadays, an unorthodox part of the scientific community sees QM as an effective theory, emerging from a more fundamental one, be it classical or unknown. This gives rise to the research line known as *emergent quantum mechanics*, whose ultimate goal would be to describe this emerging process and to characterize the fundamental phase space or theory from which it emerges. This field puts together lessons from all the work on the foundations of QM, from thermodynamics and from the vast philosophical literature on emergence and intertheoretical relationships. And it connects with statistical mechanics, one of whose goals is to describe how field theories emerge from their underlying mechanics.

1.1 Theoretical background

1.1.1 Foundations of quantum mechanics

The need to revisit the foundations of quantum mechanics comes -at least- from two problems. One of them is internal to quantum mechanics and its many aspects are summarized as *the measurement problem*. The other one concerns the incompatibility between quantum mechanics and general relativity and the impossibility of having a theory of *quantum gravity*.

In order to comment on the former, let us restate the postulates[2] of quantum mechanics:

Kinematics every isolated physical system S has an associated Hilbert space $\mathcal{H}_S$ that represents its phase space. The state of the system is completely described by a unit vector $\psi \in \mathcal{H}_S$.

Dynamics I the evolution of a closed system obeys the Schroedinger equation

$$\frac{d\psi}{dt} = -\frac{i}{\hbar} H\psi; \qquad H \text{ is the hamiltonian of } S.$$

Dynamics II a *measurement* is described[3] by an hermitian operator $M : \mathcal{H}_S \to \mathcal{H}_S$. By the spectral theorem, M admits a decomposition $M = \sum_m mP_m$, where P_m are the projectors in the subspace of eigenvalue m. The possible outcomes of the measurement are the eigenvalues m. The probability of obtaining m is given by the *Born rule* $p(m) = \psi^* P_m \psi$. The state of the system after the measurement m is $\frac{P_m \psi}{\sqrt{p(m)}}$.

Composition the phase space of a composite system AB, composed of the systems A and B, is obtained through the tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$.

Since the measurement process *suddenly* transforms the state of the system into an eigenstate of the measurement operator (thus removing any

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[2] Similar versions can be found in any textbook. Here we follow [86], for no particular reason beyond the fact that it is a great book.

[3] For simplicity, we give the finite-dimensional version of the postulate, in which *self-adjoint* and *hermitian* are equivalent, and the decomposition takes the form of a sum. Extensions of the spectral theorem for general self-adjoint operators, not even bounded, can be found e.g in [57, 54], but do not change the overall picture.

superposition in the measurement basis), it is pictured as a *wavefunction collapse*. The measurement problem is the question of how this collapse occurs, whether it occurs at all, or how to interpret it, and can be seen under different angles:

Ontological what constitutes an observer? what constitutes an observation? what if we include the observer in the system? Accepting the postulates plus a recursion argument leads to the viewpoint that "*all is* Ψ" [115]. But then the universe as a whole evolves according to the Schroedinger equation, with no collapse whatsoever.

Epistemological from the previous considerations, is the measurement result obtained during the collapse a mere illusion for the observer or is it a physical, objective process taking place? Besides, since the collapse, or observations, change the state of the observed system, what is the role of the observer in the *knowing* process? What is knowledge, then? In Schroedinger words, *"we are told that no distinction is to be made between the state of a natural object and what I know about it"* [99].

Aesthetic why do we need two dynamical postulates? Can they be unified in a more general framework? Furthermore, the measurement axiom seems *too much ad hoc* and can be unpleasing. In a way, it looks more like an effective rule than a proper physical postulate.

Intertheoretical First of all, how is the postulate compatible with special relativity? As Einstein himself noted, every measurement, not just those of entangled states, is somehow non-local[4]:

---

[4] We tend to think of non-locality as a feature of nonseparable states, and therefore a consequence of the tensor product as a composition postulate. But *"Imagine a single photon approaching a large piece of photographic film, say a square one meter on a side. Quantum mechanics might describe the state of this particle by a wave spread out over the whole square meter. But, as there is only one photon present, only one grain in the photo emulsion can be exposed, by a silver atom absorbing a quantum of light energy. After this event, the wave representing the photon's state must have collapsed to atomic dimensions. (To appreciate this change of scale, note that the area covered by one silver atom is to one square meter roughly as the dot at the end of this sentence is to the North American continent.) The photon has, so to speak, been sucked into the atom, traversing the 10 orders of magnitude separating macroscopic from microscopic in no time at all."*[119]. How did the macroscopic detector *knew* that the wave corresponded to just one particle?

> *The interpretation, according to which $|\psi|^2$ expresses the probability that this particle is found at a given point, assumes an entirely peculiar mechanism of action at a distance, which prevents the wave continuously distributed in space from producing an action in two places on the screen.* [35].

Even in Quantum Field Theory, which is supposed to be a special relativistic quantum mechanical theory, the wavefunction collapse is left as a postulate, and only the *unobserved* evolution is truly special relativistic. In Maudlin's words:

> *"In any orthodox theory the wavefunction is complete and hence must collapse, so we must consider whether collapses could be generated in a relativistically invariant way.[...] The reason that quantum field theory may seem to be evidently compatible with relativity is that the non-local influences in orthodox quantum theory are carried by wave collapse, and wave collapse is commonly ignored in physics texts"* [83].

Furthermore, if classical mechanics is to be a limiting case of quantum mechanics, how does the dynamic duality disappear in the limit?

1.1.2 Interpretations of quantum mechanics

All these questions led to a plethora of interpretations or reformulations of quantum mechanics. There is also a plethora of classifications, and in the following we review three that are particularly enlightening.

J. S. Bell divides the main interpretations in *romantic* and *unromantic* [12]. The *romantic* ones would be:

- The *many worlds interpretation*, according to which there is a universe for each possible outcome of a measurement [102].

How did the part of the wave *over here* know that the screen was detecting *the* particle *over there*?

- Bohr's *complementarity*, which establishes a fundamental scale distinction and asks us to accept the coexistence of a quantum and a classical domain, with different rules [61].
- The *mind-matter dualism* defended by Wigner and Wheeler, among others, which states that the mind does not follow by the same rules than the material world and this is where the wavefunction collapse takes place [85].

While the *unromantic* interpretations would be:

- The introduction of *nonlinear or stochastic terms* into Schroedinger's equation. Some of these approaches are nowadays known as *collapse models*, including GRW theory (see [9] for a review of these models).
- The *pilot wave picture*[5] of Bohm and de Broglie, according to which ordinary quantum mechanics can be extended and made complete and deterministic. See [33, 34] for good modern accounts. So-called *hidden variable theories* would fall in this category.
- The *pragmatic attitude* of the Copenhagen interpretation, which sees quantum mechanics as a mere tool for the computation of probabilities.

> *This pragmatic philosophy is, I think, consciously or unconsciously the working philosophy of all who work with quantum theory in a practical way . .. when so working. We differ only in the degree of concern or complacency with which we view ... out of working hours, so to speak ... the intrinsic ambiguity in principle of the theory* [12].

Tim Maudlin confronts the different interpretations to the measurement problem and concludes that the only allowable interpretations must be either *hidden variable* or *collapse* theories:

> *But at least we can be clear about the questions that must*

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[5] The Bohm-de Broglie theory gives a good account of non-relativistic quantum mechanics, reproducing all and only all of its results, with a realist ontology free from the *measurement problem.* The price to pay is a certain duality in the description of the state and, therefore, of the dynamical laws. In any case, it is clear that Bohm and his followers have been mistreated, and the fierce rejection of their ideas should be shocking for the critical physicist [50].

> *be asked of an interpretation. Is it an additional variables interpretation whose dynamics guarantee solutions to the problem of statistics and the problem of effect? Is it a collapse theory that leads to appropriate outcome states with the right probabilities, and whose fundamental terms all have clear physical significance? If the answer in each case is "no", then commit it to the flames, for it can contain nothing but sophistry and illusion* [82].

Finally, Jean Bricmont lists four possible reactions to the measurement problem:

> *A first reaction is to claim that one cannot understand the microscopic world and that one must content oneself with predicting the results of measurements, which are necessarily macroscopic, and are thus described in a "classical" (i.e., understandable) language. [...]*
>
> *A second reaction follows naturally from this idea, and consists in the hope that, by analyzing the measurement process in more detail, as a purely physical process (with no reference whatsoever to an outside "observer"), one may arrive at an understanding of what is going on.*
>
> *A third reaction is to view the quantum state as representing, not an individual system, but an ensemble of systems and having thus a role similar to probabilities in classical physics [...].*
>
> *A fourth reaction (sometimes motivated by the third) is to propose a more complete theory than quantum mechanics. One would not simply say, as in the third reaction, that particles do have properties not described by the quantum state, but one would try to say what these properties are and how they evolve in time* [21].

He goes on to show that a naive statistical interpretation, corresponding to the *third reaction*, in which all the physical observables have predefined classical values, is simply not possible. Such an interpretation is ruled out by the *no hidden variables theorems* of Kochen and Specker [66] and Bell[6]

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[6] Do not confuse this theorem with the well-known Bell inequalities.

[11]. The second reaction has to be dismissed, too, since it ultimately leads to macroscopic superposiotions (*aka Schroedinger cats*). The *first reaction* is of course the Copenhaguen interpretation, that we reject. This leaves only the *fourth reaction*, that Bricmont embraces in the form of Bohmian mechanics. In the present thesis we pursue precisely this *fourth* path, but we choose to explore connections between QM and thermodynamics instead, in a general way that will support the effective character of the former.

1.1.3 Emergent quantum mechanics

The previous considerations have led to the interpretation of quantum mechanics as an effective theory, emerging from a more fundamental one. The situation would be analogous to the relationship between classical thermodynamics and statistical mechanics, in which the former provides an accurate description of reality without referring to the *fundamental* particles.

In the context of the GR/QM antagony, similar claims have been made for GR. Indeed, if GR and QM are incompatible, it seems reasonable to consider that one of them, or both, is not fundamental.

With respect to GR, Thanu Padmanabhan has been long arguing that gravity *is* thermodynamics [90, 88, 67, 89]. His approach starts from the boltzmannian consideration that *if it can get hot, it must have microstructure* [90]. Now, thanks to the Unruh effect, spacetime events can be given a temperature. Together with symmetry considerations, this leads him to *"the interpretation of gravity as the thermodynamic limit of the kinetic theory of atoms of space"* [90].

Similarly, Erik Verlinde has been arguing that gravity is just an entropic force[7] [116, 117]. The duality, for Newtonian gravity, reads as follows:

> *Let there be given a charge distribution* ρ *which is a density. Then, static Newtonian gravity can be characterized as follows:*
>
> ***A****: There is a scalar field* ϕ *which obeys the Poisson equation* $\nabla^2\phi = 4\pi G\rho$*. A test-mass* m *in the background field of a mass* M *with field* ϕ_M *experiences a force* $\vec{F} = m\vec{\nabla}\phi_M$*.*

---

[7] An idea that granted him the Spinoza prize, the highest Dutch scientific award.

> *Here, G is some coupling constant. Verlinde's proposal instead can be cast as follows*
>
> ***B**: There are two scalar quantities S and T and a continuous set of non-intersecting surfaces $\mathcal{S}$, the 'holographic screens', whose union covers all of space $\mathbb{R}^3 = \bigcup \mathcal{S}$. The theory is defined by $2G \int_{(\mathcal{S})} \rho dV = \int_S T dA \; \forall \mathcal{S}$, and the force acting on a particle with test-mass m is given by $F\delta x = \int_S T \delta dS$, where the integral is taken over a screen that does not include the test-mass.*
>
> *Here, $(\mathcal{S})$ denotes the volume with surface $\mathcal{S}$. The volume integral $\int_{(\mathcal{S})} dV \rho$ is of course just the total mass M inside that volume, and the quantities S and T are interpreted as the entropy respectively temperature on the holographic screens. The δx is a virtual variation on the location on the particle which induces a change in the entropy on the screen* [59].

With respect to QM, perhaps the most famous emergent interpretation is that due to Nobel prize Gerard 't Hooft. He shows that the properties of quantum systems in finite-dimensional Hilbert spaces can be reproduced by classical systems: *"For any quantum system there exists at least one deterministic model that reproduces all its dynamics after prequantization"* [111]. According to him, quantum mechanics would just be a mathematical tool that averages information about more fundamental *ontic states*, which can be modeled by means of cellular automata with information loss (the recent book [113] summarizes his numerous publications on the topic). Note that all the *classical mechanics no-go theorems* (such as those compiled by David Jennings and Matthew Leifer [60]) must be taken with care, since they do not necessarily apply in an emergent context[8].

Despite being the most widely known, and having inspired many[9], 't Hooft's is not the only approach. *"There is a growing number of deterministic models of quantum mechanical objects which are based on conjectured fundamental information loss or dissipation mechanisms"* [36]. To name but a few, in the literature one finds *quantum graphity* ([68, 79], see also [80]); Adler's

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[8] In the same way that they do not apply for Bohmian mechanics. Careless critics keep citing them as a counter-argument.

[9] Notably Giuseppe Vitiello, Massimo Blasone and collaborators [16, 15, 14, 17] and Hans-Thomas Elze, who has dozens of papers on the subject (we only cite here the last one, [39]).

trace dynamics [5]; or quantumness emerging from classic systems with fluctuations [56]. Roger Penrose argues that gravity should modify *"'the very formalism of quantum mechanics"* [93].

In a related vein, although not at all emergentist, John Baez[10] studied an analogy between quantum mechanics and statistical mechanics and proposed a quantity called *Quantropy* as the analogue of entropy on the quantum mechanical side [8]. In line with this, there exists some literature exploiting *"quantum techniques for stochastic mechanics"* [7], since path integrals are somehow ubiquitous (see [64]).

It is interesting to remark that just a few months after Schroedinger published his equation, Erwin Madelung gave a hydrodynamical - effective, emergent - interpretation of it [76]. He mapped the nowadays known as Quantum Hamilton-Jacobi or Hamilton-Jacobi-Madelung equation with the Euler equation. The former equation results from writing the wavefunction explicitly as amplitude and phase, inserting it into the Schröedinger equation and separating into real and imaginary parts. The result is an equation that looks much like Euler's equation, except for the fact that it contains an *extra* term which, in Madelung's words, *"represents the internal mechanism of the continuum"*. To recover the Schroedinger equation from the Euler equation one needs to assume that $\oint_L \mathbf{v} \cdot dl = 2\pi n$, where n is an integer and L is any closed loop in space [118]. But this corresponds precisely to the old *Bohr-Sommerfeld quantization condition*, which turns out to be equivalent to the semiclassical approximation [71]. Recent references study the *Madelung equations* from a geometric perspective [97, 51, 73], which reflects a permanent interest on the topic.

1.2 Goals

The main goal of this thesis is to investigate, in the context of emergent quantum mechanics, similarities between non-relativistic quantum mechanics (QM) and some thermodynamical theories. In particular, we investigate:

- A duality between QM and viscous fluid mechanics.

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[10] Who I keep calling *my favorite mathematician alive.*

- A duality between QM and markovian irreversible thermodynamics in the linear regime.
- How a quantum mechanical theory could be constructed from a dual classical thermostatics.

1.3 Prior work

This thesis partially continues with the work of the PhD dissertations of Dagoberto Acosta [2] and Milton H. Perea [94]. In [2] (and the associated papers [3], [4]), the similarity of QM with Onsager's irreversible thermodynamics was first noticed. In [94] (and the associated papers [41], [42], [43]) that duality was developed. Both thesis were advised by P. Fernández de Córdoba and J. M. Isidro, advisers of the present thesis.

1.4 Structure of the thesis

This thesis is submitted under the *article compendium* format[11].

- Chapter 1 serves as an introduction, sets the goals and presents the necessary theoretical background for a proper understanding of this thesis. Familiarity with fluid mechanics, classical thermodynamics and non-relativistic quantum mechanics is assumed.
- Chapter 2 is based on the publication [45].
- Chapter 3 is based on the publication [44].
- Chapter 4 is based on the publication [22].
- Chapter 5 serves as a discussion of the results and a conclusion.

Chapters 2, 3 and 4 are compiled from the versions available on the arXiv with no relevant content modifications. These .tex files are the *preprint*

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[11] For the regulations, see Normativa de los estudios de doctorado en la Universitat Politècnica de València, published in *Butlletí Oficial de la Universitat Politècnica de València 54*, available at http://www.upv.es/entidades/EDOCTORADO/info/798159normalc.html

author version, and contain already all the modifications suggested by the reviewers during the publishing process. Therefore, the chapters correspond faithfully to the published articles. The only notable change is in the references, which was necessary in order to provide a single unified bibliography for this PhD thesis.

Chapter 2

Schroedinger vs. Navier-Stokes

This chapter is based on the article Schroedinger vs. Navier–Stokes, by P. Fernández de Córdoba, J.M. Isidro and J. Vazquez Molina, published in *Entropy 18* (2016) doi:10.3390/e18010034, available electronically as arXiv: 1409.7036 [math-ph], and quoted in the bibliography as ref [45].

