# On the weak form of Ekeland's Variational Principle in quasi-metric spaces


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**Abstract**

We show that a quasi-metric space is right K-sequentially complete if and only if it satisfies the property of the weak form of Ekeland's Variational Principle. This result solves a question raised by S. Cobzaş in his paper "Completeness in quasi-metric spaces and Ekeland's Variational Principle", Topology and its Applications 158 (2011), 1073-1084.



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## 1 Introduction and preliminaries

Throughout this paper the letters $\mathbb{R}$, $\mathbb{N}$ and $\omega$ will denote the set of all real numbers, the set of all positive integer numbers and the set of all nonnegative integer numbers, respectively.

Recall that if $X$ is a (nonempty) set, a function $f : X \to \mathbb{R} \cup \{\infty\}$ is said to be proper if there exists $x \in X$ such that $f(x) < \infty$.

In [7, Theorem 1.1], Ekeland proved his celebrated variational principle in the realm of complete metric spaces. Later on, Weston [14] (see also [13]) showed that an easy consequence of Ekeland's Variational Principle, the so-called weak form of Ekeland's Variational Principle, characterizes the metric completeness.

Quasi-metric versions of Ekeland's Variational Principle and its weak form have been obtained in [1, 3, 10]. In particular, Cobzaş proved in [3] the following nice result.

**Theorem 1** (*Ekeland Variational Principle* [3, Theorem 2.4]). *Suppose that (X,d) is a $T_1$ quasi-metric space and $f : X \to \mathbb{R} \cup \{\infty\}$ is a proper bounded below function. For given $\varepsilon > 0$ let $x_\varepsilon \in X$ be such that $f(x_\varepsilon) \leq \inf f(X) + \varepsilon$.*
*If $(X, d)$ is right K-sequentially complete and f is d-lsc, then for every $\lambda > 0$ there exists $z = z_{\varepsilon,\lambda} \in X$ such that*
(a) $f(z) + \frac{\varepsilon}{\lambda} d(z, x_\varepsilon) \leq f(x_\varepsilon)$;
(b) $d(z, x_\varepsilon) \leq \lambda$;
(c) $f(z) < f(x) + \frac{\varepsilon}{\lambda} d(x, z)$ for all $x \in X \backslash \{z\}$.

Taking $\lambda = 1$ in Theorem 1, Cobzaş deduced the following.

**Corollary 1** (*Ekeland Variational Principle-weak form* [3, Corollary 2.7]). *Let (X,d) be a right K-sequentially complete $T_1$ quasi-metric space. Then, for every proper bounded below d-lsc function $f : X \to \mathbb{R} \cup \{\infty\}$ and for every $\varepsilon > 0$ there exists $y_\varepsilon \in X$ such that*

(i) $f(y_\varepsilon) \leq \inf f(X) + \varepsilon$;
(ii) $f(y_\varepsilon) < f(x) + \varepsilon d(x, y_\varepsilon)$ *for all* $x \in X \backslash \{y_\varepsilon\}$.

In this note we shall prove that the converse of Corollary 1 holds. Thus, we obtain a characterization of right K-sequential completeness which, on one hand, solves a question raised in [3, Remark 2.11], and, on the other hand, generalizes the aforementioned characterization of metric completeness, due to Weston, to the quasi-metric framework. In fact, we shall prove the result in the realm of (non necessarily $T_1$) quasi-metric spaces by using suitable modifications of the lower semicontinuity of $f$ and of condition (ii) in Corollary 1. Connections with quasi-metric versions of Caristi's fixed point theorem will be also considered.

In the rest of this section we recall some notions and basic properties on the theory of quasi-metric spaces which will be used in the sequel. Our main references will be [4] and [9].

A quasi-metric on set $X$ is a function $d : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$ : (i) $x = y \Leftrightarrow d(x, y) = d(y, x) = 0$; (ii) $d(x, z) \leq d(x, y) + d(y, z)$.

A quasi-metric space is a pair $(X, d)$ such that $X$ is a set and $d$ is a quasi-metric on $X$.

Each quasi-metric $d$ on $X$ induces a $T_0$ topology $\tau_d$ on $X$ which has as a base the family of open balls $\{B_d(x, r) : x \in X, \varepsilon > 0\}$, where $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

If $\tau_d$ is a $T_1$ topology on $X$, we say that $(X, d)$ is a $T_1$ quasi-metric space.