2.1 Introduction

Interaction with an environment provides a mechanism whereby classical behaviour can emerge from a quantum system [122]. At the same time, however, dissipation into an environment can change this picture towards the opposite conclusion. Indeed certain forms of quantum behaviour have been *experimentally* shown to arise within classical systems subject to dissipation [28, 96]. Now systems in thermal equilibrium are well described by classical thermostatics, while small deviations from thermal equilibrium can be described by the classical thermodynamics of irreversible processes [87]. It is sometimes possible to model long–wavelength dissipative processes through the dynamics of *viscous* fluids. Fluid viscosity provides a relatively simple dissipative mechanism, a first deviation from ideal, frictionless behaviour. Two relevant physical quantities useful to characterise viscous fluids are shear viscosity η and the entropy per unit 3–volume, s [70]. In a turn of events leading back to the Maldacena conjecture [77] it was found that, for a wide class of thermal quantum field theories in 4 dimensions, the ratio η/s for the quark–gluon plasma must satisfy the inequality [69]

$$\frac{\eta}{s} \geq \frac{\hbar}{4\pi k_B}. \tag{2.1}$$

The predicted value of the ratio η/s for the quark–gluon plasma has found experimental confirmation [75]. The simultaneous presence of Planck's constant $\hbar$ and Boltzmann's constant k_B reminds us that we are dealing with theories that are both *quantum* and *thermal*.

One might be inclined to believe that these two properties, *quantum* on the one hand, and *thermal* on the other, are separate. One of the purposes of this paper is to show that this predisposition must be modified, at least partially, because the terms *quantum* and *thermal* are to a large extent linked (see *e.g.* [42, 67] and refs. therein). In fact, that these two properties belong together follows from the analysis of refs. [96, 122], even if the conclusions of these two papers seem to point in opposite directions.

In this article we elaborate on a theoretical framework that can accomodate the ideas of the previous paragraph. In plain words, this framework can be summarised in the statement *quantum* = *classical* + *dissipation*, although of course this somewhat imprecise sentence must be made precise. To begin with, we will restrict our analysis to quantum systems with a finite number of degrees of freedom. So we will be dealing not with theories of fields, strings and branes, but with plain quantum mechanics instead.

In the early days of quantum mechanics, Madelung provided a very intuitive physical interpretation of the Schroedinger wave equation in terms of a probability fluid [76]. Decomposing the complex wavefunction ψ into amplitude and phase, Madelung transformed the Schroedinger wave equation into an equivalent set of two: the quantum Hamilton–Jacobi equation, and the continuity equation. Further taking the gradient of the phase of ψ, Madelung arrived at a velocity field satisfying the Euler equations for an *ideal* fluid. In Madelung's analysis, the quantum potential U is interpreted as being (proportional to) the pressure field within the fluid. It is important to stress that Madelung's fluid was ideal, that is, *frictionless*. Independently of this analogy, Bohm suggested regarding the quantum potential U as a force field that the quantum particle was subject to, in addition to any external, classical potential V that might also be present [18].

There exists yet a third, so far unexplored alternative to Madelung's and Bohm's independent interpretations of the quantum potential. In this alternative, explored here, *the quantum potential is made to account for a dissipative term in the equations of motion of the probability fluid*. The velocity field no longer satisfies Euler's equation for an ideal fluid—instead it satisfies the Navier–Stokes equation for a *viscous* fluid. It is with this viscosity term in the Navier–Stokes equation, and its physical interpretation as deriving from the Schroedinger equation, that we will be concerned with in this paper.

It has long been argued that quantum mechanics must emerge from an underlying classical, deterministic theory via some coarse–graining, or information–loss mechanism [36, 37, 48, 53, 52, 111, 109, 113]; one refers to this fact as the *emergence property* of quantum mechanics [25]. Many emergent physical theories admit a thermodynamical reformulation, general relativity being perhaps the best example [88, 116]. Quantum mechanics is no exception [30, 81]; in fact our own approach [42, 43] to the emergence property of quantum mechanics exploits a neat correspondence with the classical thermodynamics of irreversible processes [87].

In this article, the dissipation that is intrinsic to the quantum description of the world will be shown to be ascribable to the viscosity η of the *quantum probability fluid* whose density equals Born's amplitude squared $|\psi|^2$. Moreover, the viscosity η will turn out to be proportional to $\hbar$, thus vanishing in the limit $\hbar \to 0$. Now mechanical action (resp. entropy) is quantised in units of Planck's constant $\hbar$ (resp. Boltzmann's constant k_B), and Eq. (2.1) contains these two quanta. (Concerning Boltzmann's constant k_B as a quantum of entropy, see refs. [72, 116]). Hence an important implication of our statement *quantum* $=$ *classical* $+$ *dissipation* is that quantum and thermal effects are inextricably linked.

Some remarks on conventions are in order; we follow ref. [70]. The viscosity properties of a fluid can be encapsulated in the viscous stress tensor σ'_{ik},

$$\sigma'_{ik} := \eta \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l} \right) + \zeta \delta_{ik} \frac{\partial v_l}{\partial x_l}, \tag{2.2}$$

where η (shear viscosity) and ζ (bulk viscosity) are positive coefficients, and the v_i are the components of the velocity field $\mathbf{v}$ within the fluid. Then the Navier–Stokes equation reads

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} + \frac{1}{\rho}\nabla p - \frac{\eta}{\rho}\nabla^2\mathbf{v} - \frac{1}{\rho}\left(\zeta + \frac{\eta}{3}\right)\nabla\left(\nabla\cdot\mathbf{v}\right) = 0. \tag{2.3}$$

Here p is the pressure, and ρ the density of the fluid. In the particular case of *irrotational* flow considered here, the Navier–Stokes equation simplifies to

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} + \frac{1}{\rho}\nabla p - \frac{\eta'}{\rho}\nabla^2\mathbf{v} = 0, \qquad \eta' := \zeta + \frac{4\eta}{3}. \tag{2.4}$$

For notational simplicity, in what follows we will systematically write η for the viscosity coefficient η' just defined, bearing in mind, however, that we will always be dealing with Eq. (2.4) instead of (2.3).

The above must be supplemented with the continuity equation and the equation for heat flow. If T denotes the temperature and κ the thermal conductivity of the fluid, then the equation governing heat transfer within the fluid reads

$$\rho T\left(\frac{\partial s}{\partial t} + (\mathbf{v}\cdot\nabla)s\right) - \sigma'_{ik}\frac{\partial v_i}{\partial x_k} - \nabla\cdot(\kappa\nabla T) = 0. \tag{2.5}$$

We will use the notations $\mathcal{I}$ and $\mathcal{S}$ for mechanical action and entropy, respectively, while the dimesionless ratios $\mathcal{I}/\hbar$ and $\mathcal{S}/2k_B$ will be denoted in italic type:

$$I := \frac{\mathcal{I}}{\hbar}, \qquad S := \frac{\mathcal{S}}{2k_B}. \tag{2.6}$$

The factor of 2 multiplying k_B, although conventional, can be justified. By Boltzmann's principle, the entropy of a state is directly proportional to the logarithm of the probability of that state. In turn, this is equivalent to Born's rule:

$$\text{(Boltzmann)} \quad \mathcal{S} = k_B \ln\left(\left|\frac{\psi}{\psi_0}\right|^2\right) \Longleftrightarrow |\psi|^2 = |\psi_0|^2 \exp\left(\frac{\mathcal{S}}{k_B}\right) \quad \text{(Born)}. \tag{2.7}$$

Above, $|\psi_0|$ is the amplitude of a fiducial state ψ_0 with vanishing entropy. Such a fiducial state is indispensable because the argument of the logarithm in Boltzmann's formula must be dimensionless. It is convenient to think of ψ_0 as being related to a 3–dimensional length scale l defined through

$$l := |\psi_0|^{-2/3}. \tag{2.8}$$

One can also think of ψ_0 as a normalisation factor for the wavefunction.

2.2 The physics of Navier–Stokes from Schroedinger

2.2.1 Computation of the viscosity

Our starting point is Madelung's rewriting of the Schroedinger equation for a mass m subject to a static potential $V = V(\mathbf{x})$,

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\nabla^2\psi - V\psi = 0, \tag{2.9}$$

by means of the substitution

$$\psi = \psi_0 \exp\left(S + \frac{\mathrm{i}}{\hbar}\mathcal{I}\right) = \psi_0 A \exp\left(\frac{\mathrm{i}}{\hbar}\mathcal{I}\right), \qquad A := \mathrm{e}^S. \tag{2.10}$$

This produces, away from the zeroes of ψ, an equation whose imaginary part is the continuity equation for the quantum probability fluid,

$$\frac{\partial S}{\partial t} + \frac{1}{m}\nabla S\cdot\nabla\mathcal{I} + \frac{1}{2m}\nabla^2\mathcal{I} = 0, \tag{2.11}$$

and whose real part is the quantum Hamilton–Jacobi equation:

$$\frac{\partial\mathcal{I}}{\partial t} + \frac{1}{2m}(\nabla\mathcal{I})^2 + V + U = 0. \tag{2.12}$$

Here

$$U := -\frac{\hbar^2}{2m}\frac{\nabla^2 A}{A} = -\frac{\hbar^2}{2m}\left[(\nabla S)^2 + \nabla^2 S\right] \tag{2.13}$$

is the quantum potential [18]. Next one defines the velocity field of the quantum probability fluid

$$\mathbf{v} := \frac{1}{m}\nabla\mathcal{I}. \tag{2.14}$$

Then the gradient of Eq. (2.12) equals

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} + \frac{1}{m}\nabla U + \frac{1}{m}\nabla V = 0. \tag{2.15}$$

The flow (2.14) is irrotational. We will sometimes (though not always) make the assumption of incompressibility, $\nabla \cdot \mathbf{v} = 0$. This reduces to the requirement that the phase $\mathcal{I}$ satisfy the Laplace equation,

$$\nabla^2 \mathcal{I} = 0. \tag{2.16}$$

We will see in Eq. (2.23) that the above Laplace equation is an equivalent restatement of the semiclassicality condition.

At this point we deviate from Madelung's reasoning and compare Eq. (2.15) not to Euler's equation for an ideal fluid, but to the Navier–Stokes equation instead, Eq. (2.4). For the correspondence to hold, we first identify $(\nabla p)/\rho$ with $(\nabla V)/m$. Second, it must hold that

$$\frac{1}{m}\nabla U + \frac{\eta}{\rho}\nabla^2 \mathbf{v} = 0. \tag{2.17}$$

That is, the gradient of the quantum potential must exactly compensate the viscosity term in the fluid's equations of motion. Thus frictional forces within the fluid are quantum in nature. Altogether, we have established the following:

Theorem 1 *Whenever condition (2.17) holds, the gradient of the quantum Hamilton–Jacobi equation, as given by Eq. (2.15), is a Navier–Stokes equation for irrotational, viscous flow:*

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} - \frac{\eta}{\rho}\nabla^2 \mathbf{v} + \frac{1}{\rho}\nabla p = 0. \tag{2.18}$$

Here the pressure p of the quantum probability fluid and the mechanical potential V are related as per

$$\frac{1}{\rho}\nabla p = \frac{1}{m}\nabla V, \tag{2.19}$$

while the density ρ of the fluid is given by

$$\rho = m|\psi|^2 = \frac{m}{l^3}\mathrm{e}^{2S} = \frac{m}{l^3}A^2. \tag{2.20}$$

Given V, m and ρ, the equation $(\nabla p)/\rho = (\nabla V)/m$ defines a vector field $\mathbf{p} = \rho\nabla V/m$, that however need not be a gradient field ∇p. We will see later (theorem 4) that, at least in the classical limit, the above equation is integrable, thus defining a scalar function p such that $\mathbf{p} = \nabla p$.

The order of magnitude of the viscosity coefficient η can be inferred from Eqs. (2.13), (2.14) and (2.17): since U is $O(\hbar^2)$ and $\mathcal{I}$ is $O(\hbar)$, we conclude:

Theorem 2 *Whenever condition (2.17) holds, the viscosity coefficient η of the quantum probability fluid is proportional to Planck's constant:*

$$\eta = \frac{1}{l^3} O\left(\hbar\right). \tag{2.21}$$

It is worthwhile stressing that Eq. (2.21) only provides an order of magnitude for η as a function of $\hbar$—namely, η is a linear function of $\hbar$. The denominator l^3 has been included for dimensional reasons, while a dimensionless factor multiplying the right–hand side of Eq. (2.21) is allowed.[1] Moreover, this dimensionless factor will generally depend on the quantum state under consideration, because both U and $\mathcal{I}$ are state–dependent. Although the viscosity of the quantum probability fluid depends, through an undetermined dimensionless factor, on the quantum state, the order of magnitude provided by Eq. (2.21) is universal.

2.2.2 Viscous states *vs.* dissipation–free states

Condition (2.17) need not be satisfied by all wavefunctions, as the functions S and $\mathcal{I}$ are already determined by the quantum Hamilton–Jacobi equation and by the continuity equation. Thus our next task is to exhibit a class of quantum–mechanical wavefunctions for which condition (2.17) is indeed satisfied, either exactly or at least approximately.

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[1] This dimensionless factor is undetermined, in the sense that our argument does not provide its precise value—not in the sense that the viscosity η is undetermined.

Exact solutions

Eq. (2.17) integrates to

$$U + \frac{\eta}{\rho}\nabla^2\mathcal{I} = C_0(t), \qquad C_0(t) \in \mathbb{R}, \tag{2.22}$$

where the integration constant $C_0(t)$ may generally depend on the time variable. Let us for simplicity set $C_0(t) = 0$. Using (2.13) and (2.20) the above becomes

$$\frac{2\eta l^3}{\hbar^2}\nabla^2\mathcal{I} = \mathrm{e}^{2S}\left[(\nabla S)^2 + \nabla^2 S\right]. \tag{2.23}$$

One can regard (2.23) as a Poisson equation $\nabla^2\Phi = \varrho$, where the role of the electric potential Φ is played by the phase $\mathcal{I}$ and that of the charge density ϱ is played by the right–hand side of Eq. (2.23). The bracketed term, $(\nabla S)^2 + \nabla^2 S$, is actually proportional to the Ricci scalar curvature of the conformally flat metric $g_{ij} = \mathrm{e}^{-S(\mathbf{x})}\delta_{ij}$, where δ_{ij} is the Euclidean metric on $\mathbb{R}^3$. Eq. (2.23) has been dealt with in ref. [1], in connection with the Ricci–flow approach to emergent quantum mechanics. For the moment we will relax the requirement that Eq. (2.17) hold exactly, and will satisfy ourselves with approximate solutions instead.

Approximate solutions

Under the assumption that ρ is spatially constant, Eq. (2.17) integrates to

$$U(\mathbf{x}, t) = C_1(t), \qquad C_1(t) \in \mathbb{R}, \tag{2.24}$$

where Eqs. (2.14) and (2.16) have been used; the integration constant $C_1(t)$ may however be time–dependent. Equivalently, one may assume that S in (2.23) is approximately constant as a function of the space variables, hence $\mathcal{I}$ is an approximate solution of the Laplace equation (2.16). Still another way of arriving at (2.24) is to assume the flow to be approximately incompressible, $\nabla\cdot\mathbf{v} \simeq 0$. Of course, $\rho = mA^2/l^3$ is generally not spatially constant. However, in the semiclassical limit, the amplitude $A = \mathrm{e}^S$ is a slowly–varying function of the space variables. Under these assumptions, Eq. (2.24) holds approximately:

Theorem 3 *In the semiclassical limit, the sufficient condition (2.17) guaranteeing the validity of the Navier–Stokes equation is equivalent to Eq. (2.24).*

We can now consider the effect of taking the semiclassical limit in the identification $(\nabla p)/\rho = (\nabla V)/m$ made in Eq. (2.19). In this limit ρ is approximately constant, and the above identification defines an integrable equation for the scalar field p. Therefore:

Theorem 4 *In the semiclassical limit, the identification* $(\nabla p)/\rho = (\nabla V)/m$ *made in Eq. (2.19) correctly defines a scalar pressure field* p *within the probability fluid.*

In the stationary case, when $\psi = \phi(\mathbf{x}) \exp(-\mathrm{i}Et/\hbar)$, the quantum potential becomes time–independent, and condition (2.24) reduces to the requirement that U be a constant both in space and in time:

$$U(\mathbf{x}) = C_2, \qquad C_2 \in \mathbb{R}. \tag{2.25}$$

Theorem 5 *In the semiclassical limit of stationary eigenfunctions, the sufficient condition (2.17) guaranteeing the validity of the Navier–Stokes equation is equivalent to Eq. (2.25).*

One expects semiclassical stationary states to possess vanishing viscosity because, having a well–defined energy, they are dissipation–free. This expectation is borne out by a simple argument: Eq. (2.17) and the (approximate) spatial constancy of U imply $\eta\nabla^2\mathbf{v} = 0$. This reduces the Navier–Stokes equation (2.4) to the Euler equation for a perfect fluid. Therefore:

Theorem 6 *All semiclassical stationary states have vanishing viscosity:* $\eta = 0$.

Thus, as far as dissipation effects are concerned, the combined assumptions of stationarity and semiclassicality lead to a dead end. Furthermore, we cannot lift the requirement of semiclassicality because stationarity alone does not guarantee that the sufficient condition (2.17) holds. Even if we per decree assign a non–semiclassical but stationary state $\eta = 0$, that state need not satisfy condition (2.17)—the very assignment of a viscosity η would be flawed.

A physically reasonable assumption to make is that viscosity must be proportional to the density of the fluid:

$$\eta = C_3 \rho. \tag{2.26}$$

Here C_3 is some dimensional conversion factor that does not depend on the space variables: $C_3 \neq C_3(\mathbf{x})$. Then Eq. (2.17) integrates to

$$U + mC_3 \left(\nabla \cdot \mathbf{v}\right) = C_4, \qquad C_4 \in \mathbb{R}. \tag{2.27}$$

When the flow is incompressible, $\nabla \cdot \mathbf{v} = 0$, and Eq. (2.27) reduces to the case already considered in Eqs. (2.24) and (2.25). Thus the proportionality assumption (2.26) provides an independent rationale for the semiclassical approximation made earlier, and viceversa. In turn, this shows that the semiclassicality condition can be recast as done in Eq. (2.16). We conclude:

Theorem 7 *In the semiclassical limit, the viscosity η is proportional to the density ρ of the quantum probability fluid. In particular, the viscosity η is approximately spatially constant for semiclassical states. Moreover, the proportionality factor C_3 in Eq. (2.26) is linear in Planck's constant $\hbar$:*

$$C_3 = \frac{\hbar}{m} f. \tag{2.28}$$

Here $f \geq 0$ is an arbitrary dimensionless factor. By what was said previously, $f = 0$ when the state considered is an energy eigenstate, while $f > 0$ on all other states. Hence f is best thought of as a function $f : \mathcal{H} \to \mathbb{R}$ on the Hilbert space $\mathcal{H}$ of quantum states.

Having exhibited the existence of approximate solutions to condition (2.17), whenever dealing with dissipation effects we will restrict our discussion to *nonstationary states*.

2.2.3 The ratio of viscosity to entropy density

We have interpreted dissipation as a quantum effect within the probability fluid. Hence the increase $\mathrm{d}s/\mathrm{d}t$ in the volume density of entropy of the probability fluid also qualifies as a quantum effect. Here we will compute $\mathrm{d}s/\mathrm{d}t$ in the semiclassical regime, both for stationary and nonstationary states.

Considering a stationary state first, we expect $\mathrm{d}s/\mathrm{d}t = 0$ because $\eta = 0$. This expectation is confirmed by the following alternative argument. We see that Eq. (2.5) reduces to

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\partial s}{\partial t} + (\mathbf{v} \cdot \nabla) s = \frac{\kappa}{\rho} \frac{\nabla^2 T}{T}, \tag{2.29}$$

because the dissipation term σ'_{ik} vanishes. On the other hand, by Boltzmann's principle (2.7) we can write the entropy $\mathcal{S}$ in terms of the amplitude $A = \mathrm{e}^S$ as

$$\mathcal{S} = 2k_B \ln \left(\left| \frac{\psi}{\psi_0} \right| \right) = 2k_B \ln A. \tag{2.30}$$

This is reminiscent of the expression for the entropy of an ideal gas as a function of its temperature, *viz.* $\mathcal{S} = gk_B \ln(T/T_0)$, with g a dimensionless number and T_0 some fixed reference temperature. Which suggests identifying the quantum–mechanical amplitude A with the thermodynamical temperature T, at least in the absence of friction—as is indeed the case for stationary states and for the ideal gas. So we set

$$A = \frac{T}{T_0}. \tag{2.31}$$

Thus $\nabla^2 A = 0$ implies $\nabla^2 T = 0$. In the semiclassical approximation, A is a slowly–varying function, and one can approximate $\nabla^2 A$ by zero. Thus substituting Eq. (2.31) into Eq. (2.29), we arrive at a counterpart to theorem 5:

Theorem 8 *In the semiclassical approximation, the entropy density of any stationary state is constant in time:* $\mathrm{d}s/\mathrm{d}t = 0$.

Our next task is to obtain an estimate for the order of magnitude of the entropy density s. This is readily provided by Eq. (2.30):

Theorem 9 *In the semiclassical approximation, the volume density of entropy s of the quantum probability fluid is proportional to Boltzmann's constant:*

$$s = \frac{1}{l^3} O\left(k_B\right). \tag{2.32}$$

As already mentioned regarding Eq. (2.21), the denominator l^3 has been included for dimensional reasons, and an undetermined, dimensionless factor multiplying the right–hand side is allowed. Finally combining Eqs. (2.21) and (2.32) together we can state:

Theorem 10 *For the quantum probability fluid in the semiclassical approximation, the order of magnitude of the ratio of viscosity to entropy density is*

$$\frac{\eta}{s} = O\left(\frac{\hbar}{k_B}\right). \tag{2.33}$$

Again an undetermined, dimensionless factor multiplying the right–hand side is allowed, but the dependence on the length scale l has dropped out.

2.2.4 Nonstationary states: emergent reversibility

Nonstationary states can be readily constructed as linear combinations of stationary eigenstates with different energy eigenvalues. The ratio η/s of the viscosity to the entropy density of a nonstationary state is important for the following reason. Any nonstationary state thermalises to a final equilibrium state. The time required for this transition is of the order of the Boltzmann time τ_B,

$$\tau_B := \frac{\hbar}{k_B T}, \tag{2.34}$$

where T is the temperature of the final equilibrium state [55]. In Eq. (2.31) we have related the temperature T to the amplitude $A = |\psi_{\rm eq}|$ of the equilibrium state wavefunction $\psi_{\rm eq}$. Therefore:

Theorem 11 *For semiclassical, nonstationary states of the quantum probability fluid, the Boltzmann time is directly proportional to the ratio* η/s *of the viscosity to the entropy density of the initial state, and inversely proportional to the amplitude of the final equilibrium state.*

Out of this analysis there arises a nice picture of the thermalisation process, whereby a nonstationary state decays into a final stationary state. In this picture we have a slow dynamics superimposed on a fast dynamics. The latter corresponds to nonstationary states; the former, to stationary states. Viscous states correspond to the fast dynamics, while dissipation–free states pertain to the slow dynamics. Time reversibility emerges as a conservation law that applies only to the emergent, slow dynamics.

2.2.5 Stationary states: emergent holography

Turning now our attention to stationary states, let us see how an emergent notion of holography arises naturally in our context. For stationary states we first set $\partial S/\partial t = 0$ in the continuity equation (2.11), then apply the semiclassicality condition (2.16), next divide through by $\hbar$ and finally switch from $\mathcal{I}$ to I as per Eq. (2.6). This establishes:

Theorem 12 *For semiclassical stationary states we have*

$$\nabla I \cdot \nabla S = O\left(l^{-2}\right). \tag{2.35}$$

For such states, Eqs. (2.25) and (2.35) are equivalent.

In the limit $l \to \infty$ we have $\nabla I \cdot \nabla S = 0$, and the foliation $I = \text{const}$[2] intersects orthogonally the foliation $S = \text{const}$. That the length scale l, in our case of semiclassical stationary states, can be regarded as being sufficiently large, follows from Eq. (2.8). Indeed a classical, perfectly localised state around $\mathbf{x} = \mathbf{x}_0$ carries a wavefunction $\delta(\mathbf{x} - \mathbf{x}_0)$, the amplitude of which is almost everywhere zero. As this localised state spreads out, ceasing to be perfectly classical, its width can be taken as an inverse measure of its localisation. In other words, the limit $\hbar \to 0$ is equivalent to the limit $l \to \infty$. Thus neglecting the right–hand side of Eq. (2.35) we arrive at:

Theorem 13 *Semiclassical stationary states provide two independent foliations of 3–dimensional space by two mutually orthogonal families of 2–dimensional surfaces, respectively defined by* $I = \text{const}$ *and by* $S = \text{const}$.

The foliation $I = \text{const}$ is well known since the early days of quantum theory. On the other hand the foliation $S = \text{const}$ was little used in mechanical contexts until the groundbreaking contributions of refs. [49, 88, 116] to the notion of *emergent spacetime*. Specifically, in ref. [116], isoentropic surfaces $S = \text{const}$ are taken to be holographic screens, while also qualifying as equipotential surfaces $V = \text{const}$ of the gravitational field. We see immediately that:

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[2] This is abuse of language. Strictly speaking, the equation $I = \text{const}$ defines only one leaf of the foliation. The foliation itself is the union of all the leaves obtained by letting the constant run over the corresponding range.

Theorem 14 *Under the above assumptions of stationarity and semiclassicality,*
i) the vector field ∇I *is parallel to the foliation* $S = \text{const}$*;*
ii) the vector field ∇S *is parallel to the foliation* $I = \text{const}$*;*
iii) whenever $\nabla I \neq 0 \neq \nabla S$*, the vector fields* ∇I *and* ∇S *define an integrable 2–dimensional distribution on* $\mathbb{R}^3$*.*

The integrability of the distribution defined by the vector fields ∇I and ∇S follows from the semiclassicality property $\nabla I \cdot \nabla S = 0$. Then Frobenius' theorem guarantees the existence of a family of 2–dimensional integral manifolds for the distribution.[3] Each leaf of this integral foliation, that we denote by $F = \text{const}$, is such that its two tangent vectors ∇S and ∇I point in the direction of maximal increase of the corresponding quantities, S and I. Therefore:

Theorem 15 *Under the above assumptions of stationarity and semiclassicality, the foliation* $F = \text{const}$ *is orthogonal to the two foliations* $S = \text{const}$ *and* $I = \text{const}$ *simultaneously.*

According to ref. [116], the leaves $S = \text{const}$ are holographic screens, enclosing that part of space that can be regarded as having emerged. We see that the leaves $I = \text{const}$ play an analogous role with respect to the time variable. Now the wavefunction contains both amplitude and phase. Hence the two foliations $S = \text{const}$ and $I = \text{const}$ must appear on the same footing—as is actually the case. Taken together, these facts can be renamed as the *holographic property of emergent quantum mechanics*. To be precise, this holographic property has been analysed here in the semiclassical regime only.

2.3 Discussion

To first order of approximation, any viscous fluid can be characterised by its viscosity coefficients and by its volume density of entropy. In this paper

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[3] A purely differential–geometric proof of this statement can be found in ref. [65]; a related theorem by Liouville, in the context of classical integrability theory, can be found in ref. [6].

we have obtained an estimate for the order of magnitude of these quantities, in the case of irrotational flow, for the quantum probability fluid. Our analysis makes decisive use of Madelung's factorisation of the quantum wavefunction into amplitude and phase. However, we deviate substantially from Madelung on the following key issue: *Madelung's probability fluid is ideal, while our is viscous*. Correspondingly, Madelung's fluid satifies Euler's equation for a perfect fluid, while ours satisfies the Navier–Stokes equation. Consequently, the pressure within the fluid is also different: in Madelung's analysis, pressure is (proportional to) the quantum potential U, while our pressure is (proportional to) the external potential V in the Schroedinger equation. In our alternative approach, *the quantum potential is responsible for the appearance of viscosity*. Thus classical friction in the fluid can be regarded as the origin of quantum effects. Moreover, the dissipation that is inherent to quantum phenomena, under the guise of viscosity in our case, is a nonstationary phenomenon.

By letting the quantum potential account for the viscosity of the probability fluid, our analysis lends support to the emergent paradigm of quantum mechanics: the resulting theory, once dissipation has been taken into account, is no longer classical but quantum. We regard viscosity as the dissipation, or information–loss mechanism, whereby the fluid described by the Navier–Stokes equation (a classical process) becomes the quantum wavefunction satisfying the Schroedinger equation (a quantum process). This mechanism illustrates the statement *quantum* = *classical* + *dissipation* made in the introductory section.

Chapter 3

The Holographic Quantum

This chapter is based on the article The Holographic Quantum, by P. Fernández de Córdoba, J.M. Isidro and J. Vazquez Molina, published in *Foundations of Physics* ***46*** (2016), pp 787–803, DOI 10.1007/s10701-015-9986-2, available electronically as arXiv:1503.07662v2 [quant-ph], and quoted in the bibliography as ref [44].

3.1 Introduction

The holographic principle [19, 106, 107] has permeated wide areas of theoretical physics over the last twenty years. Stepping outside its initial quantum–gravity framework, it reached string theory [77, 120] as well as more established domains such as QCD [63] and condensed matter theory [58], to name but a few.

Another theoretical development of recent years is the recognition that *gravity arises as an emergent phenomenon* [89, 88, 116], a fact that has far–reaching consequences for our understanding of spacetime. Added to the dissipative properties already known to be exhibited by gravity [110, 91, 103, 104], this opens the gate to the application of thermodynamics to (supposedly) nonthermal physics. Indeed, thermodynamics is the paradigm of emergent theories. It renounces the knowledge of a vast amount of detailed microscopic information, keeping just a handful of macroscopic variables such as volume, pressure and temperature—sufficient to state robust physical laws of almost universal applicability. These macroscopic variables are coarse–grained averages over the more detailed description provided by some underlying, microscopic degrees of freedom. Which brings us to yet another theoretical breakthrough of recent times that is worthy of mention: the notion of *emergence* [25].

The property of emergence has been postulated not only of gravity, but also of Newtonian mechanics [116] and of quantum mechanics [36, 111]; a key concept here is that of an *entropic force*. Equipped with thermodynamical tools as befits any emergent theory, we have in refs. [42, 43, 45] developed a framework that maps semiclassical quantum mechanics onto the classical thermodynamics of irreversible processes in the linear regime, the latter as developed by Onsager, Prigogine and collaborators [87, 95]. Within this framework, the statement often found in the literature, *quantisation is dissipation* [17], can be given a new interpretation.

In this paper we elaborate further on the above–mentioned map of semiclassical quantum mechanics onto the classical theory of linear, irreversible processes (sections 3.2 and 3.3); we call these two theories *dual* to each other. From there we move on to the nonlinear regime of the thermodynamics or, equivalently, to the quantum regime beyond the Gaussian approximation (section 3.4). Next we formulate a holographic–like principle for quantum mechanics (section 3.5) and place it in correspondence with the second law of thermodynamics (section 3.6)). The term *holographic–like* is meant to stress that, while it is true that no gravity is present in our framework, an undeniable conceptual similarity with the holographic principle of quantum gravity underlies the principle postulated here. We summarise our conclusions in section 3.7.

A word on notation is in order. Rather than using natural units, we will explicitly retain Planck's constant $\hbar$ and Boltzmann's constant k_B in our expressions, in order to better highlight the properties of the map presented here between quantum mechanics and irreversible thermodynamics. In particular, the role that $\hbar$ plays on the mechanical side of our correspondence will be played by k_B on the thermodynamical side. If we were to set $\hbar = 1 = k_B$, the fact that they are counterparts under our correspondence [27, 67] would be somewhat obscured.

3.2 Basics in irreversible thermodynamics

The following is a very brief summary of some notions of irreversible thermodynamics [87, 95] that we will make use of.

Let an irreversible thermodynamical system be characterised by its entropy function S. Assume that the thermodynamical state of the system is determined by just one extensive variable $x = x(\tau)$, where τ is time variable. We can thus write $S = S(x(\tau))$. At any instant of time, the probability P of a state is given by Boltzmann's principle,

$$k_B \ln P = S + \text{const.} \tag{3.1}$$

Let S_0 denote the maximum (equilibrium) value of S, and let us redefine the coordinate x so it will vanish when evaluated at equilibrium: $S_0 = S(x = 0)$. Irreversible thermodynamics [87] analyses the response of the system when driven away from equilibrium. For this purpose one introduces the *thermodynamical force* X,

$$X = \frac{\mathrm{d}S}{\mathrm{d}x}, \tag{3.2}$$

which measures the tendency of the system to restore equilibrium. Nonequilibrium causes fluxes to appear in the system, that is, nonvanishing time derivatives $\mathrm{d}x/\mathrm{d}\tau$ and $\mathrm{d}S/\mathrm{d}\tau$. Further one supposes that the irreversible process considered is *linear*. This amounts to the assumption that the flux is proportional to the force,

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = LX, \qquad L > 0, \tag{3.3}$$

where L is a positive constant, independent of x and τ. One also writes (3.3) under the form

$$X = R\frac{\mathrm{d}x}{\mathrm{d}\tau}, \qquad R = L^{-1} > 0, \tag{3.4}$$

where the dimensions of R are time × entropy × x^{-2}. Eq. (3.4) is often termed a *phenomenological law*. Indeed numerous dissipative phenomena, at least to first order of approximation, take on the form of a linear relation between a driving force X and the corresponding flux $\mathrm{d}x/\mathrm{d}\tau$: Ohm's law in electricity, Fourier's law of heat transfer, etc, are familiar examples. In linear irreversible thermodynamics, the time rate of entropy production is the product of those two:

$$\frac{\mathrm{d}S}{\mathrm{d}\tau} = X\frac{\mathrm{d}x}{\mathrm{d}\tau}. \tag{3.5}$$

On the other hand, Taylor–expanding the entropy around its (maximum) equilibrium value and keeping terms up to second–order we have

$$S = S_0 - \frac{1}{2}sx^2 + \ldots, \qquad s := -\left(\frac{\mathrm{d}^2S}{\mathrm{d}x^2}\right)_0 > 0. \tag{3.6}$$

Three consequences follow from truncating the expansion (3.6) at second order. First, the force X is a linear function of the coordinate x:

$$X = -sx. \tag{3.7}$$

Second, in conjunction with Boltzmann's principle (3.1), the expansion (3.6) implies that the probability distribution for fluctuations is a Gaussian in the extensive variable x:

$$P(x) = Z^{-1}\exp\left(\frac{S}{k_B}\right) = Z^{-1}\exp\left(-\frac{1}{2k_B}sx^2\right), \tag{3.8}$$

where Z is some normalisation.[1] Third, the phenomenological law (3.4) specifies a linear submanifold of thermodynamical phase space:

$$R\frac{\mathrm{d}x}{\mathrm{d}\tau} + sx = 0. \tag{3.9}$$

Fluctuations around the deterministic law given by Eq. (3.9) can be modelled by the addition of a random force F_r. This turns the deterministic equation (3.9) into the stochastic equation

$$R\frac{\mathrm{d}x}{\mathrm{d}\tau} + sx = F_r. \tag{3.10}$$

We are interested in computing the path $x = x(\tau)$ under the influence of these random forces, under the assumption that F_r has a vanishing average value. While mimicking random fluctuations, this assumption ensures that the net force continues to be given as in the deterministic Eq. (3.9). Now

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[1] We will henceforth omit all normalistion factors, bearing in mind that all probabilites are to be normalised at the end.

our aim is to calculate the probability of any path in configuration space. For this purpose we need to introduce some concepts borrowed from ref. [32].