Given a subset $A$ of a quasi-metric space $(X, d)$, we shall denote by $\overline{A}$ the closure of $A$ with respect to $\tau_d$.

If $(X, d)$ is a quasi-metric space and $f : X \to \mathbb{R} \cup \{\infty\}$ is a proper function, we say that $f$ is $d$-lsc whenever $f$ is $\tau_d$-lower semicontinuous on $X$.

A sequence $(x_n)_{n\in\mathbb{N}}$ in a quasi-metric space $(X, d)$ is said to be right K-Cauchy if for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ whenever $n_0 \leq n \leq m$ (see e.g. [4, 9, 11]).

A quasi-metric space $(X, d)$ is said to be right K-sequentially complete (right K-complete in [3]) if each right K-Cauchy sequence in $(X, d)$ is $\tau_d$-convergent in $X$.

The Sorgenfrey quasi-metric space (see e.g. [4, Example 1.1.6]) is a well-known example of a right K-sequentially complete $T_1$ quasi-metric space.

We conclude the section with the following well-known fact (see e.g. [12, Lemma 1]) which will be used in the proof of our main result.

**Lemma.** *Let $(x_n)_{n\in\mathbb{N}}$ be a right K-Cauchy sequence in a quasi-metric space $(X, d)$. If $(x_n)_{n\in\mathbb{N}}$ has a $\tau_d$-cluster point $x \in X$, then $(x_n)_{n\in\mathbb{N}}$ is $\tau_d$-convergent to $x$.*

# 2 The results

We start this section by introducing a generalization of the notion of lower semicontinuity which is inspired in the concept of sequential lower semicontinuity and that will be crucial in the rest of the paper.

Recall (see e.g. [5, Chapter 1] or [6, Chapter 1]) that a proper function $f : X \to \mathbb{R} \cup \{\infty\}$ on a topological space $(X, \tau)$ is said to be sequentially lower semicontinuous if whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in $X$ that $\tau$-converges to some $x \in X$, we have $f(x) \leq \liminf_{n\to\infty} f(x_n)$.

It is well known ([5, Proposition 1.3]) that if $(X, \tau)$ is first countable then a proper function $f : X \to \mathbb{R} \cup \{\infty\}$ is lower semicontinuous if and only if it is sequentially lower semicontinuous. Therefore, lower semicontinuity and sequential lower semicontinuity are equivalent concepts for quasi-metric spaces.

Now let $(X, d)$ be a quasi-metric space. We say that a proper function $f : X \to \mathbb{R} \cup \{\infty\}$ is nearly lower semicontinuous (nearly $d$-lsc, in short) if whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence of distinct points in $X$ that $\tau_d$-converges to some $x \in X$, we have $f(x) \leq \liminf_{n\to\infty} f(x_n)$.

Clearly, a proper function $f : X \to \mathbb{R} \cup \{\infty\}$ on a $T_1$ quasi-metric space $(X, d)$ is (sequentially) $d$-lsc if and only if it is nearly $d$-lsc.

However, this equivalence does not hold for quasi-metric spaces, in general. An easy example is as follows: Let $X = \{0, 1\}$ and let $d$ be the quasi-metric on $X$ such that $d(0, 0) = d(0, 1) = d(1, 1) = 0$ and $d(1, 0) = 1$. Then, every proper function

$f : X \to \mathbb{R} \cup \{\infty\}$ is nearly $d$-lsc, whereas the function $f$ defined on $X$ as $f(0) = 1$ and $f(1) = 0$ is not $d$-lsc.

Next we prove the main result of this paper.