The *unconditional probability density function* $f\begin{pmatrix} x \\ \tau \end{pmatrix}$, also called *one–gate function*, is defined such that the product $f\begin{pmatrix} x \\ \tau \end{pmatrix} \mathrm{d}x$ equals the probability that the random trajectory $x = x(\tau)$ pass through a gate of width $\mathrm{d}x$ around x at the instant τ. The *conditional probability density function* $f\left(\begin{array}{c|c} x_2 & x_1 \\ \tau_2 & \tau_1 \end{array}\right)$, also called the *two–gate function*, is defined such that $f\left(\begin{array}{c|c} x_2 & x_1 \\ \tau_2 & \tau_1 \end{array}\right) \mathrm{d}x_2 \, \mathrm{d}x_1$ equals the probability that a thermodynamical path pass through a gate of width $\mathrm{d}x_2$ around x_2 at time τ_2, *given* that it passed through a gate of width $\mathrm{d}x_1$ around x_1 at time τ_1. The assumption that our stochastic process (3.10) satisfies the Markov property ensures that the unconditional probability $f\begin{pmatrix} x_2 \\ \tau_2 \end{pmatrix}$ can be obtained from the conditional probability $f\left(\begin{array}{c|c} x_2 & x_1 \\ \tau_2 & \tau_1 \end{array}\right)$ by letting $\tau_1 = -\infty$ in the latter and setting a fixed value of x_1, say $x_1 = 0$. Informally speaking: Markov systems have a short–lived memory.

Let us consider a time interval (τ_1, τ_{n+1}) , which we divide into n subintervals of equal length. Then the conditional probabilities obey the Chapman–Kolmogorov equation,

$$f\left(\begin{array}{c|c} x_{n+1} & x_1 \\ \tau_{n+1} & \tau_1 \end{array}\right) = \int \mathrm{d}x_n \cdots \int \mathrm{d}x_2 \, f\left(\begin{array}{c|c} x_{n+1} & x_n \\ \tau_{n+1} & \tau_n \end{array}\right) \cdots f\left(\begin{array}{c|c} x_2 & x_1 \\ \tau_2 & \tau_1 \end{array}\right), \qquad (3.11)$$

where all $n-1$ intermediate gates at $x_2, x_3, \ldots, x_n$ are integrated over. In particular, the unconditional probability density $f\begin{pmatrix} x \\ \tau \end{pmatrix}$ propagates according to the law

$$f\begin{pmatrix} x_2 \\ \tau_2 \end{pmatrix} = \int \mathrm{d}x_1 \, f\left(\begin{array}{c|c} x_2 & x_1 \\ \tau_2 & \tau_1 \end{array}\right) f\begin{pmatrix} x_1 \\ \tau_1 \end{pmatrix}. \qquad (3.12)$$

It turns out that, for a Markovian Gaussian process, the conditional probability function $f\left(\begin{array}{c|c} x_2 & x_1 \\ \tau_2 & \tau_1 \end{array}\right)$ that solves the Chapman–Kolmogorov equation is given by [87]

$$\begin{aligned} f\left(\begin{array}{c|c} x_2 & x_1 \\ \tau_2 & \tau_1 \end{array}\right) &= \frac{s}{2k_B} \frac{\mathrm{e}^{s(\tau_2-\tau_1)/2R}}{\sqrt{\pi \, \sinh\left[s(\tau_2-\tau_1)/R\right]}} \qquad (3.13) \\ &\times \exp\left\{ -\frac{s}{2k_B} \frac{\left[\mathrm{e}^{s(\tau_2-\tau_1)/2R} \, x_2 - \mathrm{e}^{-s(\tau_2-\tau_1)/2R} \, x_1\right]^2}{2 \, \sinh\left[s(\tau_2-\tau_1)/R\right]} \right\}. \end{aligned}$$

As a consistency check we observe that, in the limit $\tau_2 \to \infty$, the conditional probability (3.13) reduces to the unconditional probability (3.8). Using the Chapman–Kolmogorov equation (3.11) one can reexpress the conditional probability (3.13) as

$$f\left(\begin{array}{c|c} x_{n+1} & x_1 \\ \tau_{n+1} & \tau_1 \end{array}\right) = \exp\left[-\frac{1}{4k_B}\int_{\tau_1}^{\tau_{n+1}} \mathrm{d}\tau\, R\left(\frac{\mathrm{d}x}{\mathrm{d}\tau} + \gamma x\right)^2\right]_{\min}, \gamma := \frac{s}{R}, \tag{3.14}$$

subject to the boundary conditions $x(\tau_1) = x_1$ and $x(\tau_{n+1}) = x_{n+1}$. Above, γ carries the dimension of inverse time, while the subscript *min* reminds us that the integral is to be evaluated along that particular path which minimises the integral.

Now $f\left(\begin{smallmatrix} x_2 \\ \tau_2 \end{smallmatrix}\right)$ can be obtained from $f\left(\begin{smallmatrix} x_2 & | & x_1 \\ \tau_2 & | & \tau_1 \end{smallmatrix}\right)$ by letting $\tau_1 = -\infty$ and $x_1 = 0$ in the latter. In order to take this limit in Eq. (3.14) we first define the *thermodynamical Lagrangian* $\mathcal{S}$ to be

$$\mathcal{S} := \frac{R}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau} + \gamma x\right)^2, \tag{3.15}$$

or, dropping a total derivative,

$$\mathcal{S} = \frac{R}{2}\left[\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 + \gamma^2 x^2\right]. \tag{3.16}$$

The dimensions of $\mathcal{S}$ are entropy per unit time. The corresponding Euler–Lagrange equation reads

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} - \gamma^2 x = 0, \tag{3.17}$$

while

$$x(\tau) = x_2 \mathrm{e}^{\gamma(\tau - \tau_2)} \tag{3.18}$$

is the particular solution to (3.17) that satisfies the boundary conditions $x(\tau = -\infty) = 0$ and $x(\tau = \tau_2) = x_2$. Thus evaluating (3.14) along this extremal path yields

$$f\left(\begin{array}{c|c} x_2 & 0 \\ \tau_2 & -\infty \end{array}\right) = f\left(\begin{array}{c} x_2 \\ \tau_2 \end{array}\right) = \exp\left[-\frac{s}{2k_B}(x_2)^2\right]. \tag{3.19}$$

This is again in agreement with Boltzmann's principle (3.1) in the Gaussian approximation (3.6). Moreover, the conditional probability density

$f\left(\begin{smallmatrix} x_2 \\ \tau_2 \end{smallmatrix}\middle|\begin{smallmatrix} x_1 \\ \tau_1 \end{smallmatrix}\right)$ admits the path–integral representation [87][2]

$$f\left(\begin{matrix} x_2 \\ \tau_2 \end{matrix}\middle|\begin{matrix} x_1 \\ \tau_1 \end{matrix}\right) = \int_{x(\tau_1)=x_1}^{x(\tau_2)=x_2} \mathrm{D}x(\tau)\, \exp\left\{-\frac{1}{2k_B}\int_{\tau_1}^{\tau_2}\mathrm{d}\tau\,\mathcal{S}\right\}. \tag{3.20}$$

In fact, a saddle–point evaluation of the path integral (3.20) is readily seen to yield the two–gate function (3.14).

The above Eqs. (3.2)–(3.20) have obvious generalisations to a case with D independent thermodynamical coordinates.

3.3 Quantum mechanics vs. irreversible thermodynamics

The attentive reader will have noticed the striking similarity between Eqs. (3.2)–(3.20) and the quantum mechanics of the harmonic oscillator. The corresponding Lagrangian is

$$\mathcal{L} = \frac{m}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - \frac{k}{2}x^2. \tag{3.21}$$

Mechanical time is denoted by the variable t; it is related to thermodynamical time τ through the Wick rotation

$$\tau = \mathrm{i}t. \tag{3.22}$$

We define as usual the angular frequency ω through $\omega^2 = k/m$. Let us for simplicity assume that the thermodynamical extensive coordinate x of the dual irreversible thermodynamics is a length. In this way no dimensionful factor is needed to reinterpret it as the coordinate of the harmonic oscillator in the mechanical dual theory. Then the Wick rotation (3.22) and the replacements[3]

$$\frac{m\omega}{\hbar} = \frac{s}{2k_B}, \qquad \omega = \gamma \tag{3.23}$$

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[2] What quantum theorists call the Feynman path integral was independently developed in ref. [87] by Onsager and collaborators, who appear to have arrived at the notion of a path integral all by themselves, without previous knowledge of Feynman's earlier work [47].

[3] Implicit in the replacements (3.23) is the assumption that the thermodynamical extensive variable x, and the mechanical variable x, both have units of length. A dimensionful conversion factor is to be understood in case the dimensions do not match.

provide us with a dictionary to establish a 1–to–1 map between the linear, irreversible thermodynamics of section 3.2 and the quantum mechanics of the harmonic oscillator.

Specifically, let us spell out the entries of this map, one by one [4]. The mechanical Lagrangian (3.21) is readily obtained from its thermodynamical counterpart (3.16) upon application of the replacements (3.22), (3.23):

$$\frac{\mathcal{S}}{2k_B} = -\frac{\mathcal{L}}{\hbar}. \tag{3.24}$$

The above also makes it clear that the thermodynamical analogue of Planck's constant $\hbar$ is twice Boltzmann's constant, $2k_B$. In this way the thermodynamical path integral (3.20) becomes its usual quantum–mechanical expression. Unconditional probabilities $f\left(\begin{smallmatrix} x \\ \tau \end{smallmatrix}\right)$ in thermodynamics become wavefunctions squared $|\psi(x,t)|^2$ in quantum mechanics. Thus the 1–gate distribution function (3.19) gives the squared modulus of the oscillator groundstate,

$$f\begin{pmatrix} x \\ \mathrm{i}t \end{pmatrix} = \exp\left(-\frac{m\omega}{\hbar}x^2\right). \tag{3.25}$$

The thermodynamical conditional probabiliy (3.13) becomes proportional to the quantum–mechanical Feynman propagator. Away from the caustics, the latter is given by

$$K\left(x_2, t_2 | x_1, t_1\right) = \sqrt{\frac{m\omega}{2\pi \mathrm{i}\hbar \sin\left(\omega(t_2 - t_1)\right)}} \tag{3.26}$$

$$\times \exp\left\{\frac{\mathrm{i}m}{2\hbar}\frac{\omega}{\sin\left(\omega(t_2-t_1)\right)}\left[(x_2^2 + x_1^2)\cos\left(\omega(t_2-t_1)\right) - 2x_2x_1\right]\right\}$$

and one actually finds

$$f\left(\begin{array}{c|c} x_2 & x_1 \\ \mathrm{i}t & 0 \end{array}\right) = \exp\left(\frac{\mathrm{i}\omega t}{2} - \frac{\Delta V}{\hbar\omega}\right)\sqrt{\frac{2m\omega}{\hbar}}\,K\left(x_2, t | x_1, 0\right), \tag{3.27}$$

where $\Delta V = V(x_2) - V(x_1)$, with $V(x) = kx^2/2$ the harmonic potential. The Chapman–Kolmogorov equation (3.11) becomes the group property of propagators, while the propagation law (3.12) exactly matches that for wavefunctions ψ under propagators K. Altogether, the promised 1–to–1 map is complete.

Our Eqs. (3.21)–(3.27) have obvious generalisations to higher dimensions. Since the concept of *equipotential submanifolds* will play a key role in our duality between quantum mechanics and irreversible thermodynamics, it will be useful to consider the lowest dimension in which equipotential manifolds are *2–dimensional surfaces*. Configuration space is then 3–dimensional, which we take to be $\mathbb{R}^3$, coordinatised by x, y, z. For simplicity we will assume the harmonic potential to be isotropic, so the harmonic force is $\mathbf{F}_h = -k(x, y, z)$. On the thermodynamical side of our correspondence, this translates into the fact that Onsager's (inverse) coefficients R_x, R_y, R_z in Eq. (3.4) are all equal, so the dissipative force acting on the system is $\mathbf{F}_d = R(\mathrm{d}x/\mathrm{d}\tau, \mathrm{d}y/\mathrm{d}\tau, \mathrm{d}z/\mathrm{d}\tau)$. We then have a thermodynamical Lagrangian

$$\mathcal{S} = \frac{R}{2}\left[\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2 + \gamma^2(x^2 + y^2 + z^2)\right] \tag{3.28}$$

and a mechanical Lagrangian

$$\mathcal{L} = \frac{m}{2}\left[\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2 - \omega^2(x^2 + y^2 + z^2)\right]. \tag{3.29}$$

The latter has the family of 2–dimensional spheres $x^2 + y^2 + z^2 = \rho^2$ as equipotential surfaces within the mechanical configuration space $\mathbb{R}^3$. We claim that the thermodynamical counterpart of this family of spheres is the family of 5–dimensional submanifolds

$$\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2 + \gamma^2(x^2 + y^2 + z^2) = \rho^2 \tag{3.30}$$

within the thermodynamical phase space $\mathbb{R}^6$; we may call the above hypersurfaces *isoentropic submanifolds*. Although we seem to have a dimensional mismatch between isoentropic submanifolds and equipotential surfaces, this mismatch disappears if we restrict to those thermodynamical trajectories that satisfy the equation of motion of the thermodynamical Lagrangian (3.28). This equation was given in (3.17) and solved in (3.18); we see that, *on shell*, the velocity $\mathrm{d}x/\mathrm{d}\tau$ is proportional to the coordinate x. This property effectively allows us to replace the term $(\mathrm{d}x/\mathrm{d}\tau)^2 + (\mathrm{d}y/\mathrm{d}\tau)^2 + (\mathrm{d}z/\mathrm{d}\tau)^2$ in Eq. (3.30) with a constant multiple of $x^2 + y^2 + z^2$. In turn, this reduces the family of 5–dimensional submanifolds (3.30) to a family of 2–dimensional spheres—exactly as in the mechanical case.

We conclude that *equipotential surfaces for the mechanical problem become isoentropic surfaces for the thermodynamical problem, and viceversa*. This is in nice agreement with the results of ref. [116] for the gravitational potential, in the context of a theory of emergent spacetime.

3.4 Beyond the harmonic approximation

While explicit expressions for our map between quantum mechanics and irreversible thermodynamics are difficult to obtain beyond the harmonic approximation considered so far, some key physical ideas can be extracted from the previous analysis and generalised to an arbitrary potential. On the thermodynamical side, this generalisation implies going beyond the Gaussian approximation made in Eq. (3.6) or, equivalently, beyond the assumption (3.7) of linearity between forces and fluxes.

Let a mechanical system be described by a Lagrangian function $\mathcal{L} = \mathcal{L}(q_i, \dot{q}_i)$. For simplicity we assume our configuration space to be $\mathbb{R}^D$; an additional $\mathbb{R}$ stands for the time axis. The mechanical time variable t, initially real, will be complexified presently.

We will equate certain spacetime concepts (on the left–hand side of the equations below) to certain thermodynamical quantities (on the right). To begin with, we observe that the two physical constants $\hbar$ and k_B allow one to regard time t and temperature T as mutually inverse, through the combination

$$\frac{1}{t} = \frac{k_B}{\hbar} T. \tag{3.31}$$

Admittedly, this observation is not new [30].

Corresponding to the mechanical system governed by the Lagrangian $\mathcal{L}(q_i, \dot{q}_i)$ there will be a thermodynamical system whose dynamics will be governed by an entropy $S = \int \mathcal{S} \mathrm{d}t$. Following our previous result (3.24), let us postulate the following differential relation between the two of them:

$$\frac{1}{\hbar} \mathcal{L} \mathrm{d}t = \frac{C}{2k_B} \mathrm{d}S = \frac{C}{2k_B} \mathcal{S} \mathrm{d}t, \qquad C \in \mathbb{C}. \tag{3.32}$$

Again, dimensionality arguments basically fix the two sides of the above relation, but leave room for a dimensionless number C. Agreement with the Wick rotation (3.22) requires that we set $C = -\mathrm{i}$. Now Eq. (3.32) overlooks the fact that the right–hand side contains the exact differential $\mathrm{d}S$, while the differential $\mathcal{L}\mathrm{d}t$ on the left–hand side is generally *not* exact. In other words, while there exists a well–defined entropy *function* $S = \int \mathcal{S}\mathrm{d}t$, the line integral $I = \int \mathcal{L}\mathrm{d}t$ generally depends on the trajectory in $\mathbb{R}^D$ being integrated along.

The mechanical action I, however, *can* define a path–independent function of the integration endpoint if we restrict to a certain class of trajectories in $\mathbb{R}^D$. Let us see how this comes about. Let $V = V(q_i)$ be the potential function of the mechanical system under consideration. The equation

$$V(q_i) = \mathrm{const} \tag{3.33}$$

defines, as the constant on the right–hand side is varied, a family of $(D-1)$–dimensional, *equipotential submanifolds* of $\mathbb{R}^D$. An elementary example, when $D = 3$, is the case of the Newtonian potential generated by a point mass located at the origin O. Then the above family of equipotential surfaces is a family of concentric spheres $\mathbb{S}_\rho$ of increasing radii $\rho > 0$, all centred at O. This family of equipotentials, singular only at O, defines a foliation of $\mathbb{R}^3 - \{O\}$, so the latter space equals the union $\cup_{\rho>0}\mathbb{S}_\rho$ of all leaves $\mathbb{S}_\rho$. This foliation can also be used to define a coordinate system on $\mathbb{R}^3 - \{O\}$. Namely, one splits $\mathbb{R}^3 - \{O\}$ into 2 tangential directions to the spheres of the foliation, and 1 normal direction. For example, the standard spherical coordinates ρ, θ, φ centred at O qualify as such a coordinate system, ρ being the normal coordinate and θ, φ the tangential coordinates.

Returning now to the general case when both D and $V(q_i)$ are arbitrary, Eq. (3.33) defines, for each particular value of the constant on the right–hand side, one equipotential leaf $\mathbb{L}_n$ of a foliation $\cup_n\mathbb{L}_n$ of $\mathbb{R}^D$. Here the subindex n stands for a certain (local) coordinate n on $\mathbb{R}^D$ that is normal to all the leaves. The $D-1$ tangential coordinates thus span the $(D-1)$–dimensional leaves $\mathbb{L}_n$, each one of them being located at a specific value of the normal coordinate n. We will assume that all the leaves $\mathbb{L}_n$ are compact.

Trajectories within $\mathbb{R}^D$ that run exclusively along this normal coordinate n, thus being orthogonal to the leaves, are such that the action integral I *does* defines a function I_n of the integration endpoint; the subindex n reminds us of the restriction to these normal trajectories. Independence of path is merely a consequence of the 1–dimensionality of the normal directions to the equipotential leaves $\mathbb{L}_n$. This is the particular class of trajectories mentioned above: along them, $\mathcal{L}\mathrm{d}t$ defines an exact differential, $\mathrm{d}I_n$. For these normal trajectories, the differential equation (3.32) makes perfect sense as an equality between two exact differentials. For these normal trajectories we can write

$$\frac{1}{\hbar}I_n - \frac{C}{2k_B}S = \text{const.} \tag{3.34}$$

Now the sought–for thermodynamics *cannot* be the standard thermodynamics of equilibrium processes as presented in any standard textbook, say, ref. [23]. Among other reasons for this not being the case, standard equilibrium thermodynamics does not include time as one of its variables. We have already in section 3.3 produced evidence that it must in fact be the *explicitly time–dependent, classical thermodynamics of irreversible processes* as developed by Onsager, Prigogine *et al* [87, 95]. We will present arguments in section 3.5, to the effect that quantum states arise through a dissipative mechanism. For completeness the thermodynamical dual to quantum mechanics must be supplemented with the relation

$$\frac{1}{T} = \frac{\partial S}{\partial U}, \tag{3.35}$$

which must always be satisfied. So we take (3.35) to *define* the internal energy U of the thermodynamical theory, given that T and S have already been defined.

3.5 Quantum states as equivalence classes of classical trajectories

A key consequence of using normal and tangential coordinates in $\mathbb{R}^D$ is that quantum states ψ, to be constructed presently, will factorise as

$$\psi = \psi_t \psi_n, \tag{3.36}$$

or sums thereof. Here, the normal wavefunction ψ_n depends exclusively on the normal coordinate n, while ψ_t is a function of the tangential coordinates. For example, in the case of the Coulomb potential, the wavefunction ψ_t would be a spherical harmonic $Y_{lm}(\theta, \varphi)$, while ψ_n would be a radial wavefunction $R_{nl}(\rho)$. This construction contains elements that are very reminiscent of those present in ref. [116]. In this latter paper, *equipotential surfaces of the gravitational potential are identified as isoentropic surfaces*. Our equipotential leaves are the counterpart of the *holographic screens* of ref. [116].

Moreover, the classical mechanics exhibits a precise mechanism whereby *different classical trajectories coalesce into a single equivalence class that can, following ref. [111], be identified as a single quantum state ψ*. So the presence of Planck's constant $\hbar$ in Eq. (3.32) obeys not just dimensional reasons—it is the sure sign of an information–loss mechanism, a dissipative processs that is truly quantum in nature.

Let us see how this dissipation comes about. In order to do this we need to explain why many different classical trajectories coalesce into one single quantum state ψ. A quantum of area on the leaf $\mathbb{L}_n$ measures L_P^2, where L_P denotes the Planck length. According to the holographic principle, at most 1 bit of information fits into this quantum of area L_P^2. One classical trajectory traversing this quantum of area corresponds to 1 bit of information. Classically one can regard the surface density of trajectories as being correctly described by a smooth distribution function: there fit some 1.4×10^{69} classical trajectories into each square meter of area on the leaf $\mathbb{L}_n$[19]. Although this is a huge number, it sets an upper limit on the potentially infinite number of classical trajectories that can traverse one quantum of area L_P^2.

The holographic principle alone would suffice to account for the lumping together of many different classical trajectories into one equivalence class. One equivalence class, or quantum state, would be comprised by all those different classical trajectories crossing one given quantum of area L_P^2.

Of course, the *actual* number of quantum particles traversing one square meter of area on the leaf $\mathbb{L}_n$ is much smaller than the above 1.4×10^{69}. The reason is simple: quantum effects become nonnegligible on matter well before quantum–gravity effects become appreciable on the geometry. Again, the existence of a (now particle–dependent) quantum of area is responsible for this. This can be seen as follows.

Let m be the mass of the particle under consideration. Its Compton wavelength $\lambda_C = \hbar/(mc)$ imposes a fundamental limitation on its position, that we can call a *quantum of length*, denoted Q_1. This Q_1, which is particle–dependent, is of a fundamentally different nature than the *geometric* quantum of length L_P. On configuration space $\mathbb{R}^D$, this gives rise to a quantum Q_{D-1} of $(D-1)$–dimensional volume within the leaf $\mathbb{L}_n$, with measure (proportional to) λ_C^{D-1}, and to a quantum of length Q_1 along the normal coordinate.

In the presence of more than one particle species with different masses, each mass m_i defines one value of the quantum $Q^{(i)}_{D-1}$. Then a quantum of volume that remains valid for all particles is the largest value of all those $Q^{(i)}_{D-1}$. This is the quantum of volume determined by the lightest particle.

Let us now elucidate how quantum states ψ can arise as equivalence classes of different classical trajectories. By Eq. (3.36) we have to account for the appearence of the normal wavefunction ψ_n and of the tangential wavefunction ψ_t.

Starting with ψ_t, let us consider all the different classical trajectories traversing any one quantum of volume Q_{D-1} within a leaf $\mathbb{L}_n$. The allowed values of the momentum carried by those trajectories are those compatible with the uncertainty principle. Since the particle has been spatially localised to an accuracy of λ_C along each tangential coordinate, the corresponding momentum can be specified to an accuracy of $\hbar/\lambda_C$. Therefore, corresponding to a spatial quantum of volume Q_{D-1} in the leaf, we have a quantum of volume $P_{D-1} = (\hbar/\lambda_C)^{D-1}$ in momentum space.

We are now in a position to state a postulate:

All the different classical trajectories traversing any quantum of volume Q_{D-1} in the leaf $\mathbb{L}_n$, and simultaneously traversing a quantum P_{D-1} in tangential momentum space, are to be regarded as different representatives of just one tangential state ψ_t.

An analogous postulate for the normal coordinate reads:

All classical trajectories traversing any quantum of length Q_1 *along the normal coordinate* n*, and simultaneously traversing the corresponding quantum* P_1 *in normal momentum space, make up one normal state* ψ_n*.*

In support of the above postulate, let us return to Eq. (3.23), where the mechanical combination $m\omega/\hbar$ has been identified with the thermodynamical quotient $s/(2k_B)$. The constant s, defined in Eq. (3.6), carries the dimensions of entropy × x^{-2}, so $s/(2k_B)$ has the dimensions x^{-2}. Thus $s/(2k_B)$ is homogeneous to the inverse square of the Compton wavelength, λ_C^{-2}.

On the other hand, the constant s (and the frequency γ in (3.23)) are all the data one needs in order to univocally specify the irreversible thermodynamics that is dual to the given quantum mechanics. The previous statement, which holds exactly true in the harmonic approximation of section 3.3, is raised to the category of a principle in the above postulate. Indeed, let us assume going beyond the harmonic approximation in mechanics. In the thermodynamical dual theory, this is equivalent to considering terms beyond quadratic in the Taylor expansion (3.6). Higher derivatives $\mathrm{d}^3S/\mathrm{d}x^3$, $\mathrm{d}^4S/\mathrm{d}x^4$, etc, evaluated at the equilibrium point, simply introduce new constants s_3, s_4, etc, which can be dimensionally accounted for in terms of just two physical constants, namely k_B and λ_C. Up to a set of *dimensionless* coefficients, all the data we need in the irreversible thermodynamics can be constructed in terms of k_B and powers of λ_C.

These arguments render our above postulate a very plausible statement. Moreover, they provide an estimate of the entropy increase (*i.e.*, of the amount of information loss) involved in the lumping together of many classical trajectories into just one quantum state. Namely, *the increase in entropy* ΔS *due to the formation of one equivalence class of classical trajectories is a positive multiple of* λ_C^2 *times the coefficient* s,

$$\Delta S = ns\lambda_C^2, \qquad n > 0, \tag{3.37}$$

where n *is a dimensionless number*. (Admittedly, our arguments leave n undetermined, although one could resort to Landauer's principle [72] in order to argue that n must be of order unity). More importantly, the surface density of entropy s can be naturally identified, via Eq. (3.37), with the entropy increase ΔS due to the formation of quantum states as equivalence classes

[110, 111]. In other words, *the dissipation that is inherent to irreversible thermodynamics has a natural counterpart in quantum mechanics*.

Having described the dissipative mechanism whereby classical trajectories organise into quantum states, we go next to a counting of the number of quantum states. Since the leaf $\mathbb{L}_n$ has been assumed compact, it encloses a finite number N_n of volume quanta Q_{D-1}. Tentatively identifying this number N_n with the (complex) dimension of the tangential Hilbert space $\mathcal{H}_t$, we immediately realise that the quantum of momentum P_{D-1} is contained an infinite number of times within tangential momentum space (this is however a *countable* number of times). Indeed the momenta may grow to arbitrarily large values. Therefore, the tangential Hilbert space $\mathcal{H}_t$ is infinite–dimensional, and separable.

On the other hand, the dimension of the normal Hilbert space $\mathcal{H}_n$ is infinite already from the start (again a countable infinity, hence $\mathcal{H}_n$ is separable). The reason for this is the noncompactness of $\mathbb{R}^D$: the normal coordinate n must cover an interval of infinite length.[4] This implies that the normal coordinate encloses an infinite (though countable) number of length quanta Q_1. Multiplication by the number of independent momentum quanta P_1 does not alter this separable, infinite–dimensionality of $\mathcal{H}_n$.

Altogether, the complete Hilbert space $\mathcal{H}$ of quantum states is the tensor product $\mathcal{H}_t \otimes \mathcal{H}_n$. However, because it singles out the normal coordinate n, one might worry that our construction depends on the particular choice of a leaf $\mathbb{L}_n$ within the foliation. Now the only possible difference between any two leaves $\mathbb{L}_{n_1}$ and $\mathbb{L}_{n_2}$ is the value of their $(D-1)$–dimensional volume. Hence the numbers of volume quanta N_{n_1} and N_{n_2} they enclose may be different—but they are both finite. This possible difference is washed away upon multiplication by the (countably infinite) number of momentum quanta P_{D-1} corresponding to each leaf. The dimension of $\mathcal{H}_t$ is therefore countably infinite regardless of the point, n_1 or n_2, along the radial coordinate—that is, regardless of which leaf is considered.[5]

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[4] In case more than just one normal coordinate is needed, this statement is to be understood as meaning the sum of all the lengths so obtained.

[5] We should remark that the assumption of compactness of the leaves $\mathbb{L}_n$ can be lifted without altering our conclusions. A noncompact leaf encloses an infinite (yet countable) number of volume quanta Q_{D-1}. Upon multiplication by an infinite (yet countable)

As explained in ref. [3], determining the tangential wavefunctions ψ_t does not require a knowledge of the specific dynamics under consideration. Instead, this tangential dependence is univocally fixed by the geometry of the leaves $\mathbb{L}_n$. In more technical terms, the wavefunctions ψ_t must provide a complete orthonormal set for a unitary, irreducible representation of the isometry group of the leaves $\mathbb{L}_n$. Moreover, as argued in ref. [3], the modulus squared $|\psi|^2$, evaluated at the value n, is proportional to the surface density of entropy flux across the leaf $\mathbb{L}_n$.

3.6 Quantum uncertainty vs. the second law

Just as Planck's constant $\hbar$ represents a coarse–graining of phase space into cells of minimal volume, or quanta of action, so does Boltzmann's constant k_B represent a *quantum of entropy*. This implies that any process must satisfy the condition

$$\Delta S = N k_B, \qquad N \in \mathbb{N}. \tag{3.38}$$

The above expresses a quantised form of the second law of thermodynamics. The extreme smallness of the numerical value of k_B in macroscopic units makes this quantisation macroscopically unobservable. In particular, unless $N = 0$, the second law becomes

$$\Delta S \geq k_B. \tag{3.39}$$

In this form, the second law is actually a rewriting of the quantum–mechanical uncertainty principle for the canonical pair E, t:

$$\Delta E \Delta t \geq \frac{\hbar}{2}. \tag{3.40}$$

Of course, this derivaton of the uncertainty relation $\Delta E \Delta t \geq \hbar/2$ is heuristic, because time is a parameter in quantum mechanics. It is only in the

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number of momentum–space quanta P_{D-1}, the dimension of the tangent Hilbert space $\mathcal{H}_t$ remains denumerably infinite. This form of holography in which the leaves are noncompact replaces the notion of *inside vs. outside* the leaf with the equivalent notion of *one side of the leaf vs. the other side*. One should not dismiss this possibility as unphysical: the constant potential, for example, can be regarded as having either compact or noncompact equipotential submanifolds.

limit $k_B \to 0$ that the second law (3.39) reduces to its classical formulation $\Delta S \geq 0$. The limit $k_B \to 0$ is the thermodynamical counterpart of the usual semiclassical limit $\hbar \to 0$ of quantum mechanics.

We conclude that the equivalence between Eqs. (3.39) and (3.40) is a consequence of our basic postulate (3.32). In other words, the second law (3.39) expresses, in the thermodynamical theory, the same statement as the uncertainty principle (3.40) expresses in the quantum–mechanical theory.

Our correspondence implies that, while one needs two canonical variables E, t in order to express the uncertainty principle in the quantum theory, just one variable S is needed in order to write the second law. An equivalent way of saying this is that *entropy is a selfconjugate variable*: one does not have to multiply it with a canonical variable (say, ξ) in order to obtain a product ξS carrying the dimensions of the quantum k_B. The variable S already carries the dimensions of its corresponding quantum k_B.

3.7 Discussion

The holographic principle of quantum gravity states that there fits at most 1 bit of information into each quantum of area L_P^2 in configuration space, where L_P is Planck's length. For quantum mechanics, in section 3.5 we have postulated that

There fits at most 1 quantum state into each quantum of volume $(\lambda_C)^{2D}$ *in phase space, whereby the Compton length* λ_C *of the particle in question extends once along each coordinate* q *and once along each conjugate momentum* p *in a* $2D$*–dimensional phase space.*

Thus our postulate is conceptually analogous to the holographic principle of quantum gravity. We should stress, however, that our postulate does not follow from, nor does it imply, the holographic principle of quantum gravity.