**Theorem 2.** *For a quasi-metric space $(X, d)$ the following conditions are equivalent:*

(1) *$(X, d)$ is right K-sequentially complete.*

(2) *For every self mapping $T$ of $X$ and every proper bounded below nearly d-lsc function $\varphi : X \to \mathbb{R} \cup \{\infty\}$ satisfying $d(Tx, x) + \varphi(Tx) \leq \varphi(x)$ for all $x \in X$, there exists $z = z_{T,\varphi} \in X$ such that $\varphi(z) = \varphi(Tz)$.*

(3) *For every proper bounded below nearly d-lsc function $f : X \to \mathbb{R} \cup \{\infty\}$ and for every $\varepsilon > 0$ there exists $y_\varepsilon \in X$ such that*

(i) $f(y_\varepsilon) \leq \inf f(X) + \varepsilon$;

(ii') *$f(y_\varepsilon) < f(x) + \varepsilon d(x, y_\varepsilon)$ for all $x \in X \setminus \overline{\{y_\varepsilon\}}$, and $f(y_\varepsilon) \leq f(x)$ for all $x \in \overline{\{y_\varepsilon\}}$.*

*Proof.* (1) $\Rightarrow$ (2). Let $T$ be a self mapping of $X$ and $\varphi : X \to \mathbb{R} \cup \{\infty\}$ a proper bounded below nearly $d$-lsc function such that $d(Tx, x) + \varphi(Tx) \leq \varphi(x)$ for all $x \in X$.

For each $x \in X$ let

$$A_x := \{y \in X : d(y, x) + \varphi(y) \leq \varphi(x)\}, \qquad \text{and} \qquad i(x) := \inf \varphi(A_x).$$

Then $\{x, Tx\} \subseteq A_x$ and $i(x) \leq \varphi(x)$ for all $x \in X$. Take $x_0 \in X$ such that $\varphi(x_0) < \infty$. There is $x_1 \in A_{x_0}$ such that $\varphi(x_1) \leq i(x_0)+1$. In particular, $\varphi(x_1) < \infty$. Similarly, there is $x_2 \in A_{x_1}$ such that $\varphi(x_2) \leq i(x_1) + 2^{-1}$. Following this process we obtain a sequence $(x_n)_{n\in\omega}$ in $X$ such that

$$(\mathrm{I}_1) \qquad x_{n+1} \in A_{x_n},$$

$$(\mathrm{I}_2) \qquad \varphi(x_n) < \infty,$$

and

$$(\mathrm{I}_3) \qquad \varphi(x_{n+1}) - 2^{-n} \leq i(x_n) \leq \varphi(x_n),$$

for all $n \in \omega$.

We are going to show that $(x_n)_{n\in\omega}$ is a right K-Cauchy sequence in $(X, d)$. By conditions $(\mathrm{I}_1)$ and $(\mathrm{I}_2)$ we have

$$(\mathrm{II}) \qquad d(x_{n+1}, x_n) \leq \varphi(x_n) - \varphi(x_{n+1}),$$

for all $n \in \omega$.

By condition (II), $(\varphi(x_n))_{n\in\omega}$ is a non-increasing sequence of real numbers bounded below, so it converges to $l := \inf_{n\in\omega} \varphi(x_n)$.

From condition (II) and the triangle inequality we deduce that

$$d(x_m, x_n) \leq \varphi(x_n) - \varphi(x_m),$$

whenever $m \geq n$. Hence, $(x_n)_{n\in\omega}$ is a right K-Cauchy sequence in $(X, d)$.

Without loss of generality, we distinguish the following two cases.

Case 1. The sequence $(x_n)_{n\in\omega}$ is eventually constant. Then, there is $n_0 \in \omega$ such that $x_n = x_{n_0}$ for all $n \geq n_0$. By condition ($\mathrm{I}_3$), $\varphi(x_{n_0}) - 2^{-n} \leq i(x_{n_0}) \leq \varphi(x_{n_0})$ for all $n \geq n_0$. Therefore, taking limit when $n \to \infty$, $i(x_{n_0}) = \varphi(x_{n_0})$. Since $Tx_{n_0} \in Ax_{n_0}$, we have $i(x_{n_0}) \leq \varphi(Tx_{n_0}) \leq \varphi(x_{n_0})$, so $\varphi(Tx_{n_0}) = \varphi(x_{n_0})$.

Case 2. $x_n \neq x_m$ for all $n, m \in \omega$ with $n \neq m$. Since $(X, d)$ is right K-sequentially complete, there exists $z \in X$ such that $\lim_{n\to\infty} d(z, x_n) = 0$.

We shall prove that $\varphi(z) = \varphi(Tz)$. To this end, we first note that $\varphi(z) \leq l$ because $\varphi$ is nearly $d$-lsc.