We can summarise our construction as follows. Let a quantum–mechanical system be given in configuration space $\mathbb{R}^D$. Let this latter space be foliated as per $\cup_n \mathbb{L}_n$, where each leaf $\mathbb{L}_n$ is an equipotential submanifold, in dimension $D-1$, of the given mechanical potential function $V(q_i)$. Assume that each leaf $\mathbb{L}_n$ encloses a finite D–dimensional volume $\mathbb{V}_n$, so $\partial \mathbb{V}_n = \mathbb{L}_n$. Then quantum states in $\mathbb{V}_n$ are equivalence classes of different classical trajectories. These equivalence classes comprise all those classical trajectories that fit into one given quantum of volume in configuration space, with the corresponding momenta inside the corresponding quantum in momentum space. No quantum particle can be located to an accuracy better than its Compton wavelength.[6] Hence a physically reasonable unit for defining this quantum of length (and thus areas and volumes) is the Compton wavelength. Configuration space is subdivided into many such elementary volume quanta, each one of them (with the corresponding quanta in momentum space) defining one different quantum state.

The quantisation of phase–space area by Planck's constant $\hbar$ proceeds along lines that are somewhat similar to ours, although not exactly identical. We recall that, semiclassically, the (symplectic) area element $\mathrm{d}p \wedge \mathrm{d}q$, divided by $\hbar$, gives the number of different quantum states fitting into that area element. However, the coordinate width $\mathrm{d}q$ may be arbitrarily squeezed, provided the momentum $\mathrm{d}p$ is correspondingly enlarged, and viceversa.

On the contrary, our construction makes use of the Compton wavelength λ_C as a fundamental quantum of length (for the specific particle considered), below which no sharper localisation is possible: there is no squeezing the particle below this lower limit. This gives rise to an arrangement of different classical trajectories into equivalence classes that, following ref. [111], we identify with quantum states. This is an irreversible, dissipative mechanism that exhibits the emergent nature of quantum mechanics. The Hilbert space of quantum states is determined as described in section 3.5.

Under our correspondence, an irreversible thermodynamics can be mapped into a quantum mechanics, and viceversa. This correspondence may be regarded as *dictionary* that allows one to switch back and forth between a *quantum–mechanical picture* and a *thermodynamical picture* of one and the same physics.

A key point to remark is the following. Thermodynamical approaches to quantum theory are well known [30, 81]. In particular, the link between (complex–time) quantum mechanics, on the one hand, and the *equilibrium*

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[6] Unless, of course, one is willing to allow for pair creation out of the vacuum, thus quitting quantum mechanics and entering field theory.

statistical mechanics of the Gibbs ensemble, on the other, has been known for long. We should stress that we have *not* dwelled on this long–established connection. Rather, the new correspondence explored here is that between (complex–time) quantum mechanics, and the *classical* thermodynamics of *irreversible* processes. *Classicality* of the thermodynamics means that $\hbar$ does not appear on the thermodynamical side of the correspondence, its role being played instead by Boltzmann's constant k_B. *Irreversibility* implies the existence of dissipation, as befits the presence of quantum effects.

Chapter 4

Entropy, topological theories and emergent quantum mechanics

4.1 Motivation

The approach of emergence of quantum mechanics has provided interesting clues into the deeper structure of the theory. The statement that *standard quantum mechanics is an emergent phenomenon* [38, 39, 112, 113] has found further support in a series of papers, some of which have been reviewed in ref. [44]. Although this is a huge topic to summarise here, let us briefly mention some key points of this approach. The underlying notion is that it provides a coarse grained version of some deeper theory, out of which quantum mechanics emerges as a kind of effective description. This effective description, in using variables that arise as averages over large collections of individual entities carrying the truly fundamental degrees of freedom, ignores the underlying fine structure. These fundamental degrees of freedom have been identified in refs. [112, 113] as those of cellular automata.

This state of affairs is reminiscent of the relation between thermodynamics (as an emergent phenomenon) and statistical mechanics (the corresponding underlying theory). Based on this analogy, we have in previous publications (see ref. [44] and refs. therein) established a bijective map that one can define between quantum mechanics, on the one hand, and the classical thermodynamics of irreversible processes, on the other [87, 95]. It must be stressed that the classical thermodynamics of irreversible processes [87, 95] is conceptually quite different from the usual *thermostatics of equilibrium* as presented in the standard textbooks [23]. Specifically, in the theory of irreversible processes, the continual production of entropy provides a rationale for the dissipation, or information loss, that has been argued to lie at the heart of quantum mechanics [112, 113]. The relevance of thermodynamical concepts to quantum theory and gravity has been emphasised recently in refs. [8, 81, 89, 92, 108].

It might thus appear that the usual quasistatic thermodynamics [23], *i.e.*, the thermostatics of equilibrium processes, possesses no quantum mechanical dual theory at all. In this letter we point out that such a conclusion is not true: the thermostatics of equilibrium processes *does* have a quantum mechanical dual, namely, *a quasistatic quantum mechanics*. Under *quasistatic* we mean that the kinetic term in the mechanical Lagrangian can be neglected compared to the potential term.

Neglecting the kinetic term in the Lagrangian function forces one to look elsewhere for the dissipative mechanism that is characteristic of quantum theory [112, 113]. In particular, such a mechanism can no longer be identified with the continual production of entropy associated with Onsager's kinetic term $L_{ij}\dot{q}^i\dot{q}^j$. The reciprocity theorem [87] ensures $L_{ij} = L_{ji}$, and dissipation requires that this matrix be positive definite; the latter two properties ensure that L_{ij} qualifies as a metric. The result of neglecting the kinetic term in the Lagrangian is a mechanics bearing some resemblance to topological field theory [13]. Indeed, once the metric represented by the kinetic term is neglected, correlation functions can no longer be metric dependent. Hence, while correlators can still depend on the topology of the underlying manifold, they can no longer depend on its metric structure. In our case the underlying manifold will be given by the equipotential submanifolds (within configuration space) of the potential function.

4.2 A quasistatic mechanics

A quasistatic mechanics is obtained by neglecting the kinetic term K in the mechanical Lagrangian $L = K - U$, and keeping only the potential term U:

$$L = -U. \tag{4.1}$$

Since our Lagrangian does not depend on the velocities $\dot{q}$, this phase space is constrained by the requirement that all momenta vanish, $p = 0$, and the Hamiltonian equals

$$H = U. \tag{4.2}$$

We can now construct the reduced phase space corresponding to this reduced configuration space, and eventually quantise it.[1] When moving along equipotential submanifolds, the particle is effectively free; whenever motion takes place between neighbouring equipotentials, forces will cause the particle's kinetic energy to increase or decrease. However, the allowed motions must be quasistatic, so even for these motions K must be negligible compared to U. In classical mechanics, motion along equipotential submanifolds, plus a vanishing kinetic energy, imply that a classical particle must forever stay at rest. Quantum mechanically, due to the uncertainty principle, a (more or less localised) free particle always carries a nonzero kinetic energy. So neglecting the kinetic energy of a quantum particle implies a large uncertainty in the position. This large uncertainty is reflected in a large spread of the corresponding wavepacket: the latter encompasses a large interval of different classically allowed positions, or states, all of which coalesce into a single quantum state. It is only in the limit of complete delocalisation in space that a quantum particle can carry zero kinetic energy.

We have just described an information loss mechanism whereby different classical states (different spatial positions on an equipotential submanifold, corresponding to different classically allowed equilibrium states) are lumped together into just one quantum state. This information loss has been argued to be a key feature of the quantum world.

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[1] For our purposes it will not be necessary to apply Dirac's theory of constrained quantisation [31].

4.3 The thermostatics dual to quasistatic mechanics

We claim that *the quasistatic quantum mechanical model described in section 4.2 possesses a dual theory: the classical thermostatics of equilibrium processes*. In what follows we will exhibit the claimed duality explicitly.

The classical thermostatics of equilibrium [23] is a theory of quasistatic processes. In particular, all kinetic energies are neglected; the processes described either are in thermal equilibrium, or at most differ infinitesimally from thermal equilibrium. This feature is in sharp contrast with the thermodynamics of irreversibility [87, 95], that we described in previous publications [44] as a thermodynamical dual of quantum mechanics, *whenever the kinetic energies involved could not be neglected*.

Next we recall that classical thermostatics is, like quantum mechanics, an emergent theory. By *emergent* we mean that classical thermostatics is the result of coarse graining over very many microscopic degrees of freedom; the resulting theory renounces the knowledge of detailed information about its constituent degrees of freedom, retaining just a handful of relevant averages such as pressure, volume and temperature. In other words, *an information loss mechanism is at work*. This situation is similar to that described in section 4.2 for the passage from classical mechanics to quantum mechanics.

In the dual thermostatics considered here, the counterpart of the mechanical action $I = \int L\mathrm{d}t$ is the entropy S. We will identify isoentropic submanifolds (of thermodynamical state space) with equipotential submanifolds (of mechanical state space). This is justified because, in the approach of emergence, *forces are (proportional to) entropy gradients*. In the particular case of the gravitational force, this identification has been put forward in ref. [116]; it coincides with the viewpoint applied in the theory of irreversibility [95] and, indeed, with the whole programme of the emergent physics paradigm. In this way the quantum mechanical exponential

$$\exp\left(-\frac{\mathrm{i}}{\hbar}I\right) \tag{4.3}$$

becomes, in the dual thermostatics,

$$\exp\left(\frac{S}{k_B}\right). \tag{4.4}$$

The correspondence between expressions (4.3) and (4.4) has been known for long, having been discussed more recently in ref. [8] from the point of view of statistical mechanics. However, we would like to stress that the theory being considered here as dual to quantum mechanics is *not* statistical mechanics, but the thermostatics of equilibrium emerging from the latter.

Finally the connection between the mechanical time variable t and the temperature T is as follows:[2]

$$\frac{\mathrm{i}}{\hbar}t \longleftrightarrow -\frac{1}{k_B T}, \tag{4.5}$$

where $\hbar$, k_B are Planck's constant and Boltzmann's constant, respectively. The double arrow is to be understood as *replace every occurrence of* $\mathrm{i}t/\hbar$ *in the mechanical theory with* $-1/k_B T$ *in the thermostatical dual, and viceversa*. Quasistatic mechanics therefore corresponds to isothermal processes in the dual thermostatics.

4.4 The quasistatic mechanics dual to thermostatics

Given some specific thermostatical systems, below we illustrate how to define their corresponding (quasistatic) quantum mechanical duals.

4.4.1 The ideal gas

An expression for the entropy of a system in terms of its thermodynamical variables is called a *fundamental equation* for the system [23]. To be specific let us consider 1 mole of an ideal gas occupying a volume V at a fixed temperature T. Its fundamental equation reads

$$S(V) = S_0 + k_B \ln\left(\frac{V}{V_0}\right), \tag{4.6}$$

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[2] This substitution is widely applied in thermal field theory, see *e.g.* ref. [62].

where S_0 is the entropy in the fiducial state specified by V_0; we take S_0 to contain a constant contribution from the fixed temperature T. The entropy depends only on the volume V; the latter, running over $(0,\infty)$, can be regarded as the thermodynamical coordinate for the *isothermal* processes of an ideal gas.

In order to construct a kinetic energy operator K for the quantum theory, the standard rule is

$$K := -\frac{\hbar^2}{2M}\nabla^2, \tag{4.7}$$

where ∇^2 is the Laplacian operator on functions. By definition, the Laplacian requires a metric g_{ij}:

$$\nabla^2 = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ik}\partial_k\right), \qquad g = |\det(g_{ij})|. \tag{4.8}$$

The fundamental equation (4.6) provides us with a clue as to which metric can be meaningfully chosen. We first observe that Eq. (4.6) is valid in 3–dimensional space, where the volume V scales like r^3; here r, θ, φ are spherical coordinates. This suggests using the Euclidean metric in $\mathbb{R}^3$,

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2, \tag{4.9}$$

and imposing the following two requirements. First, motion along the radial direction r must cause an increase or decrease of the entropy, as per the fundamental equation (4.6), with $V = 4\pi r^3/3$; second, the sphere $r = r_0$ must define an isoentropic surface for each r_0.

Further support for our argument follows from a classic result by H. Weyl:[3] *the volume V occupied by the ideal gas within Euclidean space is related, in*

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[3] We quote this result from ref. [84]: let $R \subset \mathbb{R}^3$ be a bounded region with piecewise smooth boundary, and let $V(R) = \int_R \sqrt{g}\,\mathrm{d}^3x$ denote its volume with respect to some Riemannian metric on $\mathbb{R}^3$. Then the eigenvalue equation for the Laplacian on R, $\nabla^2 f = \lambda f$, supplemented with some mild boundary conditions, has a countable infinity of real eigenvalues λ_n satisfying $0 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots$. These eigenvalues can be arranged into a partition function $Z(t)$,

$$Z(t) := \mathrm{Tr}\exp\left(t\nabla^2\right) = \sum_{n=1}^{\infty}\exp\left(t\lambda_n\right), \tag{4.10}$$

and it turns out that the small t asymptotics of $Z(t)$ is given by

$$Z(t) \simeq \frac{V(R)}{(4\pi t)^{3/2}}, \qquad t \to 0. \tag{4.11}$$

a natural way, to the spectrum of the Laplacian operator within (and on the boundary surface of) V.

We will initially define the Hilbert space $\mathcal{H}$ of *quasistatic quantum mechanics* as the space of those states that minimise the expectation value of the kinetic energy, subject to the constraint that they be normalised (plus some boundary conditions to be specified below). Thus introducing a Lagrange multiplier $-\lambda \in \mathbb{R}$, we need to solve

$$\frac{\delta}{\delta|\psi\rangle}\left(\langle\psi|K|\psi\rangle - \lambda\langle\psi|\psi\rangle\right) = 0, \qquad \langle\psi|\psi\rangle = 1. \tag{4.12}$$

Since K is selfadjoint, Eq. (4.12) leads to

$$K|\psi\rangle = \lambda|\psi\rangle, \tag{4.13}$$

so the Hilbert space $\mathcal{H}$ is initially defined as

$$\mathcal{H} := \mathrm{Ker}\left(K - \lambda_{\mathrm{min}}\right), \tag{4.14}$$

where λ_{min} is the minimal kinetic energy; we have seen that $\lambda \geq 0$. We will presently see how the inclusion of a potential function U affects the definition (4.14) of the Hilbert space.

4.4.2 Motion along isoentropic surfaces

We first analyse motion along a given isoentropic surface, which we take to be the unit sphere S^2. The angular part $\nabla^2_{S^2}$ of the Laplacian operator on $\mathbb{R}^3$ leads to the kinetic energy operator K_{S^2}:

$$K_{S^2}\psi := -\frac{\hbar^2}{2M}\nabla^2_{S^2}\psi = -\frac{\hbar^2}{2M}\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right]. \tag{4.15}$$

Within the space $L^2(S^2)$ the eigenvalues λ of Eq. (4.13) are $\hbar^2 l(l+1)/(2M)$, with $l \in \mathbb{N}$; the least kinetic energy for motion on S^2 corresponds to the zeroth spherical harmonic $Y_{00} = (4\pi)^{-1/2}$:

$$K_{S^2}Y_{00} = 0. \tag{4.16}$$

An analogous result holds within $\mathbb{R}^d$ (it is not necessary to assume that $d = 3$; it is not necessary that the metric be the Euclidean one; it is also not necessary to assume that R is a sphere). However, the Euclidean assumption is suggested by the fundamental equation (4.6), while the assumption of spherical symmetry (in no way imposed by the ideal gas) provides a welcome simplification.

The corresponding particle is completely delocalised on S^2, as befits the fact that its momentum vanishes exactly. The Hilbert space $\mathcal{H}_{S^2}$ is defined as the linear span of the spherical harmonic Y_{00}, *i.e.*,

$$\mathcal{H}_{S^2} = \mathrm{Ker}\left(\nabla^2_{S^2}\right). \tag{4.17}$$

On a compact, connected manifold, the only harmonic functions are the constants; the specific value $(4\pi)^{-1/2}$ is determined by normalisation. Although we have computed $\dim \mathcal{H}_{S^2}$ explicitly, the finite dimensionality of $\mathrm{Ker}\left(\nabla^2_{S^2}\right) \subset L^2(S^2)$ was already guaranteed on the basis of general results concerning the theory of elliptic operators on compact Riemannian manifolds [100].[4] A finite dimensional Hilbert space is a feature of many topological theories [13]: although a metric was initially required to define a Laplacian operator, the metric dependence is softened in the end, through the requirement of quasistatisticity (4.12).

Finally we can add a potential function $U = U(r)$ depending only on the radial variable r and the previous arguments remain entirely valid. We then get back to the situation described in section 4.2: a particle moving quasistatically along the equipotential submanifolds of a certain potential.

4.4.3 Motion across isoentropic surfaces

Next we analyse motion across isoentropic surfaces. The radial part ∇^2_r of the Laplacian operator on $\mathbb{R}^3$ gives rise to the kinetic energy operator K_r:

$$K_r\psi := -\frac{\hbar^2}{2M}\nabla^2_r\psi = -\frac{\hbar^2}{2M}\left(\frac{\mathrm{d}^2\psi}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}\psi}{\mathrm{d}r}\right). \tag{4.18}$$

By Eqs. (4.13) and (4.18) we need to solve

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}\psi}{\mathrm{d}r} + c^2\psi = 0, \qquad c^2 := \frac{2M\lambda}{\hbar^2} \geq 0; \tag{4.19}$$

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[4] In this particular case, one can more simply apply the Hodge theorem [101]: since the 2–sphere S^2 is a compact, orientable Riemannian manifold, we have

$$\dim \mathrm{Ker}\left(\nabla^2_{S^2}\right) = b^0(S^2) = 1,$$

where b^0 is the zeroth Betti number of the manifold in question.

a fundamental set of solutions is $\left\{\psi_{\pm}(r) = r^{-1}\exp(\pm icr)\right\}$. A vanishing kinetic energy is attained when $c = 0$. However the corresponding wavefunction, $\psi(r) = 1/r$, is neither regular at $r = 0$, nor square integrable over the interval $(0, \infty)$. Imposing regularity of $\psi(r)$ at $r = 0$ one is left with the wavefunctions

$$\psi(r) = \frac{1}{r}\sin\left(cr\right), \tag{4.20}$$

while the wavenumber $c \in \mathbb{R}$ remains undetermined. We can determine c if we recall the relation between the squared wavefunction $|\psi|^2$ and the entropy [44]:

$$|\psi|^2 = \exp\left(\frac{S}{k_B}\right). \tag{4.21}$$

Collecting different microstates into a single pure quantum state is reminiscent of Von Neumann's density matrix formulation of the entropy of a mixed quantum state. However, even a pure state embodies a probability distribution; the latter has an associated Shannon entropy. The entropy of a pure state is not monotonic in time under Schroedinger evolution; this problem remains unsolved.

Let r_0 be the radius of the fiducial sphere in Eq. (4.6). When evaluated at $r = r_0$, Eq. (4.21) becomes, by Eq. (4.20),

$$\frac{1}{r_0}\sin(cr_0) = \exp\left(\frac{S_0}{2k_B}\right). \tag{4.22}$$

Now the sine function is bounded between -1 and $+1$. This requires fine tuning the value of the fiducial entropy S_0 as a function of the fiducial radius r_0, or viceversa, if Eq. (4.22) is to have a real solution for c. The simplest choice is to formally set $S_0 = -\infty$. This choice has the added bonus that Eq. (4.22) admits real solutions for c, without the need to fine tune r_0 as a function of S_0; it corresponds to imposing the additional boundary condition $\psi(r_0) = 0$. Then the admissible eigenfunctions, with their corresponding wavenumbers $c_n \in \mathbb{R}$, are given by

$$\psi_n(r) = \sqrt{\frac{2}{r_0}}\,\frac{1}{r}\sin\left(c_n r\right), \qquad c_n = \frac{n\pi}{r_0} \qquad n = 1, 2, \ldots \tag{4.23}$$

We have normalised ψ_n within $L^2\left([0, r_0]\right)$.

The least kinetic energy is attained when $n = 1$. Therefore we define the Hilbert space $\mathcal{H}_r$ as the kernel

$$\mathcal{H}_r = \mathrm{Ker}\left(\nabla_r^2 + c_1^2\right). \tag{4.24}$$

This 1–dimensional space is generated by the wavefunction $\psi_1(r)$. More generally, the finite dimensionality of $\mathrm{Ker}\left(\nabla_r^2 + c_n^2\right) \subset L^2([0, r_0])$ for all $n = 1, 2, \ldots$ is guaranteed by the theory of elliptic operators on compact Riemannian manifolds [100].

So far, the total Hilbert space $\mathcal{H}$ is the tensor product of the spaces (4.17) and (4.24):

$$\mathcal{H} = \mathcal{H}_{S^2} \otimes \mathcal{H}_r. \tag{4.25}$$

We have up to now considered a free particle. If a potential function $U(r)$ is included, then the Hilbert space (4.24) must be redefined to be

$$\mathcal{H}_r = \mathrm{Ker}\left(-\frac{\hbar^2}{2M}\nabla_r^2 - \frac{\hbar^2}{2M}c_1^2 + U(r)\right), \tag{4.26}$$

and the latter substituted back into Eq. (4.25). The above kernel remains finite dimensional. This is because the addition of $U(r)$ does not alter the ellipticity of the Hamiltonian, hence general theorems concerning the spectrum of elliptic operators on compact Riemannian manifolds continue to apply [100]. Of course, the presence of a potential on the quantum mechanical side modifies the fundamental equation (4.6) of the corresponding thermostatics.

We close this section with some remarks.
i) The compact configuration space $[0, r_0] \times S^2$ has advantage that, due to energy quantisation, one can univocally identify a *nonvanishing* state of least kinetic energy. On the noncompact configuration space $[0, \infty) \times S^2$, the allowed energy eigenvalues run over $[0, \infty)$, and no *nonvanishing* state of least energy exists.
ii) Results analogous to those presented above would continue to hold if the free quantum particle were placed in a cubic box of volume L^3, with vanishing boundary conditions for the wavefunction on the sides of the cube. The use of Cartesian coordinates renders isoentropic surfaces (now cubes) somewhat clumsier to work with than spheres, but the expectation value of the entropy (see Eq. (4.28) below) remains metric independent, and also

the Hilbert space continues to be 1–dimensional.
iii) Analogous results would hold as well if we worked in d–dimensional Euclidean space $\mathbb{R}^d$, *viz*: finite dimensionality of the Hilbert space, and metric independence of the expectation of the entropy.

4.4.4 A metric free entropy

It is instructive to compute the expectation value of the entropy in the state (4.23). We set $V = 4\pi r^3/3$, $V_0 = 4\pi r_0^3/3$, and write the quantum mechanical operator corresponding to the classical entropy of Eq. (4.6) as

$$\hat{S}(r) = S_0 + 3k_B \ln\left(\frac{\hat{r}}{r_0}\right). \tag{4.27}$$

The carets are meant to indicate quantum operators. Subtracting the infinite constant S_0 one finds an expectation value of the entropy

$$\langle\psi_n|\hat{S}|\psi_n\rangle = 3k_B \int_0^{r_0} r^2 |\psi_n(r)|^2 \ln\left(\frac{r}{r_0}\right) \mathrm{d}r = 3k_B \left(\frac{\mathrm{Si}(2\pi n)}{2\pi n} - 1\right), \tag{4.28}$$

where $\mathrm{Si}(x) := \int_0^x t^{-1} \sin t \, \mathrm{d}t$ is the sine integral function. In particular, *all terms depending on r_0 drop out of Eq. (4.28).* This is in perfect agreement with the topological character [13] of our model: the entropy cannot depend on the radius r_0 of the fiducial sphere, because the latter requires a metric for its definition.

4.4.5 The quantum mechanical partition function

The quantum mechanical partition function $Z_{\mathrm{qm}}(t)$ is defined by

$$Z_{\mathrm{qm}}(t) = \sum_n \dim\mathcal{H}_n \, \exp\left(-\frac{\mathrm{i}}{\hbar} E_n t\right), \tag{4.29}$$

where $\mathcal{H}_n$ is the Hilbert eigenspace corresponding to the energy eigenvalue E_n. The above sum is usually divergent, but it can be made to converge by Wick rotating the time variable as per

$$Z_{\mathrm{qm}}(\tau) = \sum_n \dim\mathcal{H}_n \, \exp\left(-\frac{1}{\hbar} E_n \tau\right). \tag{4.30}$$

In the quasistatic limit, the above sum is dominated by the least energy eigenvalue, $E_{\rm min}$, and $Z_{\rm qm}(\tau)$ becomes $Z_{\rm qqm}(\tau)$, the subindex "qqm" standing for *quasistatic quantum mechanics*:

$$Z_{\rm qqm}(\tau) = {\rm dim}\mathcal{H}_{\rm min}\, \exp\left(-\frac{1}{\hbar}E_{\rm min}\tau\right). \tag{4.31}$$

Therefore

$$Z_{\rm qqm}(0) = {\rm dim}\mathcal{H}_{\rm min}, \tag{4.32}$$

and *the partition function of quasistatic quantum mechanics computes the dimension of the Hilbert space of quantum states*; also a conclusion that is reminiscent of topological models [13].

4.5 Conclusions and outlook

The application of differential and Riemannian geometry to the theory of thermodynamical fluctuations has turned out to be extremely useful [98, 20, 46]. Thus, *e.g.*, the classical thermodynamics of irreversible processes [87, 95] requires for its formulation a metric on phase space. This metric is provided by Onsager's matrix of kinetic coefficients L_{ij}. The metric enters the quantum mechanical dual theory [44] through the kinetic term in the mechanical Lagrangian.

On the contrary, the thermostatics of equilibrium processes [23] is genuinely metric free. Therefore, if thermostatics is to possess any quantum mechanical dual at all, this dual theory should be a topological theory [13], in the sense that it should be metric independent.

That the classical thermostatics of equilibrium processes should possess a quantum mechanical dual is suggested by two observations. First, by the claim that quantum mechanics is an emergent phenomenon [38, 39, 112, 113, 44, 105]. Second, by the widespread opinion that thermodynamics (be it of equilibrium [23] or nonequilibrium [87, 95]) is the paradigm of all emergent sciences. These conclusions remain unaltered even if, as argued in ref. [24], the emergent aspects of quantum mechanics can only become visible at very high energies.

Two guiding principles are at work here: the notion that forces are entropy gradients, and the requirement that all processes be quasistatic. Entropy gradients, while defining a direction for evolution, ignore microscopic structures, retaining only coarse grained averages: this is a feature of emergent phenomena. Ignoring the metric structure of the underlying manifold amounts to ignoring the kinetic term in the Lagrangian. Quantum mechanically, due to the uncertainty principle, the effects of the kinetic term cannot be cancelled completely, unless one accepts a complete delocalisation of the particle in space. The result of following these two guiding principles is a quasistatic quantum mechanics, which is dual to the classical thermostatics of equilibrium processes, and shares a number of key properties in common with topological, *i.e.*, metric free models.

After completion of this work there appeared ref. [26], where the WKB expansion of quantum mechanics is developed from the point of view of topological string theory [78]. Ref. [26] provides further evidence of the existing links between topological theories and quantum mechanics. Some of these links have been analysed in the present paper, from the alternative standpoint of the approach of emergence of quantum theory; further connections are being studied in an upcoming publication.

Chapter 5

Conclusion

Is quantum mechanics an effective, emergent theory? This is the question that motivates this thesis and the whole body of research under the umbrella of *emergent quantum mechanics.*

Current physics is far from being *A Theory of Everything.* What we have are different physical theories operating at different energy scales, which creates the need to study their intertheoretical relationships. How can Quantum Mechanics and General Relativity be made compatible? How do Thermodynamical theories emerge from their underlying mechanics? How does Classical Mechanics emerge from Quantum Mechanics? May Quantum Mechanics and/or Relativity emerge from a more fundamental theory? Is spacetime a low energy -crystallized- fluid? Is the wavefunction the result of some coarse-graining?

We have seen that the need to revisit the foundations of *the most successful physical theory* comes motivated both by its inner problems and by its intertheoretical relationship with both general relativity and classical mechanics. The idea that the solution may come from a more fundamental theory is motivated both by the probabilistic nature of its results and by the *recursive belief* that there is no *ultimate theory*, or at least no so far from the Planck scale.

The contributions of this PhD thesis, in the form of three articles [45, 44, 22], can be summarized as follows:

- Inspired by Madelung's hydrodynamical interpretation [76], and motivated by the role of dissipation in the emergent approach [36], we developed a mapping between the Quantum Hamilton-Jacobi equation and the Navier-Stokes equation. According to it, the quantum potential (that *"internal mechanism of the continuum"*), can be seen as the viscous term in the Navier-Stokes equations. The correspondence is only valid in the semiclassical approximation, but this is not so problematic since the mapping itself (just like the Madelung mapping [118]) is only invertible if the Bohr-Sommerfeld quantization condition applies for any loop. A first consequence of the mapping is that the viscosity to entropy density ratio is to leading order $\frac{\eta'}{s} = o\left(\frac{\hbar}{k_B}\right)$. This is in agreement with the theoretical [69] and experimental [75] result that, for a quark-gluon plasma, the viscosity to entropy density ratio is greater than $\frac{1}{4\pi}\frac{\hbar}{k_B}$, which suggests that the missing factor in our analysis is $\frac{1}{4\pi}$. Another consequence of the mapping is a picture in which non-stationary states thermalize into stationary states, which have zero viscosity.

- In another vein, inspired by Onsager's formulation of irreversible thermodynamics [87] and by Verlinde's work on emergent gravity [116], we give a mapping between Feynman's propagators for the harmonic oscillator and thermodynamic irreversible Markovian processes in the linear regime. In this analogy, the equipotential surfaces of the mechanical problem correspond to isentropic surfaces of the thermodynamical problem. As suggested by the holographic principle [19], we conjecture that, to every mechanical lagrangian, there will correspond a thermodynamical one. Going beyond the harmonic oscillator means going beyond the linear regime and the markovian approximation on the thermodynamical side, far from equilibrium, to a realm where not so many exact results are known yet.

- Finally, we ask ourselves: if quantum mechanics is dual to irreversible thermodynamics, is there any quantum mechanical theory that is dual to classical thermostatics? The answer seems to be *yes*. At least, we show how a Hilbert space can be constructed from a thermostatical theory defined by an equation of state (that is, an entropy function on the macrostates). The idea underlying thermostatics is that pro-

> cesses take place *very slowly*, so that kinetic energy can be neglected. The uncertainty principle prevents us from graciously removing the kinetic energy term in the quantum hamiltonian, but we can always minimize it. The result is a finite dimensional Hilbert space, obtained as the kernel of certain operator. Inspired by the holographic principle again, we define an entropy operator acting on the states of the Hilbert space, which depends on the entropy function of the thermostatics at hand. For the quantum mechanical theory dual to the *ideal gas thermostatics*, it turns out that the average value of the entropy is metric-independent, whence the connection with topological theories.

Since thermodynamics is the *queen of the emergent theories*, it is our impression that the previous mappings support the believe that quantum mechanics is also emergent, and that its current problems may be solved once the *fundamental* theory is known.

Bibliography

[1] Abraham, S., Fernández de Córdoba, P., Isidro, J., and Santander, J. A mechanics for the Ricci flow. *Int. J. Geom. Meth. Mod. Phys. 6* (2009), 759. arXiv:0810.2356 [hep-th].

[2] Acosta, D. *On the emergent aspects of quantum mechanics in relation to the thermodynamics of irreversible processes and emergent gravity.* PhD thesis, Universitat Politècnica de València, 2012. doi:10.4995/Thesis/10251/36530.

[3] Acosta, D., Fernández de Córdoba, P., Isidro, J., and Santander, J. An entropic picture of emergent quantum mechanics. *Int. J. Geom. Meth. Mod. Phys. 9*, 1250048 (2012). arXiv:1107.1898 [hep-th].

[4] Acosta, D., Fernández de Córdoba, P., Isidro, J., and Santander, J. Emergent quantum mechanics as a classical, irreversible thermodynamics. *Int. J. Geom. Meth. Mod. Phys. 10*, 1350007 (2013). arXiv:1206.4941 [math-ph].

[5] Adler, S. L. *Quantim theory as an emergent phenomenon.* Cambridge University Press, 2004.

[6] Arnold, V. *Mathematical Methods of Classical Mechanics.* Springer, 1989.

[7] Baez, J. C., and Biamonte, J. Quantum techniques for stochastic mechanics, 2012. arXiv:1209.3632 [quant-ph].

[8] Baez, J. C., and Pollard, B. S. Quantropy. *Entropy 17* (2013), 772. arXiv:1311.0813 [quant-ph].

[9] Bassi, A., Lochan, K., Satin, S., Singh, T. P., and Ulbricht, H. Models of wave-function collapse, underlying theories, and experimental tests. *Reviews of modern physics 85* (2013), 471.

[10] Batterman, R., Ed. *The Oxford Handbook of Philosophy of Physics.* Oxford University Press, 2013.

[11] Bell, J. S. On the problem of hidden variables in quantum mechanics. *Reviews of modern physics 38* (1966).

[12] Bell, J. S. Six possible worlds of quantum mechanics. *Proceedings of Nobel Symposium 65* (1989), 359–373.

[13] Birmingham, D., Blau, M., Rakowski, M., and Thompson, G. Topological field theory. *Phys. Rep. 209* (1991), 129.

[14] Blasone, M., Jizba, P., and Kleinert, H. Quantum behavior of deterministic systems with information loss: Path integral approach. *Annals of Physics 320* (2005), 468. arXiv:quant-ph/0504200.

[15] Blasone, M., Jizba, P., and Scardigli, F. Can quantum mechanics be an emergent phenomenon? *Journal of Physics: Conference Series 174*, 012034 (2009).

[16] Blasone, M., Jizba, P., Scardigli, F., and Vitiello, G. Dissipation and quantization for composite systems. *Phys. Lett. A 373* (2009), 4106. arXiv:0905.4078 [quant-ph].

[17] Blasone, M., Jizba, P., and Vitiello, G. Dissipation and quantization. *Phys. Lett. A 287* (2001), 205. arXiv:hep-th/0007138.

[18] Bohm, D. A suggested interpretation of the quantum theory in terms of “hidden” variables i. *Phys. Rev. 85* (1952), 166.