Let $n \in \omega$ be fixed. Given $\varepsilon > 0$ there exists $m > n$ such that $d(z, x_m) < \varepsilon$. Then

$$\begin{aligned} d(z, x_n) &\leq d(z, x_m) + d(x_m, x_n) < \varepsilon + \varphi(x_n) - \varphi(x_m) \\ &\leq \varepsilon + \varphi(x_n) - \varphi(z). \end{aligned}$$

(The last inequality holds because $\varphi(z) \leq l \leq \varphi(x_m)$).

Since $\varepsilon$ was arbitrarily chosen, we deduce that $d(z, x_n) \leq \varphi(x_n) - \varphi(z)$, and thus $z \in A_{x_n}$ for all $n \in \omega$. Consequently $i(x_n) \leq \varphi(z) \leq \varphi(x_n)$ for all $n \in \omega$. Since, by condition ($\mathrm{I}_3$), $l = \lim_{n\to\infty} i(x_n)$, we conclude that $l = \varphi(z)$.

Let show now that $\varphi(Tz) = l$, which will imply $\varphi(z) = \varphi(Tz)$.

Indeed, since $z \in A_{x_n}$ we obtain

$$d(Tz, x_n) \leq d(Tz, z) + d(z, x_n) \leq \varphi(z) - \varphi(Tz) + \varphi(x_n) - \varphi(z),$$

for all $n \in \omega$. This implies that $Tz \in A_{x_n}$, and thus $i(x_n) \leq \varphi(Tz) \leq \varphi(x_n)$ for all $n \in \omega$, which for $n \to \infty$ yields $\varphi(Tz) = l$.

(2) $\Rightarrow$ (3). We shall proceed by contradiction. Suppose that there exist a proper bounded below nearly $d$-lsc function $f : X \to \mathbb{R} \cup \{\infty\}$ and an $\varepsilon > 0$ such that the conclusion of (3) fails. Putting

$$A := \{y \in X : f(y) \leq \inf f(X) + \varepsilon\},$$

it follows that for every $y \in A$ one (or both) of the following conditions holds:

(c1) there exists $x_y \in X \backslash \overline{\{y\}}$ such that $f(y) \geq f(x_y) + \varepsilon d(x_y, y)$.

(c2) there exists $x_y \in \overline{\{y\}} \backslash \{y\}$ such that $f(y) > f(x_y)$.

Note that if $x_y$ satisfies (c1), then $d(x_y, y) > 0$, so $f(y) > f(x_y)$.

Therefore, in both cases, $x_y \in A$ whenever $y \in A$.

Now fix an $y_0 \in A$ and define $T : X \to X$ by $Tx = y_0$ for all $x \in X \backslash A$, and $Ty = x_y$ for all $y \in A$.

Define now $\varphi : X \to \mathbb{R} \cup \{\infty\}$ by $\varphi(x) = \infty$ for all $x \in X \backslash A$, and $\varphi(y) = \varepsilon^{-1} f(y)$ for all $y \in A$.

In fact we make a selection - take for instance $x_y$ satisfying (c1) if both of the conditions (c1) and (c2) are satisifed.

It is clear that $\varphi$ is proper and bounded below.

Moreover, it is nearly $d$-lsc. Indeed, let $(x_n)_{n\in\mathbb{N}}$ be a sequence of distinct points in $X$ that $\tau_d$-converges to some $x \in X$. Then, for each $n \in \mathbb{N}$, $\varphi(x_n) = \infty$ or $\varphi(x_n) = \varepsilon^{-1} f(x_n)$. If $x \in A$, from the fact that $f(x) \leq \liminf_{n\to\infty} f(x_n)$, it immediately follows that $\varphi(x) \leq \liminf_{n\to\infty} \varphi(x_n)$. If $x \in X \backslash A$, from the near $d$-lower semicontinuity of $f$ we deduce that $x_n \in X \backslash A$ eventually, so there is $n_0 \in \mathbb{N}$ such that $\varphi(x) = \varphi(x_n) = \infty$ for all $n \geq n_0$.

Next we show that $d(Tx, x) + \varphi(Tx) \leq \varphi(x)$ for all $x \in X$.