[19] Bousso, R. The holographic principle. *Rev. Mod. Phys. 74* (2002), 825. arXiv:hep-th/0203101.

[20] Bravetti, A., López-Monsalvo, C., and Nettel, F. Conformal gauge transformations in thermodynamics. *Entropy 17* (2015), 6150. arXiv:1506.07160 [math-ph].

[21] Bricmont, J. *Making sense of quantum mechanics.* Springer, 2016.

[22] Cabrera, D., Fernández de Córdoba, P., Isidro, J., and Vazquez Molina, J. Entropy, topological theories and emergent quantum mechanics. *Entropy 19*, 87 (2016). arXiv:1611.07357 [math-ph].

[23] Callen, H. *Thermodynamics and an Introduction to Thermostatics.* Wiley, New York, 1985.

[24] Calmet, X. Quantum mechanics, gravity and modified quantization relations. *Phil. Trans. R. Soc. A 373*, 20140244 (2015). arXiv:1504.07768 [hep-th].

[25] Carroll, R. *On the Emergence Theme of Physics.* World Scientific, 2010.

[26] Codesido, S., and Mariño, M. Holomorphic anomaly and quantum mechanics, 2016. arXiv:1612.07687 [hep-th].

[27] Cohen-Tannoudji, G. An interpretive conjecture for physics beyond the standard models: Generalized complementarity. arXiv:1402.0823 [gr-qc].

[28] Couder, Y., Protière, S., Fort, E., and Boudaoud, A. Dynamical phenomena: Walking and orbiting droplets. *Nature 437* (2005), 208.

[29] Crowther, K., Ed. *Effective Spacetime. Understanding Emergence in Effective Field Theory and Quantum Gravity.* Springer, 2016.

[30] de Broglie, L. La thermodynamique cachée des particules. *Ann. Inst. Poincaré (A) Physique Théorique 1* (1964), 1.

[31] Dirac, P. A. M. *Lectures on Quantum Mechanics.* Dover, New York, 2001.

[32] Doob, J. *Stochastic Processes.* Wiley, New York, 1953.

[33] Dürr, D., Goldstein, S., and Zanghi, N. *Quantum Physics Without Quantum Philosophy.* Springer, Berlin Heidelberg, 2012.

[34] Dürr, D., and Teufel, S. *Bohmian Mechanics. The Physics and Mathematics of Quantum Theory.* Springer, Berlin Heidelberg, 2009.

[35] Einstein, A. Proceedings of the 1927 solvay conference. Available at p441 of Bacciagaluppi, G. and Valentini, A., *Quantum Mechanics at the Crossroads. Reconsidering the 1927 Solvay Conference*, Cambridge University Press, Cambridge, 2009.

[36] Elze, H.-T. Symmetry aspects in emergent quantum mechanics. *J. Phys. Conf. Ser. 171*, 012034 (2009).

[37] Elze, H.-T. Discrete mechanics, time machines and hybrid systems. *EPJ Web of Conferences 58*, 01013 (2013). arXiv:1310.2862 [quant-ph].

[38] Elze, H.-T. Multipartite cellular automata and the superposition principle. *Int. J. Quant. Info. 14*, 1640001 (2016). arXiv:1604.04201 [quant-ph].

[39] Elze, H.-T. Quantum models as classical cellular automata. To appear in *Journal of Physics: Conference Series*, 2017. arXiv:1701.02252 [quant-ph].

[40] Esfeld, M., Deckert, D. A., and Oldofredi, A. *A guide to the philosophy of cosmology.* Oxford University Press, forthcoming publication, ch. What is matter? The fundamental ontology of atomism and structural realism. arXiv:1510.03719 [physics.hist-ph].

[41] Fernández de Córdoba, P., Isidro, J., and Perea, M. H. Emergence from irreversibility. *Journal of Physics: Conference Series 442*, 012033 (2013).

[42] Fernández de Córdoba, P., Isidro, J., and Perea, M. H. Emergent quantum mechanics as a thermal ensemble. *Int. J. Geom. Meth. Mod. Phys.*, 1450068 (2014). arXiv:1304.6295 [math-ph].

[43] Fernández de Córdoba, P., Isidro, J., Perea, M. H., and Vazquez Molina, J. The irreversible quantum. *Int. J. Geom. Meth. Mod. Phys. 12*, 1550013 (2015). arXiv:1311.2787 [quant-ph].

[44] Fernández de Córdoba, P., Isidro, J., and Vazquez Molina, J. The holographic quantum. *Foundations of Physics 46* (2016), 787–803. arXiv:1503.07662 [quant-ph].

[45] Fernández de Córdoba, P., Isidro, J., and Vazquez Molina, J. Schroedinger vs. Navier–Stokes. *Entropy 18*, 34 (2016). arXiv:1409.7036 [math-ph].

[46] Fernández de Córdoba, P., and Isidro, J. M. Generalised complex geometry in thermodynamical fluctuation theory. *Entropy 17* (2015), 5888. arXiv:1505.06647 [math-ph].

[47] Feynman, R. Space–time approach to non–relativistic quantum mechanics. *Rev. Mod. Phys. 20* (1948), 367.

[48] Finster, F. The fermionic projector, entanglement and the collapse of the wavefunction. *J. Phys. Conf. Ser. 206*, 012024 (2011). arXiv:1011.2162 [quant-ph].

[49] Finster, F., Grotz, A., and Schiefeneder, D. *Quantum Field Theory and Gravity. Conceptual and Mathematical Advances in the Search for a Unified Framework*. Birkhäuser, 2012, ch. Causal Fermion Systems: A Quantum Space-Time Emerging from an Action Principle, p. 157. arXiv:1102.2585 [math-ph].

[50] Freire, O. J. Science and exile: David Bohm, the cold war, and a new interpretation of quantum mechanics. *Historical Studies in the Physical and Biological Sciences 36* (2005), 1.

[51] Fusca, D. The Madelung transform as a momentum map, 2016. arXiv:1512.04611v2 [math.SG].

[52] Gallego Torromé, R. On the emergence of quantum mechanics, diffeomorphism invariance and the weak equivalence principle from deterministic cartan-randers systems. arXiv:1402.5070 [math-ph].

[53] Gallego Torromé, R. A Finslerian version of ’t Hooft deterministic quantum models. *J. Math. Phys. 47*, 072101 (2006). arXiv:math-ph/0501010.

[54] Gitman, D. M., Tyutin, I. V., and Voronov, D. L. *Self-adjoint Extensions in Quantum Mechanics.* Birkhäuser, New York, 2012.

[55] Goldstein, S., Hara, T., and Tasaki, H. The approach to equilibrium in a macroscopic quantum system for a typical nonequilibrium subspace. arXiv:1402.3380 [cond-mat.stat-mech].

[56] Groessing, G. From classical Hamiltonian flow to quantum theory: Derivation of the Schroedinger equation. *Foundations of Physics Letters 17* (2004), 343. arXiv:quant-ph/0311109.

[57] Hall, B. C. *Quantum Theory for Mathematicians.* Springer, New York, 2013.

[58] Hartnoll, S. Lectures on holographic methods for condensed matter physics. *Class. Quant. Grav. 26*, 224002 (2009). arXiv:0903.3246 [hep-th].

[59] Hossenfelder, S. Comments on and comments on comments on Verlinde's paper "On the origin of gravity and the laws of Newton". arXiv:1003.1015 [gr-qc].

[60] Jennings, D., and Leifer, M. No return to classical reality. *Contemporary Physics 57* (2016).

[61] Kalckar, J., Ed. *Niels Bohr Collected Works. 7. Foundations of Quantum Physics II (1933-1958).* Elsevier, 1996, ch. Part II: Complementarity: Bedrock of the Quantal Description, p. 248.

[62] Kapusta, J., and Gale, C. *Finite-Temperature Field Theory: Principles and Applications.* Cambridge University Press, Cambridge, 2006.

[63] Kim, Y., Shin, I., and Tsukioka, T. Holographic QCD: Past, present, and future. *Prog. Part. Nucl. Phys. 68* (2013), 55. arXiv:1205.4852 [hep-ph].

[64] Kleinert, H. *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets.* World Scientific, Berlin, 1990.

[65] Kobayashi, S., and Nomizu, K. *Foundations of Differential Geometry.* Wiley, New York, 1996.

[66] Kochen, S., and Specker, E. P. The problem of hidden variables in quantum mechanics. *Journal of Mathematics and Mechanics 17* (1967), 59.

[67] Kolekar, S., and Padmanabhan, T. Indistinguishability of thermal and quantum fluctuations. arXiv:1308.6289 [gr-qc].

[68] Konopka, T., Markopoulou, F., and Severini, S. Quantum graphity: a model of emergent locality. *Phys. Rev. D 77*, 104029 (2008). arXiv:0801.0861 [hep-th].

[69] Kovtun, P., Son, D., and Starinets, A. Viscosity in strongly interacting quantum field theories from black hole physics. *Phys. Rev. Lett. 94*, 111601 (2005). arXiv:hep-th/0405231.

[70] Landau, L., and Lifshitz, E. *Fluid Mechanics (2ed in English).* Pergamon Press, Oxford, 1987.

[71] Landau, L., and Lifshitz, E. M. *Quantum Mechanics. Non-relativistic theory (3ed in English).* Pergamon Press, 1977.

[72] Landauer, R. Irreversibility and heat generation in the computing process. *IBM Journal of Research and Development 5* (1961), 183.

[73] Lessel, B. Wasserstein-geometry as a natural language for quantum hydrodynamics. Talk available in the Perimeter institute recorded seminar archive. http://pirsa.org/16050049/.

[74] Licata, I., Ed. *Beyond peaceful coexistence. The emergence of space, time and quantum.* Imperial College Press, 2016.

[75] Luzum, M., and Romatschke, P. Conformal relativistic viscous hydrodynamics: Applications to RHIC results at $\sqrt{s_{NN}} = 200$ gev. *Phys. Rev. C 78*, 034915 (2008). arXiv:0804.4015 [nucl-th].

[76] Madelung, E. Quantentheorie in hydrodynamischer form. *Z. Phys. 40* (1927), 322. Translation by Delphenich, D. H.

[77] Maldacena, J. The large N limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys. 2* (1998), 231. arXiv:hep-th/9711200.

[78] Mariño, M. *Chern–Simons Theory, Matrix Models, and Topological Strings.* Oxford Univesity Press, Oxford, 2005.

[79] Markopoulou, F. The computing spacetime. In *How the World Computes. Turing Centenary Conference and 8th Conference on Computability in Europe, CiE 2012* (2012), p. 472. arXiv:1201.3398 [gr-qc].

[80] Markopoulou, F., and Smolin, L. Quantum theory from quantum gravity. *Phys. Rev. D 70*, 124029 (2004). arXiv:gr-qc/0311059.

[81] Matone, M. 'Thermodynamique cachée des particules' and the quantum potential". *Ann. Fond. Broglie 37* (2012), 177. arXiv:1111.0270 [hep-ph].

[82] Maudlin, T. Three measurement problems. *Topoi 14* (1995), 7–15.

[83] Maudlin, T. *Quantum Non-Locality and Relativity (3ed).* Wiley-Blackwell, Chichester (UK), 2011.

[84] McKean, H., and Singer, I. Curvature and the eigenvalues of the Laplacian. *J. Diff. Geom. 1* (1967).

[85] Mehra, J., Ed. *The Collected Works of Eugene Paul Wigner. Part B Historical, Philosophical, and Socio-Political Papers.* Springer, 1995, ch. Remarks on the Mind-Body Question, p. 247.

[86] Nielsen, M. A., and Chuang, I. L. *Quantum Computation and Quantum Information, 10th Anniversary Edition.* Cambridge University Press, New York, 2010.

[87] Onsager, L., and Machlup, S. Fluctuations and irreversible processes. *Phys. Rev. 91* (1953), 1505.

[88] Padmanabhan, T. General relativity from a thermodynamic perspective. arXiv:1312.3253 [gr-qc].

[89] Padmanabhan, T. Thermodynamical aspects of gravity: New insights. *Rep. Prog. Phys. 73*, 046901 (2010). arXiv:0911.5004 [gr-qc].

[90] Padmanabhan, T. Gravity ~~and~~ is thermodynamics. *Current Science 109*, 12 (2015), 2236. arXiv:1512.06546 [gr-qc].

[91] Penrose, R. *The Road to Reality.* Jonathan Cape, London, 2004.

[92] Penrose, R. Black holes, quantum theory and cosmology. *J. Phys. Conf. Ser. 174*, 012001 (2009).

[93] Penrose, R. On the gravitization of quantum mechanics 1: Quantum state reduction. *Found. Phys. 44* (2014), 557.

[94] Perea, M. H. *On the semiclassical limit of emergent quantum mechanics, as a classical thermodynamics of irreversible processes in the linear regime.* PhD thesis, Universitat Politècnica de València, 2015. doi:10.4995/Thesis/10251/54840.

[95] Prigogine, I. *Introduction to Thermodynamics of Irreversible Processes.* Interscience, New York, 1961.

[96] Raftery, J., Sadri, D., Schmidt, S., Türeci, H., and Houck, A. Observation of a dissipation–induced classical to quantum transition. *Phys. Rev. X 4*, 031043 (2014). arXiv:1312.2963 [quant-ph].

[97] Reddiger, M. The Madelung picture as a foundation of geometric quantum theory, 2017. arXiv:1509.00467v4 [quant-ph].

[98] Ruppeiner, G. Riemannian geometry in thermodynamic fluctuation theory. *Rev. Mod. Phys. 67* (1995), 605.

[99] Schroedinger, E. Die gegenwärtige situation in der quantenmechanik. *Naturwissenschaften 23* (1935). translation by J.D. Trimmer available in The present situation in quantum mechanics, *Proceedings of the American Philosophical Society* 124, 323–338 (1984).

[100] Schwarz, A. *Quantum Field Theory and Topology.* Springer-Verlag, Berlin, 1993.

[101] Schwarz, A. *Topology for Physicists.* Springer-Verlag, Berlin, 1994.

[102] Seligman, N., and Graham, N., Eds. *The Many Worlds Interpretation of Quantum Mechanics.* Princeton University Press, Princeton, 1973.

[103] Smolin, L. On the nature of quantum fluctuations and their relation to gravitation and the principle of inertia. *Class. Quant. Grav. 3* (1986), 347.

[104] Smolin, L. Quantum gravity and the statistical interpretation of quantum mechanics. *Int. J. Theor. Phys. 25* (1986), 215.

[105] Smolin, L. Quantum mechanics and the principle of maximal variety. *Foundations of Physics 46* (2016), 736. arXiv:1506.02938 [quant-ph].

[106] Stephens, C., 't Hooft, G., and Whiting, B. Black hole evaporation without information loss. *Class. Quant. Grav. 11* (1994), 621. arXiv:gr-qc/9310006.

[107] Susskind, L. The world as a hologram. *J. Math. Phys. 36*, 6377 (1995). arXiv:hep-th/9409089.

[108] Svozil, K. Space and time in a quantized world. *Int. J. Theor. Phys. 54* (2015), 4376.

[109] 't Hooft, G. The fate of the quantum. arXiv:1308.1007 [quant-ph].

[110] 't Hooft, G. Quantum gravity as a dissipative deterministic system. *Class. Quant. Grav. 16* (1999), 3263. arXiv:gr-qc/9903084.

[111] 't Hooft, G. Emergent quantum mechanics and emergent symmetries. *AIP Conf. Proc. 957* (2007), 154. arXiv:0707.4568 [hep-th].

[112] 't Hooft, G. The mathematical basis for deterministic quantum mechanics. *J. Phys. Conf. Ser. 67*, 012015 (2007). arXiv:quant-ph/0604008.

[113] 't Hooft, G. *The Cellular Automaton Interpretation of Quantum Mechanics.* Springer, 2016. arXiv:1405.1548 [quant-ph].

[114] Tegmark, M. Shut up and calculate, 2007. arXiv:0709.4024 [physics.pop-ph].

[115] Vaidman, L. All is ψ. *Journal of Physics: Conference Series 701*, 012020 (2016).

[116] Verlinde, E. On the origin of gravity and the laws of newton. *JHEP 29* (2011). arXiv:1001.0785[hep-th].

[117] Verlinde, E. Emergent gravity and the dark universe, 2016. arXiv:1611.02269 [hep-th].

[118] Wallstrom, T. C. Inequivalence between the Schrodinger equation and the Madelung hydrodynamic equations. *Physical Review A 49* (1994), 1613.

[119] Wick, D. *The Infamous Boundary. Seven Decades of Heresy in Quantum Physics.* Copernicus, an imprint of Springer-Verlag, New York, 1995.

[120] Witten, E. Anti–de Sitter space and holography. *Adv. Theor. Math. Phys. 2* (1998), 253. arXiv:hep-th/9802150.

[121] Yin, Z., and Li, T. Bringing quantum mechanics to life: from Schrödinger's cat to Schrödinger's microbe. *Contemporary Physics* (2017). DOI: 10.1080/00107514.2016.1261860.

[122] Zurek, W. Decoherence, einselection, and the quantum origins of the classical. *Rev. Mod. Phys. 75* (2003), 715. arXiv:quant-ph/0105127.