For each $x \in X \backslash A$, we have $d(Tx, x) + \varphi(Tx) < \infty = \varphi(x)$.

Now let $y \in A$. If $x_y \in X \backslash \overline{\{y\}}$, we obtain (recall that $x_y \in A$),

$$d(Ty, y) + \varphi(Ty) = d(x_y, y) + \varepsilon^{-1} f(x_y) \leq \varepsilon^{-1}(f(y) - f(x_y)) + \varepsilon^{-1} f(x_y) = \varphi(y).$$

If $x_y \in \overline{\{y\}} \backslash \{y\}$, we obtain (recall again that $x_y \in A$),

$$d(Ty, y) + \varphi(Ty) = d(x_y, y) + \varepsilon^{-1} f(x_y) = \varepsilon^{-1} f(x_y) = \varphi(y).$$

Thus, we have shown that $d(Tx, x) + \varphi(Tx) \leq \varphi(x)$ for all $x \in X$. However $\varphi(x) \neq \varphi(Tx)$ for all $x \in X$. This contradiction concludes the proof.

(3) $\Rightarrow$ (1). We shall proceed by contradiction. Suppose that $(X, d)$ is not right K-sequentially complete. By the above lemma, there exists a right K-Cauchy sequence $(x_n)_{n\in\omega}$ in $(X, d)$ without $\tau_d$-cluster points, and we can assume, without loss of generality, that $x_n \neq x_m$ for all $n, m \in \mathbb{N}$ with $n \neq m$, and that $d(x_{n+1}, x_n) < 2^{-(n+1)}$ for all $n \in \mathbb{N}$.

Define $f : X \to \mathbb{R}$ as $f(x_n) = 2^{-(n-1)}$ for all $n \in \mathbb{N}$, and $f(y) = 2$ for all $y \in X \backslash \{x_n : n \in \mathbb{N}\}$. Then $f$ is nearly $d$-lsc. Indeed, let $(y_k)_{k\in\mathbb{N}}$ be a sequence of distinct points in $X$ that $\tau_d$-converges to some $y \in X$. Then, there is $k_0 \in \mathbb{N}$ such that $y_k \in X \backslash \{x_n : n \in \mathbb{N}\}$ for all $k \geq k_0$. Hence $f(y_k) = 2$ for all $k \geq k_0$, and consequently $f(y) \leq f(y_k)$ for all $k \geq k_0$.

Now define $A := \{x \in X : f(x) \leq \inf f(X) + 1\}$. Then $A = \{x_n : n \in \mathbb{N}\}$, and, for each $n \in \mathbb{N}$, we have

$$f(x_{n+1}) + d(x_{n+1}, x_n) < 2^{-n} + 2^{-(n+1)} < 2^{-(n-1)} = f(x_n),$$

which contradicts condition (ii') from (3) for $f$ as defined above and $\varepsilon = 1$. The proof is complete. ■

**Corollary 2.** *For a $T_1$ quasi-metric space $(X, d)$ the following conditions are equivalent:*

(1) $(X, d)$ *is right K-sequentially complete.*

(2) *For every self mapping $T$ of $X$ and every proper bounded below $d$-lsc function $\varphi : X \to \mathbb{R} \cup \{\infty\}$ satisfying $d(Tx, x) + \varphi(Tx) \leq \varphi(x)$ for all $x \in X$, there exists $z = z_{T,\varphi} \in X$ such that $z = Tz$, i.e., $T$ has a fixed point.*

(3) *For every proper bounded below $d$-lsc function $f : X \to \mathbb{R} \cup \{\infty\}$ and for every $\varepsilon > 0$ there exists $y_\varepsilon \in X$ such that*

(i) $f(y_\varepsilon) \leq \inf f(X) + \varepsilon$;

(ii) $f(y_\varepsilon) < f(x) + \varepsilon d(x, y_\varepsilon)$ *for all* $x \in X \backslash \{y_\varepsilon\}$.

*Proof.* (1) ⇒ (2). By Theorem 2, (1) ⇒ (2), there exists $z \in X$ such that $\varphi(Tz) = \varphi(z)$. Hence $d(Tz, z) = 0$. Since $(X, d)$ is $T_1$ we conclude that $Tz = z$.

(2) ⇒ (3) and (3) ⇒ (1) follow directly from the aforementioned fact that a proper function $f : X \to \mathbb{R} \cup \{\infty\}$ on a $T_1$ quasi-metric space $(X, d)$ is $d$-lsc if and only if it is nearly $d$-lsc, and Theorem 2, (2) ⇒ (3) and (3) ⇒ (1), respectively. ■

**Remark.** (1) ⇒ (2) in Corollary 2, which was proved by Cobzaş [3, Theorem 2.12] in a multivalued version, provides the celebrated Caristi's fixed point theorem [2] in case that $(X, d)$ is a metric space with $\varphi : X \to [0, \infty)$. Note that, in that case, the equivalence (1) ⇔ (2) in Corollary 2, provides the well-known characterization of the metric completeness obtained by Kirk in [8].

The following example shows that condition "nearly $d$-lsc" cannot be replaced with "$d$-lsc" in Theorem 2.

**Example 1.** Let $d$ be the quasi-metric on $\mathbb{N}$ given as $d(n, n) = 0$ for all $n \in \mathbb{N}$; $d(2n - 1, 2m - 1) = 0$ for all $n, m \in \mathbb{N}$ with $n > m$; $d(2n, 2m - 1) = 0$ for all $n, m \in \mathbb{N}$ with $2n > 2m - 1$, and $d(n, m) = 1$ otherwise. Then $(\mathbb{N}, d)$ is not right K-sequentially complete because $(2n - 1)_{n \in \mathbb{N}}$ is a right K-Cauchy sequence that is not $\tau_d$-convergent in $X$. Now let $f : \mathbb{N} \to \mathbb{R} \cup \{\infty\}$ be a proper bounded below $d$-lsc function and let $\varepsilon > 0$. Since $d(2n, 2n - 1) = 0$ and $f$ is $d$-lsc, we have $f(2n) \leq f(2n - 1)$ for all $n \in \mathbb{N}$. Hence, there exists $k \in \mathbb{N}$ such that $f(2k) < \inf f(X) + \varepsilon$. Thus $f(2k) < f(n) + \varepsilon = f(n) + \varepsilon d(n, 2k)$ for all $n \in \mathbb{N} \backslash \{2k\}$. Consequently $f$ satisfies conditions (i) and (ii') from (3) in Theorem 2, with $y_\varepsilon = 2k$ (observe that $\overline{\{2k\}} = \{2k\}$).

We conclude with an easy example which shows that (1) $\Rightarrow$ (2) in Corollary 2, cannot be generalized to right K-sequentially complete quasi-metric spaces. It also shows that condition (3) in Theorem 2 cannot be replaced with condition (3) in Corollary 2, not even for $f$ $d$-lcs.

**Example 2.** Let $X$ be the set of all ordinals less than the first uncountable ordinal number $\omega_1$. Let $d$ be the quasi-metric on $X$ given as $d(x,y) = 0$ if $y \leq x$, and $d(x,y) = 1$ otherwise. Clearly $(X,d)$ is right $K$-sequentially complete because every non-eventually constant right $K$-Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ is $\tau_d$-convergent to the element of $X$, $\sup\{x_n : n \in \mathbb{N}\}$. Define $T : X \to X$ as $Tx = x+1$ for all $x \in X$. Then $T$ has no fixed point. However, $d(Tx,x) = 0$ for all $x \in X$, so $d(Tx,x) = \varphi(x) - \varphi(Tx)$ for all $x \in X$, where the function $\varphi$ is the zero function of $X$. Therefore, the implication (1) $\Rightarrow$ (2) in Corollary 2, cannot be generalized to right K-sequentially complete quasi-metric spaces. Finally, define $f(x) = 0$ for all $x \in X$. Obviously, $f$ is a (proper) bounded below $d$-lsc function on $X$. Take $\varepsilon = 1$. Then $f(x) < \inf f(X) + \varepsilon$ for all $x \in X$. However, given $x \in X$, one has $d(z,x) = 0$ whenever $x < z$, so $f(x) = f(z) + \varepsilon d(z,x)$ whenever $x < z$. Consequently condition (3) of Theorem 2 cannot be replaced by condition (3) of Corollary 2.

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